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ON SLIPPAGE TESTS—(I)[1] A GENERALIZATION OF NEYMAN-PEARSON'S LEMMA

By Irving J. Hall[2] and Akio Kudô[3]

Iowa State University

1. Introduction. An important class of multiple decision problems is the class of slippage problems. Although they can be viewed as one area of general decision function theory [17] these problems have been mainly treated in a manner similar to the treatment of problems in hypotheses testing [8].

Slippage problems were first introduced by Mosteller [11] as a problem of testing homogeneity of a number of populations against the slippage alternatives that exactly one of the populations is different. Paulson [13], while treating the slippage problem of normal mean, was the first to formulate the problem satisfactorily. Some of the later papers along this line proved optimum properties of procedures already existing.

The results of Kudo [6] proved that an outlier test proposed by Pearson and Chandrasekar [14] in 1936 and investigated by Nair [12], Grubbs [4], and Smirnoff [15], was "optimum" within a certain class of tests and similarly the results of Truax [16] proved that a test for homogeneity of normal variances proposed by Cochran [2] in 1941 was "optimum" in a certain sense. Others papers include [3], [7], [9].

A prototype of classical slippage problems is as follows. We assume we have a populations with densities $f(x; \theta_i)$ $(i = 1, \cdots, a)$ and we wish to test the hypothesis $H_0 : \theta_1 = \cdots = \theta_a$ against a alternatives $H_i : \theta_1 = \cdots = \theta_i - \Delta = \cdots = \theta_a$ with a zero-one type of loss function where $\Delta > 0$. Let D_i be the decision to accept H_i and $\Pr(D_i \mid H_j)$ be the probability of taking D_i when H_j is true. We impose the requirements that

$$\Pr(D_0 \mid H_0) \geqq 1 - \alpha \quad \text{where} \quad \alpha \,\varepsilon\, (0, 1)$$

and

$$\Pr(D_i \mid H_i) \quad \text{is independent of } i,$$

or more explicitly speaking $\Pr(D_i \mid H_i)$ is a function of Δ but does not depend on i. We call the first a size condition and the second a symmetry condition.

It seems that these two conditions have been regarded as being insufficient for ensuring an explicit solution and thus the condition of invariance on the procedures (e.g., invariant under change of scale and shift of location) was im-

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[1] This research was supported by National Science Foundation Grants GP-3918 and GM-6149.

[2] Presently at Sandia Corporation, Albuquerque, New Mexico.

[3] Presently at Kyushu University, Fukuoka, Japan.

posed. When the invariance condition is imposed, a transformation group $G = \{g\}$ defined on the sample space must leave the problem invariant and the decision function $\varphi(x)$ satisfies the relation $\varphi(x) = \varphi(gx)$. These conditions are also imposed by Karlin and Truax [5] when they considered slippage problems employing a more general loss function.

The main purpose of this paper is to discuss slippage problems without assuming the condition of invariance of the above type.

For the sake of simplicity we concentrate our attention to a zero-one loss function. In this situation we should pay attention to the probability of correct decisions which allows us to use the terminology of hypothesis testing quite analogously.

In Section II a generalization of the Neyman-Pearson lemma to slippage problems is given. As in the case of hypothesis testing this allows us to solve problems where the null hypothesis is simple and the alternatives are simple and in some cases where the alternatives are composite. When the null hypothesis is composite the generalization of the Neyman-Pearson lemma can be used in some cases to obtain a uniformly most powerful test by introducing an adequate least favorable probability distribution over the space of the null hypothesis.

For some slippage problems a uniformly most powerful test does not exist; however, under the conditions of Theorem 2 we can be assured of the existence of a uniformly most powerful test.

It seems that a number of papers on slippage problems has been the repeated application of Corollary 2 after imposing the condition of invariance, and thus reducing the problem to one on the space of a maximal invariant.

2. Generalization of Neyman-Pearson Lemma and results. Suppose we have a sample space $\mathfrak{X}$, a σ-field $\mathfrak{L}$, with a countable number of generators, and a σ-finite measure μ, where $a + 1$ densities $p_i(x)$, $(i = 0, 1, \cdots, a)$ with respect to μ are defined. We consider a decision problem involving $a + 1$ hypothesis H_i that X has density $p_i(x)(i = 0, 1, \cdots, a)$.

We assume that there is a transformation group G on $\mathfrak{X}$ isomorphic to Π where Π is the permutation group on $(1, 2, \cdots, a)$ or its subgroup transitive on $(1, 2, \cdots, a)$. As G is finite, there is a right invariant probability measure ν on it. We further assume $\mu(A) = \mu(gA)$ for all $A \varepsilon \mathfrak{L}$ and $g \varepsilon G$ and $p_i(x) = p_{\pi_g i}(gx)$ for all $g \varepsilon G$ where π_g is the permutation $(1, \cdots, a) \rightarrow (\pi_g 1, \cdots, \pi_g a)$ corresponding to g, and $\pi_g 0 = 0$.

For the sake of demonstration, we consider the slippage problem of normal mean as treated by Paulson [13] where we have a normal populations with common variance. There are $a + 1$ hypothesis concerning the means, namely H_0 : all the means are equal, and H_i : all the means are equal except for the ith one which is larger than the others $(i = 1, \cdots, a)$. We take n observations from each population. The minimal sufficient statistic has $a + 1$ components; the first a being the sample means and the last component the sum of squares of all the observations. In this case G is the permutation group of the first a components, and G has $a!$ elements and the probability measure ν assigns $1/a!$ to each element. All the other assumptions are easily seen to be satisfied.

Another example is as follows: We consider uniform distributions on $a + 1$ unit circles C_i $(i = 0, 1, \cdots, a)$ with the centers $(0, 0)$ for C_0, and

$$(r \cos 2\pi(i - 1)/a, \; r \sin 2\pi(i - 1)/a) \text{ for } C_i \; (i = 1, \cdots, a).$$

In this case the group consists of rotations around the origin by the multiple of $2\pi/a$.

A decision function $\varphi(x) = (\varphi_0(x), \cdots, \varphi_a(x))$ is of *size* α if

$$(1) \qquad \int \varphi_0(x)P_0(x)\, d\mu(x) \geqq 1 - \alpha, \; \alpha \, \varepsilon \, (0, 1)$$

$\varphi(x)$ is of *exact size* α if equality holds. $\varphi(x)$ is *symmetric in power* if

$$(2) \qquad \int \varphi_1(x)p_1(x)\, d\mu(x) = \cdots = \int \varphi_a(x)p_a(x)\, d\mu(x).$$

The common value will be called the power of φ.

$\varphi(x)$ is the most powerful symmetric size α(MPSSα) decision function if it maximizes each term in (2) subject to condition (1) and (2). The above terminologies are concordant with the traditional ones in slippage problems. If $\varphi(x)$ satisfies

$$\varphi_i(x) = \varphi_{\pi_g i}(gx) \quad i = 0, \cdots, a \quad \text{for all} \quad g \, \varepsilon \, G$$

we say that it is invariant with respect to G. It is emphasized here that it is different from the traditional assumption of invariance, $\varphi(x) = \varphi(hx)$: where h, for instance, represents a change of location and/or scale. There is a relationship between decision functions which are symmetric in power and invariant decision functions as seen by the following lemma.

LEMMA 1. *If φ is invariant then φ is symmetric in power. Also if there exists a (an exact) size α test, φ, then there exists a (an exact) size α test, ψ, which is invariant, and if φ is symmetric in power then ψ can be chosen to have the same power as φ.*

PROOF. The first statement is immediate. Define $\psi_i(x) = \int \varphi_{\pi_g i}(gx)\, d\nu(g)$, $(i = 0, \cdots, a)$, then $\psi(x)$ is invariant. If φ is of (exact) size α, then

$$\int \psi_0(x)p_0(x)\, d\mu(x) = \int\int \varphi_0(gx)p_0(x)\, d\mu(x)\, d\nu(g) (=) \geqq 1 - \alpha.$$

Also if φ is symmetric in power then

$$P(i, \psi) = \int \psi_i(x)p_i(x)\, d\mu(x) = \int\int \varphi_{\pi_g i}(gx)p_i(x)\, d\mu(x)\, d\nu(g) = P(i, \varphi).$$

Let Φ denote the class of decision functions which are of size α and let Ψ denote the class of decision functions which are symmetric in power, and let $P(i, \varphi)$ be the probability of making the correct decision when H_i is correct. Then Lemma 2 shows that a kind of restricted minimax solution, as in [8] can be found in $\Phi \cap \Psi$.

LEMMA 2. *There exists a φ' in $\Phi \cap \Psi$ such that*

$$(3) \qquad \sup_{\varphi \varepsilon \Phi} \inf_{i=1,\cdots,a} P(i, \varphi) = \inf_{i=1,\cdots,a} P(i, \varphi').$$

PROOF. The existence of φ' in Φ satisfying (3) follows from the weak compactness of Φ which can be proved in a manner which is exactly similar to Theorem 3 of Appendix [8]. That a solution can be found in $\Phi \cap \Psi$ can be proved as follows. Let φ' satisfy (3) and define ψ the same as in Lemma 1, then $P(i, \psi)$ is independent of i, $(i = 1, \cdots, a)$. Thus

$$\inf_{i=1,\cdots,a} P(i, \psi) = P(i, \psi) = \int\int \varphi'_{\pi_g i}(gx)p_i(x)\, d\mu(x)\, d\nu(g)$$
$$= a^{-1}\sum_{j=1}^{a} P(j, \varphi') \geqq \inf_{i=1,\cdots,a} P(i, \varphi').$$

The following theorem is an extension of the Neyman-Pearson Lemma in hypotheses testing to slippage problems, whose proof is essentially similar to that of Bahadur-Goodman Theorem [1], [10].

THEOREM 1. *Consider a rule* φ *of the following form:*

$$\varphi_0(x) = 1, \xi(x), 0 \quad \text{if} \quad \max_i p_i(x) <, =, > Cp_0(x),$$
$$(4) \qquad \varphi_j(x) = \eta_j(x), 0 \quad \text{if} \quad p_j(x) =, < \max p_i(x),$$
$$i = 1, \cdots, a,$$

where $\xi(x)$ *and* $\eta_j(x)$ *are arbitrary, subject to the condition that* φ *is a decision function and* C *is a constant.*

Let $\hat{\varphi}$ *be any other rule.*

(i) *If* $E_0\hat{\varphi}_0(X) \geqq E_0\varphi_0(X)$ *then* $\sum_{i=1}^{a} E_i\hat{\varphi}_i(X) \leqq \sum_{i=1}^{a} E_i\varphi_i(X)$.

(ii) *If* $E_0\hat{\varphi}_0(X) \geqq E_0\varphi_0(X)$ *and* $\sum_{i=1}^{a} E_i\hat{\varphi}_i(X) = \sum_{i=1}^{a} E_i\varphi_i(X)$, *then* $\hat{\varphi}$ *has the form* (4) a.e. *Furthermore,* $E_0\hat{\varphi}_0(X) = E_0\varphi_0(X)$ *unless*

$$E_j\varphi_0(X) = 0, \qquad j = 1, \cdots, a.$$

(iii) *For every* $\alpha \varepsilon (0, 1)$ *there is a rule of the form* (4) *with* $\xi(x)$ *constant, say* ξ_α, *such that* $E_0\varphi_0(X) = 1 - \alpha$.

PROOF.

(i) Let

$$(5) \quad L(\varphi, C) = E_0[C\varphi_0(X)] + \sum_{i=1}^{a} E_i[\varphi_i(X)] = \int [\varphi_0(x)Cp_0(x) + \sum_{i=1}^{a} \varphi_j(x)p_i(x)]\, d\mu(x)$$

consider $L(\varphi, C) - L(\hat{\varphi}, C) = \int g(x)\, d\mu(x)$. It is easily verified that $g(x) \geqq 0$ a.e.μ. Hence

$$\sum_{i=1}^{a} E_i[\varphi_i(X) - \hat{\varphi}_i(X)] \geqq CE[\varphi_0(X) - \varphi_0(X)] \geqq 0.$$

(ii) Let $f_0 = Cp_0(x)$ and $f_i = p_i(x)$, A denote the set of integers $(0, 1, \cdots, a)$, $\mathcal{S}$ all non-void subsets of A and $J(C)$ be an element of $\mathcal{S}$ such that

$$J(C) = \{j_i, \cdots, j_k\} \text{ if } \max_{i \varepsilon A} f_i = f_{j_1} = \cdots = f_{j_k}.$$

Suppose $\hat{\varphi}$ is any other rule such that (ii) is true and define

$$R_J = \{x : \max_{i \varepsilon A} f_i = f_{j_i} = \cdots = f_{j_k}, J(C) = \{j_i = \cdots, = j_k\}$$

and for some $j \varepsilon J(C)$, $\hat{\varphi}_j(X) \neq 0\}$.

Suppose $\mu(R_J) > 0$ for some J and consider $L(\varphi, C) - L(\hat{\varphi}, C)$;

$$L(\varphi, C) - L(\hat{\varphi}, C) = \sum_{J\varepsilon\mathcal{S}} \int_{R_J} \sum_{j=0}^{a} [\varphi_j(X) - \hat{\varphi}_j(X)]f_j(X)\, d\mu(X)]$$

where $L(\hat{\varphi}, C)$ is defended in (5). It can be shown that the integral, $g_j(x)$ for any J is positive. Therefore,

$$\sum_{i=1}^{a} E_i(\varphi_i(X) - \hat{\varphi}_i(X)) > E_0(\hat{\varphi}_0(X) - \varphi(X)) \geqq 0,$$

which gives a contradiction and the first part of (ii) is proved.

To prove the second part of (ii) suppose there exists a rule $\hat{\varphi}$ as in (ii) such that $E_0\hat{\varphi}_0(X) > E_0\varphi_0(X)$ and $E_j\hat{\varphi}_0(X) > 0$ for some j $(j = 1, \cdots, a)$. Define:

$$(6) \qquad \begin{aligned} \psi_0(x) &= 0, (1 - \lambda)\hat{\varphi}_0(x) && \text{if} \quad \hat{\varphi}_0(x) =, >0, \quad \lambda \varepsilon (0, 1), \\ \psi_i(x) &= 1, \hat{\varphi}_i(x) + (\lambda/a)\hat{\varphi}_0(x) && \text{if} \quad \hat{\varphi}_i(x) =, <1. \end{aligned}$$

Now

$$\begin{aligned} E_0\psi_0(X) &= \int_{\{x:\varphi_0=0\}} \hat{\varphi}_0(x)p_0(x)\, d\mu(x) + (1 - \lambda)\int_{\{x:\varphi_i>0\}} \hat{\varphi}_0(x)p_0(x)\, d\mu(x) \\ &= (1 - \lambda)E_0\hat{\varphi}_0(x) = E_0\varphi_0(X) \text{ for some } \lambda \varepsilon (0, 1). \end{aligned}$$

Also,

$$\begin{aligned} \sum_{i=1}^{a} E_i\psi_i(X) &= \sum_{i=1}^{a}\int_{\{x:\varphi_i=1\}} \hat{\varphi}_i(x)p_i(x)\, d\mu(x) \\ &\quad + \sum_{i=1}^{a}\int_{\{x:\varphi_i<1\}} [\hat{\varphi}_i(x) + (\lambda/a)\hat{\varphi}_0(x)]p_i(x)\, d\mu(x) \\ &= \sum_{i=1}^{a} E_i\hat{\varphi}_i(X) + \lambda/a \sum_{i}^{a} E_i\hat{\varphi}_0(X), \end{aligned}$$

and since the last term is positive, we have our result.

(iii) This can be proved in a manner similar to the proof of Theorem 1 in Chapter 3 [8].

Corollary 1. *For any α there is a* MPSS *α decision function.*

Proof. Consider a rule as given in Theorem 1. This can be made invariant by taking $\eta_j(x) = (1 - \xi_\alpha)/k(x)$ where $k(x)$ is the number of times $\max_{i=1,\cdots,a} p_i(x)$ is attained, and this is a MPSS α decision function.

It is quite revealing, and also amusing, to apply this theorem to the example of $a + 1$ unit circles stated in the beginning of this section. There is no problem when these $a + 1$ circles do not intersect each other. If they do, and moreover the probability content of $C_0 \cap (\cup_{i=1}^{a} C_i)$ is less than α, then there is a test with size less than α and with the maximum power. In this case, an increase in size no longer contributes to an increase in power.

Corresponding to the theory of hypotheses testing, H_0 and H_i $(i = 1, \cdots, a)$ may well be composite and the notion of uniformly most powerful symmetric size α decision function can be introduced.

As in the case of hypothesis testing, Theorem 1 can be applied to derive a uniformly most powerful decision function when H_i and/or H_0 are composite. In case H_0 is composite, Theorem 1 is applicable by introducing an adequate least favorable probability distribution over the space of the null hypothesis.

Example 1. We assume here that we have n random observations

$(x_{i1}, \cdots, x_{in})(i = 1, \cdots, a)$ from each of a $N(\theta_i, \sigma)$ populations (σ known) and we wish to decide if all the means are equal or if one mean is larger than the rest. Namely, $H_0 : \theta_1 = \cdots = \theta_a = \theta$ and $H_i : \theta_1 = \cdots = \theta_i - \Delta = \cdots = \theta_a = \theta$ where $\Delta > 0$ and both θ and Δ are unknown. Consider the subproblem where Δ and θ are specified under H_i $(i = 1, \cdots, a)$. H_0 is made simple by placing a degenerate a priori distribution on θ at $\theta_0 = \theta + \Delta/a$.

Theorem 1 now gives the most powerful symmetric size α decision function, φ, of this reduced problem where

(7) $$\varphi_0 = 1 \quad \text{if} \quad \max_i (\bar{x}_i - \bar{\bar{x}})\sigma^{-1} < C \qquad (i = 1, \cdots, a),$$

(8) $$\varphi_j = 1 \quad \text{if} \quad \max_i (\bar{x}_i - \bar{\bar{x}})\sigma^{-1} = \bar{x}_j - \bar{\bar{x}}\sigma^{-1} > C,$$

where $\bar{x}_i = \sum_{j=1}^{n} x_{ij}/n$, $\bar{\bar{x}} = \sum_{i=1}^{a} \bar{x}_i/a$ and C is chosen to satisfy the size conditions. Since φ is independent of the specified value of θ and Δ under H_i it is the uniformly MPSSα decision function.

A uniformly most powerful symmetric size α decision function does not always exist. However, under certain conditions, we can be assured of the existence of a uniformly most powerful symmetric size α decision function as seen by Corollary 2. Corollary 2 is essentially included in the paper by Karlin and Truax [5], but is not stated explicitly and we shall state it here for the sake of clarity and completeness.

Let $p(x; \theta)$ be a family of densities of a random variable X which are indexed by a parameter θ where θ is a point in Euclidean space. Assume the existence of a curves in the parameter space which are given by $\theta = \theta_i(\tau)$, $0 < \tau < \infty$ $(i = 1, \cdots, a)$ with the common starting point $\theta_0 = \theta_i(0)$. Assume also that (a) $p(x; \theta_i(\tau))/p(x; \theta_0)$ is nondecreasing in a real valued function $T_i(x)$ $(i = 1, \cdots, a)$ (b) $p(x; \theta_i(\tau)) >, =, < p(x; \theta_j(\tau))$ for all τ as $T_i(x) >, =, < T_j(x)$.

Conditions (a) and (b) are essentially the requirements that the density $p(x; \theta_i(\tau))$ has monotone likelihood ratio on the curves $\theta = \theta_i(\tau)$.

THEOREM 2. *Consider a rule* φ *of the form*

(9) $$\varphi_0(x) = 1, \xi(x), 0 \quad \text{if} \quad \max_i T_i <, =, > C,$$
$$\varphi_j(x) = \eta_j(x), 0 \qquad \text{if} \quad T_j =, < \max_i P_i(x),$$

where $\xi(x)$ *and* $\eta_j(x)$ *are arbitrary and* C *is a constant. If* $\hat{\varphi}$ *is any other rule with* $E_0\hat{\varphi}_0(X) \geqq 1 - \alpha$, *then*

$$\sum_{j=1}^{a} E_j\hat{\varphi}_j(X) \leqq \sum_{j=1}^{a} E_j\varphi_j(X).$$

This is proved by using Theorem 1.

If we now assume a transformation group G on $\mathfrak{X}$ as we did at the beginning of this section which is isomorphic to the permutation group Π on $(1, \cdots, a)$ such that $p(x; \theta_i(\tau)) = p(gx; \theta_{\pi_g i}(\tau))$ for all $\tau \varepsilon (0, \infty)$, all $g \varepsilon G$, all $X \varepsilon \mathfrak{X}$, and $\mu(A) = \mu(gA)$ for all $g \varepsilon G$, and that $T_i(x) = T_{\pi_g i}(gx)$ for all $g \varepsilon G$ $(i = 1, \cdots, a)$, we get the following corollary.

COROLLARY 2. *Under the conditions above, for any* $\alpha \varepsilon (0, 1)$ *there is a rule of the form* (9) *with* $\eta_j(x)$ *constant, which is* MPSS α *uniformly in* τ *for testing* $H_0 : \theta = \theta_0$ *against* $H_j : \theta = \theta_j(\tau)$ $(j = 1, \cdots, a)$.

It seems that a number of results in slippage problems are the repeated application of Corollary 2 after assuming invariance in a suitable manner. $\mathfrak{X}$ is then the space of a maximal invariant.

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