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ON A GENERALIZED SAVAGE STATISTIC WITH APPLICATIONS TO LIFE TESTING[1]

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0. Summary. Let there be two samples $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n (N = m + n)$ from two populations with continuous cdf's $F(x)$ and $G(y)$. Let the first i ordered observations (out of N combined observations) contain m_i x's and n_i y's $(m_i + n_i = i)$ where m_i and n_i are random numbers. To test

$$(0.1) \qquad H_0 : F = G$$

against alternative that they are different we propose the statistic

$$(0.2) \qquad S_r^{(N)} = \sum_{i=1}^{r} a_i z_i + (m - m_r)(N - r)^{-1}\left(\sum_{r+1}^{N} a_i\right) - \tfrac{1}{2}(m + n)$$

based on the first r ordered observations only where

$$a_i^N = a_i = \sum_{j=N-i+1}^{N} 1/j,$$

and

$$z_i = 1, \quad \text{if the } i\text{th ordered observation is an } x_i,$$
$$= 0, \quad \text{otherwise.}$$

The statistic is the asymptotically most powerful rank test for censored data under the Lehmann alternative and is equivalent to the Savage statistic [14] when $r = N$. It is also known to maximize the minimum power over IFRA (or IFR) distributions asymptotically. Exact and large sample properties of $S_r^{(N)}$ are studied and a k-sample extension of it is also considered. Various tables are also provided to facilitate the use of the $S_r^{(N)}$ statistic.

1. Introduction. Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be two independent samples of sizes m and n respectively from two populations with continuous cumulative distribution functions (cdf's) $F(x)$ and $G(y)$, where F and G belong to the same family F of distribution functions indexed by a parameter θ. Let all the $m + n = N$ observations be ordered in a sequence and we want to test the hypothesis

$$(1.1) \qquad H_0 : F = G$$

against the alternative that they are different based on (at most) the first r out of the combined sample of N observations. That is we have a right censored sample of size at most r. Such a problem arises naturally in many fields as for

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example, in problems of life-testing where we are interested in comparing the mean life of items produced by two physical processes or in problems of biological assay where we can not afford to wait until all the observations are available.

To test the above hypothesis Sobel [15], [16] proposed two statistics whose large sample properties were studied by Basu [2], [3]. In [3] it was shown that in the exponential case a modified version of the statistic proposed by I. R. Savage [14] performs best. Also it is well known (see for example Savage [14], Capon [7], Hájek [11] and Basu and Woodworth [5]) that the Savage statistic is the asymptotically locally most powerful rank test under the Lehmann alternative which include, as special cases, the exponential and the Weibull distribution—the two most commonly used models in life testing. Recently Doksum [9] has shown that the Savage statistic maximizes the minimum power over the class of distributions with increasing failure rate averages (IFRA), or over the class of distributions with increasing failure rates (IFR), asymptotically. (For definition of IFRA and IFR distribution see Birnbaum, Esary and Marshall [6], Barlow and Proschan [1], p. 23.)

In view of the above findings it seems desirable to study the generalized Savage statistic to be defined later (based on only the first r ordered observations $r \leqq N$). Our study closely parallels Sobel's work [16] in which a generalized Wilcoxon statistic has been studied.

In Section 2 we have defined $S_r^{(N)}$, a generalization of the Savage statistic, based on the first r observations only. The exact and asymptotic distribution of $S_r^{(N)}$ is given in Section 3. A curtailed form of $S_r^{(N)}$, suitable for life testing problems, is discussed in Section 4 and in Section 5 we compare the $S_r^{(N)}$ test with other life tests on the basis of their curtailed forms. Finally, in Section 6 a k-sample extension of the $S_r^{(N)}$ statistic is also considered.

Two other generalizations of the Savage statistic have been proposed previously by Gastwirth [10] and Rao, Savage and Sobel [13]. However, Gastwirth did not consider any explicit form for small samples and $S_r^{(N)}$ is shown to be asymptotically equivalent to his statistic. On the other hand, while $S_r^{(N)}$ and $R_r^{(N)}$, the statistic proposed by Rao, Savage and Sobel, perform comparably in small samples, the large sample properties of $R_r^{(N)}$ are not known. The above reasons also justify the introduction of the $S_r^{(N)}$ statistic.

2. Definition of the generalized Savage statistic. Let the first i ordered observations (out of the N combined observations) contain m_i x's and n_i y's $(m_i + n_i = i)$ where m_i and n_i are random numbers $(i = 1, 2, \cdots, r)$. Also let

$$(2.1) \qquad z_i = 1, \quad \text{if the } i\text{th ordered observation is an } x,$$
$$= 0, \quad \text{otherwise} \quad (i = 1, 2, \cdots, N).$$

Then to test the hypothesis (1.1) we propose the statistic $S_r^{(N)}$ where

$$(2.2) \quad S_r^{(N)} = \sum_{i=1}^{r} a_i z_i + (m - m_r)(N - r)^{-1}\left(\sum_{r+1}^{N} a_i\right) - \tfrac{1}{2}(m + n), \qquad (r < N)$$

and

$$a_i \equiv a_i^N = \sum_{j=N-i+1}^{N} j^{-1}. \tag{2.3}$$

It is obvious that for $r = N$, $S_r^{(N)}$ reduces to the Savage statistic S where, by convention, $(m - m_r)(N - r)^{-1} = 0$.

Table I gives the weight function a_i^N which may be used for computing the statistic $S_r^{(N)}$ for $N \leqq 20$.

TABLE I

Weights a_i^N used in computing the statistic $S_r^{(N)}$

i	N = 1	2	3	4	5	6	7	8	9	10
1	1.0000	0.5000	0.3333	0.2500	0.2000	0.1667	0.1429	0.1250	0.1111	0.1000
2		1.5000	0.8333	0.5833	0.4500	0.3667	0.3095	0.2679	0.2361	0.2111
3			1.8333	1.0833	0.7833	0.6167	0.5095	0.4345	0.3790	0.3361
4				2.0833	1.2833	0.9500	0.7595	0.6345	0.5456	0.4790
5					2.2833	1.4500	1.0929	0.8845	0.7456	0.6456
6						2.4500	1.5929	1.2179	0.9956	0.8456
7							2.5929	1.7179	1.3290	1.0956
8								2.7179	1.8290	1.4290
9									2.8290	1.9290
10										2.9290

i	N = 11	12	13	14	15	16	17	18	19	20
1	0.0909	0.0833	0.0769	0.0714	0.0667	0.0625	0.0588	0.0556	0.0526	0.0500
2	0.1909	0.1742	0.1603	0.1484	0.1381	0.1292	0.1213	0.1144	0.1082	0.1026
3	0.3020	0.2742	0.2512	0.2317	0.2150	0.2006	0.1880	0.1769	0.1670	0.1582
4	0.4270	0.3854	0.3512	0.3226	0.2984	0.2775	0.2594	0.2435	0.2295	0.2170
5	0.5699	0.5104	0.4623	0.4226	0.3893	0.3609	0.3363	0.3150	0.2962	0.2795
6	0.7365	0.6532	0.5873	0.5337	0.4893	0.4518	0.4197	0.3919	0.3676	0.3462
7	0.9365	0.8199	0.7301	0.6587	0.6004	0.5518	0.5106	0.4752	0.4445	0.4176
8	1.1865	1.0199	0.8968	0.8016	0.7254	0.6629	0.6106	0.5661	0.5279	0.4945
9	1.5199	1.2699	1.0968	0.9682	0.8682	0.7879	0.7217	0.6661	0.6188	0.5779
10	2.0199	1.6032	1.3468	1.1682	1.0349	0.9307	0.8467	0.7773	0.7188	0.6688
11	3.0199	2.1032	1.6801	1.4182	1.2349	1.0974	0.9896	0.9023	0.8299	0.7688
12		3.1032	2.1801	1.7516	1.4849	1.2974	1.1562	1.0451	0.9549	0.8799
13			3.1801	2.2516	1.8182	1.5474	1.3562	1.2118	1.0977	1.0049
14				3.2516	2.3182	1.8807	1.6062	1.4118	1.2644	1.1477
15					3.3182	2.3807	1.9396	1.6618	1.4644	1.3144
16						3.3807	2.4396	1.9951	1.7144	1.5144
17							3.4396	2.4951	2.0477	1.7644
18								3.4951	2.5477	2.0977
19									3.5477	2.5977
20										3.5977

An interesting feature of the statistic $S_r^{(N)}$ is that once all the x's (or all the y's) are available the value of $S_r^{(N)}$ remains unchanged (as it should be intuitively) as can be easily seen that if $m_r = m$ or $n_r = n$ for some $r = r_0$ then $S_r^{(N)} = S_{r_0}^{(N)}$ for all $r \geqq r_0$.

3. Distribution of $S_r^{(N)}$. In this section we shall find the exact and the large sample distribution of $S_r^{(N)}$. To this end it seems desirable to have some idea about the extreme values of $S_r^{(N)}$. Clearly for fixed N the a_i^N's are increasing functions of i $(1 \leqq i \leqq N)$. We can use the above fact to prove the following:

LEMMA 3.1. *The minimum value of $S_r^{(N)}$ is attained if the first $\nu = \min(m, r)$ observations are all x's $(m_\nu = \nu)$ and is given by*

$$S_r^{(N)}(\min) = \sum_1^m a_i - \tfrac{1}{2}(m+n) \qquad (r \geqq m)$$

$$(3.1) \qquad = \sum_1^r a_i + (m-r)(N-r)^{-1}\left(\sum_{r+1}^N a_i\right) - \tfrac{1}{2}(m+n) \qquad (r < m).$$

The maximum value of $S_r^{(N)}$ can be obtained by interchanging the x's with the y's in the above statement.

PROOF. Since the a_i's are increasing functions of i and $\sum_{r+1}^N a_i/(N-r) > a_j$ $(j = 1, 2, \cdots, r)$ it is clear that $S_r^{(N)}$ will be minimized by minimizing $\{(m - m_r)/(N-r)\}. \sum_{r+1}^N a_i$ and making $\sum_1^r a_i z_i$ as small as possible. The proof for the maximum value of $S^{(N)}$ also follows similarly. It can be easily seen that for $m = n$, $\max(S_r^{(N)}) + \min(S_r^{(N)}) = 0$ for all r.

The exact distribution of $S_r^{(N)}$ under the null hypothesis can be easily found for small values of m, n and r. Since under the null hypothesis the probability of any sequence $(z_1, z_2, \cdots, z_r)$ is given by $\binom{N-r}{m-m_r}/\binom{N}{m}$ (for proof see Rao, Savage and Sobel [13]).

It should be noted that for $m = n$ the distribution of $S_r^{(N)}$ is symmetrical for any $r \leqq N$, since for any sequence $Z = (z_1, z_2, \cdots, z_r)$ we can find a dual sequence $Z^* = (1 - z_1, 1 - z_2, \cdots, 1 - z_r)$ by interchanging the x's and y's. In Table II using above results we have tabulated the tails of the exact distribution of $S_r^{(N)}$ for $m = n = r = 4\ (1)\ 8$, $m = n = 4$, $r = 5, 6$, $m = n = 5$, $r = 6, 7, 8$ and $m = n = 6$, $r = 7$. Because of the symmetry of the distributions it is enough to tabulate only one half of the table, that is, only the positive values of $S_r^{(N)}$ (say). For the special case $r = N$, some of these tables are also given in Savage [14].

For large $N(r/N \to p > 0$ as $N \to \infty)$ the asymptotic normality of $S_r^{(N)}$ both under the null and the non null hypothesis follows from the Chernoff-Savage theorem [8] as $S_r^{(N)}$ is asymptotically equivalent to the Gastwirth form [10] of the Savage statistic as can be seen from the following:

THEOREM 3.1. *$S_r^{(N)}$ is asymptotically equivalent to the Gastwirth modification of the Savage statistic.*

PROOF. Since

$$(3.2) \qquad \lim_{N\to\infty} a_i = \lim_{N\to\infty} \sum_{j=1}^i (N-j+1)^{-1} = \int_0^u (1-x)^{-1}\,dx = -\log(1-u)$$

TABLE II

Tail Probabilities of $S_r^{(N)}$ under H_0 for different values of m, n and r

$m = 4, n = 4, r = 4$

z	$S_4^{(8)}$	Cum. Prob.
0100	1.1716	.8714
0010	1.3384	.9286
0001	1.5382	.9857
0000	2.5382	1.0000

$m = n = 4, r = 5$

z	$S_5^{(8)}$	Cum. Prob.
00101	1.0881	.8857
00011	1.2881	.9286
10000	1.7787	.9429
01000	1.9216	.9671
00100	2.0882	.9714
00010	2.2882	.9857
00001	2.5382	1.0000

$m = n = 4, r = 6$

z	$S_6^{(8)}$	Cum. Prob.
010010	1.5882	.9000
001010	1.7548	.9143
100001	1.7787	.9286
010001	1.9216	.9429
000110	1.9548	.9571
001001	2.0882	.9714
000101	2.2882	.9857
000011	2.5382	1.0000

$m = n = 5, r = 5$

z	$S_5^{(10)}$	Cum. Prob.
00011	1.0615	.8968
10000	1.6826	.9167
01000	1.7937	.9365
00100	1.9187	.9563
00010	2.0616	.9762
00001	2.2282	.9960
00000	3.2282	1.0000

$m = n = 5, r = 6$

z	$S_6^{(10)}$	Cum. Prob.
001010	1.5187	.8968
010001	1.5937	.9127
000110	1.6616	.9286
001001	1.7187	.9444
000101	1.8616	.9603
000011	2.0282	.9762
100000	2.4826	.9802
010000	2.5937	.9841
001000	2.7187	.9881
000100	2.8616	.9921
000010	3.0282	.9960
000001	3.2282	1.0000

$m = n = 5, r = 7$

z	$S_7^{(10)}$	Cum. Prob.
1100000	1.5981	.8968
0001011	1.6115	.9087
1010000	1.7231	.9127
0000111	1.7781	.9246
0110000	1.8342	.9286
1001000	1.8660	.9325
0101000	1.9771	.9365
1000100	2.0326	.9405
0011000	2.1021	.9444
0100100	2.1437	.9484
1000010	2.2326	.9524
0010100	2.2687	.9563
0100010	2.3437	.9603
0001100	2.4116	.9643
0010010	2.4687	.9682
1000001	2.4826	.9722
0100001	2.5937	.9762
0001010	2.6116	.9801
0010001	2.7187	.9841
0000110	2.7782	.9881
0001001	2.8616	.9921
0000101	3.0282	.9960
0000011	3.2282	1.0000

$m = n = 5, r = 8$

z	$S_8^{(10)}$	Cum. Prob.
00011100	1.8282	.8968
01100001	1.8342	.9008
10010001	1.8660	.9048
10000110	1.8992	.9087
00101010	1.9353	.9127
01010001	1.9770	.9167
01000110	2.0103	.9206
10001001	2.0326	.9246
00011010	2.0782	.9286
00110001	2.1021	.9325
00100110	2.1353	.9365
01001001	2.1437	.9405
10000101	2.2326	.9444
00101001	2.2687	.9484
00010110	2.2782	.9524

TABLE II (*continued*)

z	$S_8^{(10)}$	Cum. Prob.
$m = n = 5, r = 8$ (*continued*)		
01000101	2.3437	.9563
00011001	2.4116	.9603
00001110	2.4448	.9643
00100101	2.4687	.9683
10000011	2.4826	.9722
01000011	2.5937	.9762
00010101	2.6116	.9802
00100011	2.7187	.9841
00001101	2.7782	.9881
00010011	2.8616	.9921
00001011	3.0282	.9960
00000111	3.2282	1.0000

z	$S_6^{(12)}$	Cum. Prob.
$m = n = r = 6$		
010001	1.4403	.8950
000110	1.5087	.9113
001001	1.5403	.9275
000101	1.6515	.9437
000011	1.7765	.9600
100000	2.3494	.9664
010000	2.4403	.9729
001000	2.5403	.9794
000100	2.6515	.9859
000010	2.7765	.9924
000001	2.9193	.9989
000000	3.9193	1.0000

z	$S_7^{(12)}$	Cum Prob.
$m = n = 6, = r = 7$		
0110000	1.7279	.8950
1001000	1.7482	.9004
0101000	1.8391	.9058
1000100	1.8732	.9113
0011000	1.9391	.9167
0100100	1.9641	.9221
1000010	2.0160	.9275
0010100	2.0641	.9329
0100010	2.1069	.9383
0001100	2.1753	.9437
1000001	2.1827	.9491
0010010	2.2069	.9545
0100001	2.2736	.9600
0001010	2.3181	.9654
0010001	2.3736	.9708
0000110	2.4431	.9762
0001001	2.4848	.9816
0000101	2.6098	.9870
0000011	2.7526	.9924

z	$S_7^{(12)}$	Cum. Prob.
$m = n = 6, r = 7$ (*continued*)		
1000000	3.1827	.9935
0100000	3.2736	.9946
0010000	3.3736	.9957
0001000	3.4848	.9968
0000100	3.6098	.9978
0000010	3.7526	.9989
0000001	3.9193	1.0000

z	$S_7^{(14)}$	Cum. Prob.
$m = n = r = 7$		
0011000	1.8479	.8998
0100100	1.8646	.9059
1000010	1.8987	.9120
0010100	1.9479	.9181
0100010	1.9757	.9242
1000001	2.0237	.9304
0001100	2.0388	.9365
0010010	2.0590	.9426
0100001	2.1007	.9487
0001010	2.1499	.9548
0010001	2.1840	.9610
0000110	2.2499	.9671
0001001	2.2749	.9732
0000101	2.3749	.9793
0000011	2.4860	.9854
1000000	3.0237	.9875
0100000	3.1007	.9895
0010000	3.1840	.9915
0001000	3.2749	.9936
0000100	3.3749	.9956
0000010	3.4860	.9977
0000001	3.6110	.9997
0000000	4.6110	1.0000

z	$S_8^{(16)}$	Cum. Prob.
$m = n = r = 8$		
00100101	1.6346	.8992
01000011	1.6632	.9036
00001110	1.6838	.9079
00010101	1.7115	.9123
00100011	1.7346	.9166
00001101	1.7949	.9210
00010011	1.8115	.9253
00001011	1.8949	.9297
00000111	1.9858	.9340
11000000	2.1689	.9362
10100000	2.2403	.9384
01100000	2.3070	.9406
10010000	2.3172	.9427
01010000	2.3839	.9449

TABLE II (*continued*)

$m = n = r = 8$ (*continued*)			$m = n = r = 8$ (*continued*)		
z	$S_8^{(16)}$	Cum. Prob.	z	$S_8^{(16)}$	Cum. Prob.
10001000	2.4006	.9471	00100001	2.8407	.9819
00110000	2.4553	.9493	00001010	2.8899	.9841
01001000	2.4673	.9514	00010001	2.9176	.9862
10000100	2.4915	.9536	00000110	2.9808	.9884
00101000	2.5387	.9558	00001001	3.0010	.9906
01000100	2.5582	.9580	00000101	3.0919	.9928
10000010	2.5915	.9601	00000011	3.1919	.9949
00011000	2.6156	.9623	10000000	3.7025	.9956
00100100	2.6296	.9645	01000000	3.7692	.9962
01000010	2.6582	.9667	00100000	3.8406	.9968
10000001	2.7026	.9688	00010000	3.9175	.9974
00010100	2.7065	.9710	00001000	4.0009	.9981
00100010	2.7296	.9732	00000100	4.0918	.9987
01000001	2.7693	.9754	00000010	4.1918	.9993
00001100	2.7899	.9775	00000001	4.3029	.9999
00010010	2.8065	.9797	00000000	5.3029	1.0000

and

$$\begin{aligned}\lim_{N\to\infty} a(N-r)^{-1} &= \lim_{N\to\infty} (N-r)^{-1}[\textstyle\sum_{i=1}^{N} a_i - \sum_{i=1}^{r} a_i] \\ &= (1-p)^{-1} \lim_{N\to\infty} [1 - \textstyle\sum_{i=1}^{r} a_i/N] \\ &\qquad \cdot(\because \textstyle\sum_{i=1}^{N} a_i = N) \\ &= (1-p)^{-1} \lim_{N\to\infty} [1 - N^{-1} \textstyle\sum_{j=1}^{r} (r-j+1) \\ &\qquad \cdot(N-j+1)^{-1}] \\ &= (1-p)^{-1}[1 - \textstyle\int_0^p (p-x)(1-x)^{-1}\,dx] \\ &= 1 - \log(1-p).\end{aligned} \tag{3.3}$$

The result follows by comparing (3.2) and (3.3) with the weight function given in [10].

To make use of the normal probability integral we need to find the mean and variance of $S_r^{(N)}$ under the null hypothesis. To this end we have the following:

THEOREM 3.2. *Denoting by* $E_0(\cdot)$ *and* $\sigma_0^2(\cdot)$ *the mean and variance under* H_0 *we have*

$$E_0(S_r^{(N)}) = \tfrac{1}{2}(m-n) \tag{3.4}$$

and

$$\sigma_0^2(S_r^{(N)}) = mn(N(N-1))^{-1}\{\textstyle\sum_1^r a_i^2 + a^2(N-r)^{-1} - N\} \tag{3.5}$$

where $a = \sum_{r+1}^{N} a_i$.

PROOF. Proof follows easily since $S_r^{(N)}$ can be written as

$$S_r^{(N)} = \sum_1^r a_i z_i + a(N - r)^{-1} \sum_{r+1}^N z_i - \tfrac{1}{2}(m + n)$$
$$= \sum_{i=1}^N l_i z_i - \tfrac{1}{2}(m + n),$$

where

$$l_i = a_i, \qquad 1 \leqq i \leqq r,$$
$$= a(N - r)^{-1}, \qquad r + 1 \leqq i \leqq N.$$

And it is well known that under H_0

$$E_0(\sum l_i z_i) = mN^{-1} \sum l_i$$

and

$$\sigma_0^2(\sum l_i z_i) = mn(N(N - 1))^{-1} \sum_1^N (l_i - \bar{l})^2.$$

Table III gives some idea about the accuracy of the normal approximation of the two sided test statistic $|S_r^{(N)}|$ for various values of m, n, r ($m = n$) and for the 5% level of significance. α' gives the size of the critical region based on the normal approximation when the exact size based on $|S_r^{(N)}|$ is .05, P_R denoting the randomization probability needed to achieve the actual size .05 based on the $|S_r^{(N)}|$ statistic. It should be noted that we have not made any correction for continuity which normally should improve upon the approximation.

4. The test based on $S_r^{(N)}$ and its curtailed form. An interesting feature of the test based on the statistic $S_r^{(N)}$ is that it might be possible to terminate the test even before all the r observations are available and predict accurately the out-

TABLE III

Comparison of Exact tests based on $|S_r^{(N)}|$ with Normal Approximation

$\alpha = .05$

m	n	r	Critical Value $\lvert S_r^{(N)}\rvert$	P_R(1)	α'(2)
4	4	4	1.5378	.1875	.1142
5	5	5	2.0616	.0600	.0614
5	5	6	2.0280	.0756	.0910
5	5	8	2.5937	.2999	.0548
6	6	6	2.5403	.6833	.0358
6	6	7	2.4431	.2200	.0588
7	7	7	2.3748	.7048	.0602
8	8	8	2.7692	.2054	.0478

(1) P_R denotes the "randomization probability" to achieve $\alpha = .05$ when the test statistic $|S_r^{(N)}|$ is used.
(2) α' is the size of the critical region for the same critical value $|S_r^{(N)}|$ when the normal approximation is used.

come based on all the r observations. This is particularly true of value in destructive testing since the earlier we reach a decision the more we save on the experimental cost and time. This feature can best be illustrated by an example.

Consider the case $m = n = r = 5$. Let X and Y refer to the failure times of two sets of items put on test. In this case $S_5^{(10)}$ is symmetrically distributed around zero under the null hypothesis and we use an equal tailed test based on $|S_5^{(10)}|$. Eight sequences with the largest values of $|S_5^{(10)}|$ are shown in Table IV.

The proposed test is to reject H_0 for large values of $|S_5^{(10)}|$. For a critical region of exact size $\alpha = .05$ we reject H_0 when $|S_5^{(10)}| > 2.0616$, accept H_0 when $|S_5^{(10)}| < 2.0616$ and randomize when $|S_5^{(10)}| = 2.0616$, that is, we reject H_0 with randomization probability $P_R = .02749$.

It is clear that the results of the test may be determined before 5 failures are observed and hence the test can be put in a curtailed form, that is we can terminate the test as soon as the decision to accept or reject the H_0 is reached. Table V gives the stopping sequences in a curtailed test allowing for randomization. Since the test is symmetric we restrict the tabulation to x sequences only.

It can be easily verified that if the first observation is an x very little can be said about the possible outcome of $|S_5^{(10)}|$. However, if in addition the second observation is a y, no matter what are the outcomes of subsequent failures the maximum value of $|S_5^{(10)}|$ will be less than 2.0616, the critical value.

It is interesting to study the expected length $E_0(N_f)$ of the stopping sequence and the expected time to terminate the test under H_0. We shall discuss these points later. Using some results of Sobel [16] and some results given in the next

TABLE IV

Test based on $|S_5^{(10)}|$ *for* $m = n = r = 5$

Sequence z	Dual sequence z^*	$\lvert S_5^{(10)} \rvert$	$P_0(z) + P_0(z^*) = 2P_0(z)$	Cumulative Probability
$xxxxx$	$yyyyy$	3.2282	1/126	.0076
$xxxxy$	$yyyyx$	2.2282	5/126	.0476
$xxxyx$	$yyyxy$	2.0616	5/126	.0873
$xxyxx$	$yyxyy$	1.9187	5/126	.1270

TABLE V

Test based on $|S_5^{(10)}|$ *in curtailed form*

Stopping sequence z	$2P_0(z)$	$\lvert S_5^{(10)}\rvert$	Action
$xxxx$	6/126	$\lvert S_5^{(10)}\rvert > 2.0616$	Reject H_0
$xxxyx$	5/126	$\lvert S_5^{(10)}\rvert = 2.0616$	Reject H_0 with probability .0275
xy	70/126		
xxy	35/126	$\lvert S_5^{(10)}\rvert < 2.0616$	Accept H_0
$xxxyy$	10/126		

section it can be shown that for this particular example

$$E_0(N_f) = 344/126 = 2.76 \quad \text{and} \quad E_0(T)/\theta = .310317$$

where θ is the parameter through which the two populations differ and $E_0(T)$ is the expected termination time.

Another interesting feature of the curtailed sequence follows from the following:

LEMMA 4.1. *For any curtailed sequence of length $d \leqq r$ the value of $S_d^{(N)}$ obtained by using* (2.2) *with r replaced by d is the conditional expectation of $S_r^{(N)}$ under H_0 given the source of the first d failures.*

PROOF.

$$(4.1) \quad E(S_r^{(N)} \mid (z_1 \cdots z_d)) = \sum_{i=1}^{d} a_i z_i + \sum_{d+1}^{r} a_i E(z_i \mid (z_1, \cdots z_d)) + (\sum_{r+1}^{N} a_i)(N-r)^{-1} \sum_{r+1}^{N} E(z_i \mid (z_1 \cdots z_d)) - \tfrac{1}{2}(m+n).$$

But $E(z_i \mid (z_1, \cdots z_d)) = (m - m_d)(N-d)^{-1}$. Hence

$$(4.2) \quad \begin{aligned} E(S_r^{(N)} \mid (z_1, \cdots, z_d)) &= \sum_{1}^{d} a_i z_i + (m - m_d)(N-d)^{-1} \\ &\quad \cdot [\sum_{d+1}^{r} a_i + \sum_{r+1}^{N} a_i] - \tfrac{1}{2}(m+n) \\ &= S_d^{(N)}. \end{aligned}$$

The above lemma shows that for increasing r, $S_r^{(N)}$ forms a martingale.

5. Comparison of nonparametric curtailed life tests. Since the statistic $S_r^{(N)}$ is asymptotically equivalent to the Gastwirth form [10] of the Savage statistic, $S_r^{(N)}$ can be shown to possess all the standard large sample properties. Moreover elsehwere ([2], [3]) Basu has computed the asymptotic relative efficiencies of the $S_r^{(N)}$ statistic with respect to other "r out of N" statistics, which show the superiority of the $S_r^{(N)}$ test in life testing situations. However, since in many life testing problems r will be usually small or of moderate size, it seems desirable to compare the performance of $S_r^{(N)}$ statistic with other statistics which are considered suitable for life testing. Sobel has already made some comparisons among several competitive tests on the basis of their curtailed forms when the parent populations are exponential. For a discussion of these tests we refer to Sobel's paper [16] whose notations we shall use. Let the density function of $F(x)$ and $G(x)$ be, under the null hypothesis H_0,

$$(5.1) \qquad f_0(x) = \theta^{-1} e^{-x/\theta}, \qquad x > 0.$$

And let under the alternative hypothesis H_1

$$(5.2) \qquad g_\theta(y) = 2\theta^{-1} e^{-y/\theta}, \qquad y \geqq \theta \ln 2,$$

and under the alterantive hypothesis H_2

$$(5.3) \qquad g_\theta(y) = \tfrac{1}{2}\theta^{-1} e^{-y/2\theta}, \qquad y > 0.$$

Thus H_1 and H_2 correspond to two situations commonly encountered in life testing problems. Denoting by $P_i(S_d)$, $E_i(N_f)$ and $E_i(T)$ the probability of any sequence $(z_1, \cdots, z_d)$ of length d under H_i, expected number of observations needed for the curtailed $|S_r^{(N)}|$ test to reach a decision under H_i and the expected time to terminate the curtailed form of the test $(i = 0, 1, 2)$ Sobel has compared various tests in terms of the above quantities. In this section we shall consider $P_i(S_d)$, $E_i(N_f)$ and $E_i(T)$ for the $S_r^{(N)}$ test for the special cases $(m = n = 5, r = 6)$, $(m = n = 5, r = 8)$, $(m = n = r = 6)$ and $(m = n = r = 7)$ using the formulas given by Sobel. However for H_2 we have the following simpler expressions.

LEMMA 5.1. *Given a sequence* $S_d \equiv (z_1, \cdots, z_d)$ *of length* $d(m_d + n_d = d, m_d \leqq m, n_d \leqq n, d \leqq r)$ *we have under* H_2

$$(5.4)\quad P_2(S_d) = m!n!((m - m_d)!(n - n_d)!)^{-1}2^{m_d} \cdot \prod_{\alpha=1}^{d} [2(m - m_d) + (n - n_d) + \alpha + \textstyle\sum_{\beta=0}^{\alpha-1} z_{d-\beta}]^{-1}$$

and

$$(5.5)\quad E_2(T \mid S_d) = 2P_2(S_d) \cdot \sum_{\alpha=1}^{d} [2(m - m_d) + (n - n_d) + \alpha + \textstyle\sum_{\beta=0}^{\alpha-1} z_{d-\beta}]^{-1}$$

where $E_2(T \mid S_d)$ *is the contribution of the stopping sequence* S_d *(that is, the term to be added) to* $E(T \mid H_2)$ *so that* $E(T \mid H_2) = \sum \{E_2(T \mid S_d)P_2(S_d)\}$ *where the summation is taken over all admissible stopping sequences* S_d $(d \leqq r)$.

PROOF. The expression $P_2(S_d)$ directly follows from an expression given by Rao, Savage and Sobel [13]. The second part also follows by substituting $f(w_i) = 2(1 - G(w_i))g(w_i)$ and integrating the variables $(w_d, w_{d-1}, \cdots, w_2, w_1)$ one at a time and in the order $w_d, w_{d-1}, \cdots, w_1$ in the expression

$$E_2(T \mid S_d) = m!n!((m - m_d)!(n - n_d)!)^{-1}\int_{(0<w_1<w_2<\cdots<w_d<\infty)} w_d \cdot \prod_{i=1}^{d} [f(w_i)]^{z_i}[g(w_i)]^{1-z_i}[1 - F(w_d)]^{m-m_d}[1 - G(w_d)]^{n-n_d}\, dw_1 \cdots dw_d.$$

Table VI shows the results of computations involving the various quantities described above. Here we have compared several statictics in terms of $E_i(N_f)$, $E_i(T)$ and the power function $P\{\text{correct decision} \mid H_i\} = P(\text{CD} \mid H_i)$ under specific alternatives H_i $(i = 1, 2)$. To facilitate disucssion we have also included corresponding results for the $R_r^{(N)}$ statistic proposed in [13] for the cases $m = n = 5, r = 6$ and $m = n = r = 6$. Looking at Table VI and comparing it with Sobel's Table IV [16] it seems clear that even in small samples the $S_r^{(N)}$ test performs as good as any one of the tests discussed by Sobel. In particular, the curtailed forms of the $S_r^{(N)}$ and the $R_r^{(N)}$ statistic are comparable in their performances.

6. K-sample extension. In this section we shall consider a k-sample extension of the two-tailed test based on $S_r^{(N)}$. Let X_{ij} $(j = 1, 2, \cdots, n_i, i = 1, 2, \cdots, k)$

TABLE VI

Performance characteristics of six curtailed tests for $\alpha = .05$

Test Statistics	Critical Value	Randomization Prob. P_R	Max No. of failures N_f	H_0		H_1			H_2		
				$E_0(N_f)$	$E_0(T)/\theta$	$P(CD \mid H_1)$	$E_1(N_f)$	$E_1(T)/\theta$	$P(CD \mid H_2)$	$E_2(N_f)$	$E_2(T)/\theta$
(1) $S_6^{(10)}$ $m = n = 5$, $r = 6$	2.0280	.07560	6	4.722	.60595	.17374	5.199	1.09108	.19916	5.195	.96467
$S_8^{(10)}$ $m = n = 5$, $r = 8$	2.5937	.29988	8	4.349	.55436	.22664	5.397	1.14918	.17258	4.644	.87051
$S_6^{(12)}$ $m = n = r = 6$	2.5403	.68330	6	3.348	.32755	.32013	5.144	.91682	.12913	3.633	.50463
$S_7^{(12)}$ $m = n = 6$, $r = 7$	2.4431	.22000	7	4.844	.49791	.33207	5.548	.96346	.13653	5.089	.74380
(2) $R_6^{(10)}$ $m = n = 5$, $r = 6$	2.028	.07500	6	4.722	.60959	.17160	5.199	1.09934	.13076	4.928	1.93001
$R_6^{(12)}$ $m = n = r = 6$	1.540	.68333	6	3.348	.32756	.32013	5.144	.93309	.12914	3.633	1.12741

(1) $S_r^{(N)}$ is the generalized Savage statistic.
(2) $R_r^{(N)}$ is the statistic considered by Rao, Savage and Sobel in [13].

be k independent samples of sizes $n_1, n_2, \cdots, n_k$ respectively from k populations with continuous cumulative distribution functions $F_1, F_2, \cdots, F_k$ respectively. We assume that the F_i's belong to a family $\mathfrak{F}$ of distribution functions indexed by a parameter θ. (The proposed test is particularly suitable for $\mathfrak{F}$ to be the family of Lehmann alternatives or the family of IFR or IFRA distributions.) As before let us assume that only the first r ordered observations out of the combined sample of size $N = \sum_{i=1}^{k} n_i$ are available. Let

$$Z_\alpha^{(i)} = 1, \quad \text{if the } \alpha\text{th ordered observation is from the } i\text{th sample,} \qquad (6.1)$$
$$= 0, \quad \text{otherwise } (\alpha = 1, 2, \cdots, N; i = 1, 2, \cdots, k),$$

and

$$S_i = \sum_{\alpha=1}^{r} a_\alpha z_\alpha^{(i)} + (n_i - n_{ir})(N - r)^{-1} a \qquad (i = 1, 2, \cdots, k) \qquad (6.2)$$

where a_α and a have the same meaning as before and $n_{ir} = \sum_{\alpha=1}^{r} z_\alpha^{(i)}$ is the cumulative number of observations from the ith sample among the first r observations. To test the null hypothesis

$$H_0 : F_1 = F_2 = \cdots = F_k \qquad (6.3)$$

against the alternative that they are different we then propose the statistic

$$A_r^{(N)} = G \sum_{i=1}^{k} n_i^{-1} (S_i - n_i)^2 \qquad (6.4)$$

based on only the first r ordered observations where

$$G^{-1} = (\sum_{\alpha=1}^{r} a_\alpha^{\ 2} + a^2 \ (N - r)^{-1} - N)/(N - 1). \tag{6.5}$$

Clearly for $k = 2$ the above test is equivalent to the symmetrical two-tailed test based on $S_r^{(N)}$.

We next compute the exact mean of $A_r^{(N)}$ under the null hypothesis. To this end we can easily show, using some results of Basu [4], that

$$E_0(S_i) = n_i, \tag{6.6}$$

$$\sigma_0^2(S_i) = n_i(N - n_i)/(GN) \qquad (i = 1, 2, \cdots, k) \tag{6.7}$$

and

$$\sigma_0(S_i, S_j) = -n_i n_j/(GN) \qquad (i, j = 1, 2, \cdots, k; i \neq j). \tag{6.8}$$

It easily follows then that

$$E_0(A_r^{(N)}) = (k - 1) \tag{6.9}$$

irrespective of the value of r.

We next want to find the asymptotic distribution of $A_r^{(N)}$ as $N \to \infty (r/N \to p > 0, n_i/N \to \lambda_i > 0)$. Asymptotic ditsribution of $A_r^{(N)}$ follows from Puri's [12] results, since in Puri's notation $A_r^{(N)}$ can be considered as an L-statistic. Thus $A_r^{(N)}$ asymptotically follows the χ^2 distribution with $(k - 1)$ degrees of freedom. That is, under H_0 $A_r^{(N)}$ follows the central χ_{k-1}^2 distribution and in the non null case $A_r^{(N)}$ follows the non-central χ_{k-1}^2 distribution.

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