The Annals of Mathematical Statistics
1968, Vol. 39, No. 6, 2029–2037

ON SLIPPAGE TESTS—(II) SIMILAR SLIPPAGE TESTS[1]

By Irving J. Hall[2], Akio Kudô[3] and Neng-che Yeh[4]

Iowa State University and Kyushu University

1. Introduction. This is a continuation of the previous paper of Hall and Kudô [1], and all the notations and nomenclature are the same as in the previous paper. The purpose of this paper is to explore the possibility of applying the concept of similarity in hypotheses testing to slippage tests.

The authors are grateful to the referee, who suggested the strengthened version of the result in the original manuscript.

2. Similarity in exponential family of distributions. In accordance with hypotheses testing we can define a similar size α decision function.

In this section we consider some general aspects of uniformly most powerful symmetric similar size α decision functions. A decision function is said to be *similar size* if the expectation of $\varphi_0(x)$ is equal to $1 - \alpha$ whenever H_0 is true.

Let S be distributed according to the exponential family with parameter space $\Omega = \{\theta\}$ which can be divided into $a + 1$ disjoint subsets $\Omega = \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_a$ such that $\Omega_0 \cup \Omega_i$ is covered by a family of disjoint curves originating from Ω_0, $\Omega_i = \{\theta_i(\gamma, \sigma): 0 < \gamma < \infty, \sigma \varepsilon \Omega_0\}$, $\theta_i(0, \sigma) = \sigma$ and $\theta_i(\gamma, \sigma) \varepsilon \Omega_i$ for all $\gamma \varepsilon (0, \infty)$ and σ so that the parameter can be expressed as $\theta_i(\gamma, \sigma)$ or (i, γ, σ) for $\theta \varepsilon \Omega_i$ and σ for $\theta \varepsilon \Omega_0$.

We assume that U is the minimal sufficient statistic, for Ω_0, (U, T_i) for $\Omega_0 \cup \Omega_i$, S for $\Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_a$ and that the density of S wrt μ can be expressed as

$$(1) \quad dP^S(s)/d\mu(s) = dP^S_{i,\gamma,\sigma}(s)/d\mu(s) = C(i, \gamma, \sigma) \exp [\alpha(i, \gamma, \sigma)U(s) + \beta(\gamma, \sigma)T_i(s)].$$

As before we assume there is a group $G = \{g\}$ of transformations on S isomorphic to the permutation group of $(1, 2, \cdots, a)$ itself or to its subgroup transitive on $(1, 2, \cdots, a)$ and $\mu(A) = \mu(gA)$. Let the number of elements in G be N. In addition we assume

A.1. $T_i(s) = T_{\pi_g i}(gs)$.

A.2. $U(gs_1) = U(gs_2)$ for all g if and only if $U(s_1) = U(s_2)$.

This enables us to define $G_u = \{g_u\}$, a transformation group defined on the space of U. G_u is, of course, finite and its number of elements is denoted by M.

Received 3 January 1966; revised 15 April 1968.

[1] This research was supported in part by National Science Foundation Grants GP-3918 and GM-6149. The work started while Hall and Kudô were at Iowa State University, and completed after Kudô went back to Kyushu University in cooperation with Yeh.

[2] Presently at Sandia Corporation, Albuquerque, New Mexico.

[3] Presently at Kyushu University, Fukuoka, Japan.

[4] Presently at Kyushu University and on leave from Tamkang College of Arts and Sciences, Taipei, Taiwan, China.

Let $\hat{G}$ be a transformation group on Ω_0, to which G is homomorphic, and let $\hat{g}$ be the element of $\hat{G}$ corresponding to g. We also assume

A.3. $C(i, \gamma, \sigma) = C(\pi_g i, \gamma, \hat{g}\sigma)$.

A.4. $\alpha(i, \gamma, \sigma)U = \alpha(\pi_g i, \gamma, \hat{g}\sigma)g_u U$.

A.5. $\beta(0, \sigma) = 0$, and $\beta(\gamma, \sigma)$ is non-decreasing in γ and $\beta(\gamma, \sigma) = \beta(\gamma, \hat{g}\sigma)$.

A.6. U is complete for Ω_0.

A.7. When $\theta \varepsilon \Omega_0$, the conditional distribution $P_\theta^{S \mid U}$ of S given $U = g_u u$ remains the same for all g_u.

Let $\bar{G} = \{\bar{g}\}$ be a group of transformations on Ω defined by $\bar{g}(i, \gamma, \sigma) = (\pi_g i, \gamma, \hat{g}\sigma)$.

LEMMA 1. *The distribution of S satisfies $P_{i,\gamma,\sigma}^S(A) = P_{\bar{g}(i,\gamma,\sigma)}^S(gA)$, for all $g \varepsilon G$, namely, G induces $\bar{G}$.*

This follows from A.1, A.3, A.4 and A.5.

LEMMA 2. *The marginal distribution of U satisfies $P_{i,\gamma,\sigma}^U(B) = P_{\bar{g}(i,\gamma,\sigma)}^U(\bar{g}_u B)$ namely G_u also induces $\bar{G}$.*

PROOF. By Assumption 2, $gU^{-1}(B) = U^{-1}(g_u B)$ and by Lemma 1

$$P_{i,\gamma,\sigma}^U(B) = P_{i,\gamma,\sigma}^S(U^{-1}(B)) = P_{\bar{g}(i,\gamma,\sigma)}^S(gU^{-1}(B))$$
$$= P_{\bar{g}(i,\gamma,\sigma)}^S(U^{-1}(g_u B)) = P_{\bar{g}(i,\gamma,\sigma)}^U(g_u B).$$

LEMMA 3. *The conditional distribution of S given $U = u$ satisfies*

$$P_{i,\gamma,\sigma}^{S \mid u}(A \mid u) = P_{\bar{g}(i,\gamma,\sigma)}^{S \mid u}(gA \mid g_u U),$$

namely, G induces a group $((i, \gamma, \sigma), u) \to (g(i, \gamma, \sigma), g_u u)$ when $((i, \gamma, \sigma), u)$ is taken as a parameter of the conditional distribution.

PROOF. By Lemma 1,

$$P_{i,\gamma,\sigma}^S(S \varepsilon A, U(S) \varepsilon B) = P_{\bar{g}(i,\gamma,\sigma)}^S(S \varepsilon gA, U(S) \varepsilon g_v B)$$

or

$$\int_B P_{i,\gamma,\sigma}^{S \mid u}(A \mid u)\, dP_{i,\gamma,\sigma}^U(u) = \int_{g_u B} P_{\bar{g}(i,\gamma,\sigma)}^{S \mid u}(gA \mid u)\, dP_{\bar{g}(i,\gamma,\sigma)}^U(u)$$
$$= \int_B P_{\bar{g}(i,\gamma,\sigma)}^{S \mid u}(gA \mid g_u U)\, dP_{\bar{g}(i,\gamma,\sigma)}^U(u).$$

By Lemma 2, and by the uniqueness of the conditional probability, the result follows.

As the distribution of S is exponential, the conditional distribution of S given U is also exponential and we have

LEMMA 4.

$$dP_{i,\gamma,\sigma}^{S \mid u}(s \mid u) = C_u(i, \gamma, \sigma) \exp\,[\beta(\gamma, \sigma)T_i(s)]h(s; u)\, d\lambda_u(s)$$

where

(a) $C_u(i, \gamma, \sigma) = C_{g_u u}(\bar{g}(i, \gamma, \sigma))$,

(b) $h(s; u) = h(gs; g_u u)$,

(c) $\lambda_u(A) = \lambda_u(gA) = \lambda_{g_u u}(A)$.

PROOF. The conditional density can be written in the form of

$$dP^{S\mid u}_{i,\gamma,\sigma}(s \mid u) = K_u(i, \gamma, \sigma) \exp [\beta(\gamma, \sigma)T_i(s)]\, d\nu_u(s).$$

As U is sufficient for Ω_0, $K_u(0, 0, \sigma)$ does not depend on σ, which we write as K_u. By Lemma 3 and A.5, we have

$$K_u \nu_u(A) = K_{g_u u} \nu_{g_u u}(gA). \tag{2}$$

Letting $C_u(i, \gamma, \sigma) = K_u(i, \gamma, \sigma)/K_u$, and applying Lemma 3, we have (a). Since $K_{g_u u}\nu_{g_u u}(A)$ is absolutely continuous with respect to

$$\lambda_u(A) = M^{-1} \textstyle\sum_{g_u \varepsilon G_u} K_{g_u u}\nu_{g_u u}(A) \quad \text{for all} \quad g_u \varepsilon G_u,$$

there is a measurable function $h(s; u)$ by Radon-Nikodym theorem such that

$$K_{g_u u}\nu_{g_u u}(A) = \textstyle\int_A h(s; g_u u)\lambda_u(s). \tag{3}$$

It is straightforward to verify (c) by the definition of $\lambda_u(A)$, and (b) can be verified by (1), (2) and (c).

LEMMA 5. *$h(s; u)$ of Lemma 4 satisfies $h(s; u) = h(s; g_u u)$ for all $g_u \varepsilon G_u$.*

PROOF. For $\theta \varepsilon \Omega_0$, we have from A.7

$$P^{S\mid u}(A \mid u) = P^{S\mid u}(A \mid g_u u) \quad \text{for all} \quad g_u \varepsilon G_u,$$

which implies

$$\textstyle\int_A h(s; u)\, d\lambda_u(s) = \int_A h(s; g_u u)\, d\lambda_u(s)$$

and the result follows.

LEMMA 6. *$G_u(i, \gamma, \sigma)$ is free from i.*

PROOF. Consider a sum

$$\begin{aligned} 1 &= aN^{-1} \textstyle\sum_{g:\, \pi_g 1 = i} \int dP^{S\mid u}_{\bar g(i,\gamma,\sigma)}(s \mid g_u u) \\ &= aN^{-1} \textstyle\sum_{g:\, \pi_g 1 = i} \int C_{g_u u}(\bar g(1, \gamma, \sigma)) \exp [\beta(\gamma, \hat g\sigma) T_{\pi_g 1}(s)] h(s{:}g_u u)\, d\mu(s) \\ &= C_u(1, \gamma, \sigma) \textstyle\int \exp [\beta(\gamma, \sigma)T_i(s)] h(s; u)\, d\mu(s) \\ &= C_u(i, \gamma, \sigma) \textstyle\int \exp [\beta(\gamma, \sigma)T_i(s)] h(s; u)\, d\mu(s), \end{aligned}$$

which implies $C_u(i, \gamma, \sigma) = C_u(1, \gamma, \sigma)$.

In the following, $E_i(\cdot\,; \gamma, \sigma)$ denotes the expectation by $P^S_{i,\gamma,\sigma}$, $E_i(\cdot \mid u; \gamma, \sigma)$ the conditional expectation when $U = u$. $E_0(\cdot\,; \sigma)$ and $E_0(\cdot \mid u)$ are the same for $\theta \varepsilon \Omega_0$. (Because of the sufficiency of U for Ω_0, the conditional expectation given U is free from σ.)

DEFINITIONS. We define sets of decision functions $\Phi_1, \cdots, \Phi_5$ by the following conditions

$\Phi_1 : E_i(\varphi_i\,; \gamma, \sigma) = E_{\pi_g i}(\varphi_{\pi_g i}\,; \tau, \hat g\sigma)$. Such φ are called symmetric in power.

$\Phi_2 : E_0(\varphi_0 : \sigma)$ is independent of σ. Such φ are called similar.

$\Phi_3 : E_0(\varphi_0 \mid u)$ is independent of U. This is called the conditional size of φ.

$\Phi_4 : E_i(\varphi_i \mid U; \gamma, \sigma) = E_{\pi_g i}(\varphi_{\pi_g i} \mid g_u U; \gamma, \hat{g}\sigma)$. Such φ are called symmetric in conditional power.

$\Phi_5 : \varphi_i(s) = \varphi_{\pi_g i}(gs)$. Such φ are called invariant.

LEMMA 7. $$\Phi_2 = \Phi_3 .$$

Note that those decision functions in Φ_3 may be said to have Neyman structure with respect to U in accordance with the theory of hypothesis testing.

LEMMA 8. *For any decision function belonging to $\Phi_1 \cap \Phi_2$ or $\Phi_3 \cap \Phi_4$ there exists one in $\Phi_5 \cap \Phi_2$ or $\Phi_3 \cap \Phi_5$, which has the same size, or conditional size and power or conditional power, respectively.*

Let S be distributed with the density

$$C(i, \gamma, \sigma) \exp [\alpha(i, \gamma, \sigma)U(s) + \beta(\gamma, \sigma)T_i(s)]$$

with respect to μ, and assumptions A.1, $\cdots$, A.7 be satisfied. Consider a rule φ of the form:

$$\text{(4)} \qquad \begin{aligned} \varphi_0(s) &= 1, \xi(s), 0 \quad \text{if} \quad \max_i T_i(s) <, =, > C(s), \\ \varphi_j(s) &= \eta_j(s), 0 \quad \text{if} \quad T_j(s) =, < \max_i T_i(s). \end{aligned}$$

We have the following theorems.

THEOREM 1. (a) *For any other rule $\hat{\varphi}$ if $E_0(\hat{\varphi}_0 \mid u) \geqq E_0(\varphi_0 \mid u)$ then*

$$\sum_{g\varepsilon G} E_{\pi_g i}(\varphi_{\pi_g i} \mid g_u u; \gamma, \hat{g}\sigma) \geqq \sum_{g\varepsilon G} E_{\pi_g i}(\hat{\varphi}_{\pi_g i} \mid g_u u; \gamma, \hat{g}\sigma)$$

for all i, γ, σ and u.

(b) *For any α, there is a rule φ of the form (4) with $\xi(s)$ being a function of u only and $E_0(\varphi \mid u) = 1 - \alpha$ for all u. Thus this φ belongs to Φ_3.*

(c) *Further φ can be made symmetric in conditional power, so that φ belongs to Φ_4.*

PROOF. (a) By Lemmas 4 and 6, we have

$$dP^{S \mid u}_{i,\gamma,\sigma}(s \mid u) = C_u(\gamma, \sigma) \exp [\beta(\gamma, \sigma)T_i(s)]h(s; u)\, d\lambda_u(s).$$

By applying Theorem 2 [1], we have, $E_0(\varphi_0(s) \mid u) \geqq E_0(\varphi_0(s) \mid u)$ which implies

$$\sum_{j=1}^{a} E_j[\varphi_j(s) \mid u; \gamma, \sigma] \geqq \sum_{j=1}^{a} E_j[\varphi_j(s) \mid u; \gamma, \sigma].$$

On the other hand, by Lemmas 4 and 5, we have

$$\begin{aligned} dP^{S \mid u}_{g(i,\gamma,\sigma)}(s \mid g_u u) &= C_{g_u u}(\gamma, \hat{g}\sigma) \exp [\beta(\gamma, \hat{g}\sigma)T_{\pi_g i}(s)]h(s; g_u u)\, d\lambda_{g_g u}(s) \\ &= C_u(\gamma, \sigma) \exp [\beta'(\gamma, \sigma)T_{\pi_g i}(s)]h(s; u)\, d\lambda_u(s). \end{aligned}$$

and hence

$$\begin{aligned} &aN^{-1}\sum_{g\varepsilon G:\pi_g 1=i} dP^{S \mid u}_{g(1,\gamma,\sigma)}(s \mid g_u u) \\ &\quad = aN^{-1}\sum_{g\varepsilon G:\pi_g 1=i} C_u(\gamma, \sigma) \exp [\beta(\gamma, \sigma)T_i(s)]h(s; u)\, d\lambda_u(s) \\ &\quad = C_u(\gamma, \sigma) \exp [\beta(\gamma, \sigma)T_i(s)]h(s; u)\, d\lambda_u(s) = dP^{S \mid u}_{i,\gamma,\sigma}(s \mid u) \end{aligned}$$

and

$$\sum_{j=1}^{a} E_j[\varphi_j(S) \mid u; \gamma, \sigma] = \sum_{j=1}^{a} \int \varphi_j(s)\, dP^{S\mid u}_{j(1,\gamma,\sigma)}(s \mid u)$$
$$= \sum_{j=1}^{a} \int \varphi_j(s) aN^{-1} \sum_{g\varepsilon G:\pi_g i = j} dP^{S\mid u}_{\bar{g}(1,\gamma,\sigma)}(s \mid g_u u)$$
$$= aN^{-1} \sum_{g\varepsilon G} E_{\pi_g i}[\varphi_{\pi_g i}(S) \mid g_u u;\ \hat{g}\sigma].$$

As the same relation holds good for $\hat{\varphi}$, we have the proof.

(b) Fix u, γ, σ and consider $dP_0^{S\mid u}$, $dP^{S\mid u}_{(i,\gamma,\sigma)}$, $\cdots$, $dP^{S\mid u}_{(a,\gamma,\sigma)}$, then we get the result by applying (iii) of Theorem 1 [1].

(c) By the same argument as in Corollary 1 [1], φ can be made to be invariant, and thus it is symmetric in conditional power.

THEOREM 2. (a) *For any α, there is a rule of the form* (4), $\varphi(s)$ *in* Φ_2 *with* $E_0(\varphi_0) = 1 - \alpha$. *Furthermore, it can be made symmetric in power.*

(b) *For any other rule $\hat{\varphi}$ with $E_0(\hat{\varphi}_0 \mid u) \geqq E_0(\varphi_0 \mid u)$ for all u*

$$(5)\quad \sum_{g\varepsilon G} E_{\pi_g i}[\varphi_{\pi_g i}(S); \gamma, \hat{g}\sigma] \geqq \sum_{g\varepsilon G} E_{\pi_g i}[\hat{\varphi}_{\pi_g i}(S); \gamma, \hat{g}\sigma] \quad \textit{for all } i, \gamma \textit{ and } \sigma.$$

(c) (5) *holds true for any other rule $\hat{\varphi}$ in Φ_2 with*

$$E_0(\hat{\varphi}_0) \geqq E_0(\varphi_0).$$

PROOF. (a) Theorem 1 guarantees the existence of a rule for each u with conditional size α, which is a measurable function of $T_1, \cdots, T_a$ for fixed u. This can be viewed as a function of S, whose measurability can be proved in exactly similar manner to that in Section 4.4 of Lehmann [3]. The second part is a consequence of (c) of Theorem 1. This leads us to the completion of the proof of (a).

(b)
$$\sum_{g\varepsilon G} E_{\pi_g i}[\varphi_{\pi_g i}(S): \gamma, \hat{g}\sigma]$$
$$= \sum_{g\varepsilon G} \int E_{\pi_g i}[\varphi_{\pi_g i}(S) \mid u; \gamma, \hat{g}\sigma]\, dP^U_{\bar{g}(i,\gamma,\sigma)}(u)$$
$$= \sum_{g\varepsilon G} \int E_{\pi_g i}[\varphi_{\pi_g i}(S) \mid g_u v; \gamma, \hat{g}\sigma]\, dP^U_{i,\gamma,\sigma}(v)$$
$$= \sum_{g\varepsilon G} \int E_{\pi_g i}[\varphi_{\pi_g i}(S) \mid g_u u; \gamma, \hat{g}\sigma]\, dP^U_{i,\gamma,\sigma}(u)$$
$$\geqq \sum_{g\varepsilon G} \int E_{\pi_g i}[\hat{\varphi}_{\pi_g i}(S) \mid g_u u; \gamma, \hat{g}\sigma]\, dP^U_{i,\gamma\ \sigma}(u)$$
$$= \sum_{g\varepsilon G} E_{\pi_g i}[\varphi_{\pi_g i}(S); \gamma, \hat{g}\sigma].$$

(c) This follows from Lemma 7.

COROLLARY 1. *For any α, there is a similar size α decision function which is the uniformly most powerful among all the decision functions which are similar size α and symmetric in power.*

The above result is not convenient in applications. The following theorem corresponds to the Theorem 1 of Section 5.1 in Lehmann [3], and is useful for applications.

THEOREM 3. *Assume the same conditions as those in Theorem 2. Let $H(x, y)$ be a measurable function, increasing in y for fixed x. Suppose $V_i = H(U, T_i)$*

$(i = 1, \cdots, a)$ are independent of U when $\theta \varepsilon \Omega_0$, then the decision function (3) can be written as

$$(6) \qquad \varphi_0(s) = 1, \xi, 0, \quad \text{if} \quad \max_i V_i <, =, > C,$$
$$\varphi_j(s) = (1 - \varphi_0(s))/k(s), 0, \quad \text{if} \quad \max_i V_i =, > V_j,$$

where $k(s)$ is the number of times $\max_i V_i$ is attained, and C and ξ are constants depending only on the size condition α.

3. Examples.

EXAMPLE 1. Slippage of normal variance [5]. Assume that we have n random observations $(x_{j1}, \cdots, x_{jn})$ $(j = 1, \cdots, a)$ from each of a $N(\theta_j, \sigma_j^2)$ populations and we wish to test $H_0: \sigma_1 = \cdots = \sigma_a = \sigma$ against $H_j: \sigma_1 = \cdots = \sigma_j(1 + \gamma)^{-\frac{1}{2}} = \cdots = \sigma_a = \sigma (j = 1 \cdots a)$ where $\gamma > 0$ and σ, γ, and $(\theta_1, \cdots, \theta_a)$ are unknown and free.

The densities under H_0 and H_j are, respectively, of the form

$$f_0(x; \theta, \sigma, 0) = c(0, 0, \omega) \exp [-\tfrac{1}{2}\sigma^{-2} \textstyle\sum_{i=1}^a \sum_{j=1}^n x_{ij}^2 + n\sigma^{-2} \sum_{i=1}^a \theta_i \bar{x}_i]$$
$$f_j(x; \theta, \sigma, \gamma) = c(j, \gamma, \omega) \exp [-\tfrac{1}{2}\sigma^{-2} \textstyle\sum_{i=1}^a \sum_{k=1}^n x_{ik}^2$$
$$+ n \textstyle\sum_{i=1}^a \theta_i \sigma_i^{-2} \bar{x}_i + \tfrac{1}{2}\sigma^{-2}(\gamma/(1 - \gamma)) \sum_{k=1}^n x_{jk}^2]$$

where $c(0, 0, \omega) = ((2\pi)^{-\frac{1}{2}}\sigma^{-1})^{na} \exp(-\frac{1}{2}n\sigma^{-2} \sum_{i=1}^a \theta_i^2)$

$$c(j, \gamma, \omega) = ((2\pi)^{-\frac{1}{2}}\sigma^{-1})^{na}(1/(1 + \gamma))^n \exp [-\tfrac{1}{2}n\sigma^{-2} \textstyle\sum_{i=1}^a \theta_i^2 + \tfrac{1}{2}n\theta_j^2\sigma^{-2}(\gamma/(1 + \gamma))]$$

and $\omega = \sigma^2, \theta_1, \theta_2, \cdots, \theta_a)$.

If we let G be the permutation group introduced in the beginning of [1], and put

$$U = [\textstyle\sum_{i=1}^a \sum_{k=1}^n x_{ik}^2, \bar{x}_1, \cdots, \bar{x}_a],$$
$$T_j = \textstyle\sum_{k=1}^n x_{jk}^2 \qquad (j = 1, \cdots, a),$$

then the conditions of Theorem 2 are satisfied.

Let

$$V_j = [\textstyle\sum_{k=1}^n x_{jk}^2 - n\bar{x}_j^2][\sum_{i=1}^a \sum_{k=1}^n x_{ik}^2 - n \sum_{i=1}^a \bar{x}_i^2]^{-1} \qquad (j = 1, \cdots, a).$$

Under H_0 the distribution of V_j $(j = 1, \cdots, a)$ does not depend on the parameters and are jointly independent of U and hence by Theorem 3 the uniformly most powerful symmetric similar size α decision function can be written as (5), i.e.

$$\varphi_0 = 1, \xi, 0 \quad \text{if} \quad \max_i V_i <, =, > C,$$
$$\varphi_j = (1 - \varphi_0)k^{-1}, 0 \quad \text{if} \quad \max_i V_i =, > V_j,$$

where k is the number of times $\max_i V_i$ is attained.

This result is identical to the solution derived by Traux [5], who imposed in addition to the condition of similarity and symmetry the following type of invariance: $\varphi(x) = \varphi(hx)$ for all $h \varepsilon H$, where H consists of transformations from x_{ij} to $ax_{ij} + b_i$, where $a > 0$, $-\infty < b_i < \infty$, $i = 1, \cdots, a$ and $j = 1, \cdots, n$.

The following examples seem not to have been considered before and the derivation of solutions seems somewhat cumbersome when we impose the assumption of invariance of the decision function with respect to change of location and/or scale.

EXAMPLE 2. Let $\{(x_{ik}, y_{ik}); k = 1, \cdots, n\}$ $(i = 0, 1)$ be random samples from bivariate normal distributions with means θ_0 and θ_1 respectively and common variance covariance matrix $\sigma^2 I$ where

$$\theta_0' = (\theta_{01}, \theta_{02}), \qquad \theta_1' = (\theta_{11}, \theta_{12})$$

and consider the problem

$$H_0: \theta_1 = \theta_0, \qquad H_j: \theta_1 = \theta_0 + \gamma\delta_j, \qquad j = 1, \cdots, a,$$

where

$$\delta_j = \begin{bmatrix} \cos 2\pi(j-1)/a \\ \sin 2\pi(j-1)/a \end{bmatrix}, \qquad \gamma > 0,$$

and θ_0, θ_1, γ and σ are unknown and free.

The densities under H_0 and H_j are, respectively, of the form

$$\begin{aligned} f_0(x, y; \theta_0, \sigma^2, 0) = c(0, 0, \omega) \exp [& -\tfrac{1}{2}\sigma^{-2} \textstyle\sum_{i=0}^{1} \sum_{j=1}^{n} (x_{ij}^2 + y_{ij}^2) \\ & + n\theta_{01}\sigma^{-2}(\bar{x}_0 + \bar{x}_1) + n\theta_{02}\sigma^{-2}(\bar{y}_0 + \bar{y}_1)], \\ f_j(x, y; \theta_0, \sigma^2, \gamma) = c(j, \gamma, \omega) \exp [& -\tfrac{1}{2}\sigma^{-2} \textstyle\sum_{i=0}^{1} \sum_{j=1}^{n} (x_{ij}^2 + y_{ij}^2) \\ & + n\theta_{01}\sigma^{-2}(\bar{x}_0 + \bar{x}_1) + n\theta_{02}\sigma^{-2}(\bar{y}_0 + \bar{y}_1) \\ & + \gamma n\sigma^{-2}(\bar{x}_1 \cos (2\pi(j-1)/a) + \bar{y}_1 \sin (2\pi(j-1)/a)] \end{aligned}$$

where $c(j, \gamma, \omega) = ((2\pi\sigma^{-\frac{1}{2}})^{2n} \exp [n\gamma\sigma^{-2}(\sum_{k=1}^{2} \theta_{0k}^2) - \frac{1}{2}n\gamma\sigma^{-2}(\theta_{01} \cos (2\pi)j - 1)/a) + \theta_{02} \sin (2\pi(j-1)/a)]$ and $\omega = (\sigma, \theta_{01}, \theta_{02})$.

Let

$$s_0 = \textstyle\sum_{i=0}^{1} \sum_{k=1}^{n} (x_{ik}^2 + y_{ik}^2), \qquad U = (s_0, \bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1)$$

and

$$T_j = \bar{x}_1 \cos 2\pi(j-1)/a + \bar{y}_1 \sin 2\pi(j-1)/a.$$

Consider a group of rotations $\{g_l\}$ of (x_{ik}, y_{ik}) given by the orthogonal matrices

$$\begin{bmatrix} \cos (2\pi l/a) & \sin (2\pi l/a) \\ -\sin (2\pi l/a) & \cos (2\pi l/a) \end{bmatrix} \qquad (l = 0, 1, \cdots, a-1).$$

It is readily seen that all the assumptions of Theorem 3 are satisfied. In particular we note that $\pi_{g_l} i = i - l \pmod{a}$, and all the groups, G, G_u, Π and $\hat{G}$ are cyclic groups of order a. The optimum decision function is given by

$$\varphi_0 = 1, \xi, 0 \quad \text{if} \quad \max_{i=1,\cdots,a} V_i <, =, > C,$$

$$\varphi_j = (1 - \varphi_0)k^{-1}, 0 \quad \text{if} \quad \max_{i=1,\cdots,a} V_i =, > V_j,$$

where

$$V_i = [(\bar{x}_1 - \bar{x}_0) \cos 2\pi(i - 1)/a + (\bar{y}_1 - \bar{y}_0) \sin 2\pi(i - 1)/a]/s$$

and $s^2 = s_0 - n(\bar{x}_0^2 + \bar{x}_1^2 + \bar{y}_0^2 + \bar{y}_1^2)$.

The generalization of this example to the p-variation situation is immediate.

EXAMPLE 3. Let y_i $(i = 1, \cdots, a)$ be $n \times 1$ vectors such that $y_i = X\beta_i + e_i$ where X is a known $n \times p$ matrix with rank $p \leqq n$ and e_i is distributed as $N(0, \sigma^2 I)$.

Consider the problem

$$H_0: \beta_j = \beta, \qquad j = 1, \cdots, a,$$

$$H_j: \beta_k = \beta, \qquad j \neq k \ (k = 1, \cdots, a) \quad \text{and} \quad \beta_j = \beta + \gamma\delta$$

where $\delta' = (0, \cdots, 0, 1)$, $\gamma > 0$, and β, γ, and σ^2 are unknown and free.

Under H_0 and H_j the densities are of the form,

$$f_0(y; \beta, \sigma^2, 0) = C' \exp\{-\tfrac{1}{2}\sigma^{-2}\textstyle\sum_{i=1}^a y_i'y_i + \sigma^{-2}\beta'X'\sum_{i=1}^a y_i\},$$

$$f_j(y; \beta, \sigma^2, \gamma) = C''f_0 \cdot \exp\{\sigma^{-2}\gamma x_p'y_j\},$$

where x_p is the last column vector in X and C'' is independent of j $(j = 1, \cdots, a)$.

Let G be the group consisting of permutations of the y_i's,

$$U = (\textstyle\sum_{i=1}^a y_i'y_i, X'\sum_{i=1}^a y_i) \qquad \text{and} \qquad T_j = x_p'y_j.$$

It can be verified that the best linear unbiased estimate $\hat{\gamma}$ of γ under H_j can be written as

$$\hat{\gamma}_j = (1 - 1/a)x_p'x_p[T_j - x_p'(a^{-1}\textstyle\sum y_i)].$$

Theorem 3 can now be used to obtain the solution

$$\varphi_0 = 1, \xi, 0 \quad \text{if} \quad \max_i V_i <, =, > C;$$

$$\varphi_j = (1 - \varphi_0)k^{-1}, 0 \quad \text{if} \quad \max_i V_i =, > V_j;$$

$$V_i = \hat{\gamma}_i/[\textstyle\sum_{i=1}^a y_i'y_i - a^{-1}(\sum y_i)'M(\sum y_i)]^{\frac{1}{2}}, \qquad M = X(X'X)^{-1}X'.$$

A generalization of this example to the situation where β_i is split into two parts: $\beta_i' = (\beta_i'^{(0)}, \beta_i'^{(1)})$ and one considers $a + 1$ hypotheses H_i $(i = 0, 1, \cdots, a)$ of the same type of $\beta_i^{(1)}$ is also straightforward.

Note that when $p = 1$ and the elements of X are all 1 this reduces to the case of Paulson [4] and further when $n = 1$ this reduces to the case of Kudô [2].

Theorem 2 is also applicable to the slippage problems with discrete distributions such as Poisson, hypergeometric, etc.

4. Note on similarity and invariance. A transformation group $\{h\}$ of S is said to leave the problem invariant if the induced group of transformations $\{\bar{h}\}$ on Ω satisfies $\bar{h}(\Omega_0) = \Omega_0$, and $\bar{h}(i, \gamma, \sigma) = (i, \gamma', \sigma')$ for all i.

THEOREM 4. *All optimum decision functions are invariant under a transformation group which leaves the problem invariant. Indeed, if the uniformly most powerful symmetric similar size α decision function exists uniquely* a.e. (μ) *then it is almost invariant under a transformation group which leaves the problem invariant.*

This can be proved in a manner similar to Theorem 6 of chapter 6 in [3].

REFERENCES

[1] HALL, I. J. and KUDÔ, A. (1968). On slippage tests—(I). A generalization of Neyman-Pearson's lemma. *Ann. Math. Statist.* **39** 1693–1699.

[2] KUDÔ, A. (1956a). On the testing of outlying observations. *Sankhyā* **17** 67–76.

[3] LEHMANN, E. L. (1959). *Testing Statistical Hypotheses.* Wiley, New York.

[4] PAULSON, E. (1952). An optimum solution to the k-sample slippage problem for the normal distribution. *Ann. Math. Statist.* **23** 610–616.

[5] TRUAX, D. R. (1953). An optimum slippage test for the variance of k normal distributions. *Ann. Math. Statist.* **24** 669–674.