The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 595–600

A NOTE ON TESTS FOR MONOTONE FAILURE RATE BASED ON INCOMPLETE DATA

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0. Abstract. Certain tests of constant failure rate versus failure rate increasing on the average are unbiased when complete samples of observations are available, as pointed out by Bickel and Doksum in the *Annals of Mathematical Statistics* (1968). In the present note, unbiasedness is proved when incomplete samples of failure data are available. A similar result is obtained for monotone tests of constant versus increasing failure rate. Finally, a table of percentiles is given to facilitate application of the total time on test statistic for testing constant failure rate versus failure rate increasing on the average.

1. Introduction and summary. Let $0 \equiv X_{(0)} \leqq X_{(1)} \leqq X_{(2)} \leqq \cdots \leqq X_{(n)}$ be the order statistics of a (complete) random sample from a population with distribution F and density f such that $F(0) = 0$. Bickel and Doksum (1968) consider the problem of testing

$$H_0 : F(t) = 1 - e^{-\lambda t} \quad t \geqq 0, \quad \lambda > 0$$

versus

$$H_1 : F \text{ IFR (i.e., } -\log [1 - F(t)] \text{ convex on } [0, \infty)).$$

Let $D_i = (n - i + 1)(X_{(i)} - X_{(i-1)})$, $i = 1, 2, \cdots, n$. They consider tests based on statistics of the form

$$(\textstyle\sum_{i=1}^n a_i D_i / \sum_{i=1}^n D_i)$$

where $a_1 \geqq a_2 \geqq \cdots \geqq a_n$. The test, ϕ_a, rejects H_0 when $\sum_{i=1}^n a_i D_i / \sum_{i=1}^n D_i \geqq c_{\alpha,a,n}$. They compute the asymptotic relative efficiency of various such tests relative to selected parametric alternatives. Such tests were shown to be unbiased against IFRA (for increasing failure rate average) alternatives by Barlow and Proschan (1966) and hence *a fortiori* for IFR alternatives. [See also Birnbaum, Esary and Marshall (1965) for justification of the IFRA assumption.]

The purpose of this note is to show that analogous tests designed to treat incomplete samples of failure data are also unbiased against IFRA alternatives. Let X_i be the time to failure of the ith item in a sample of size n. Let L_i be a given truncation time for the ith item and let

$$Z_i = \min (X_i, L_i), \qquad i = 1, 2, \cdots, n.$$

Received 25 April 1968.

[1] Research partially supported by the Office of Naval Research Contract Nonr-3656(18) with the University of California and by the Boeing Scientific Research Laboratories.

Let $0 \equiv Z_{(0)} \leqq Z_{(1)} \leqq \cdots \leqq Z_{(k)}$ be the first k observed failure times. Note that "withdrawals" may occur between $Z_{(i)}$ and $Z_{(i+1)}$ and that k is, in general, a random variable. Let $n(u)$ be the (random) number of items on test at time u.

We define a test, ϕ_a^*, (a modification of ϕ_a) which rejects H_0 in favor of

$$H_1': F \text{ IFRA (i.e., } -\{\log [1 - F(t)]\}/t \text{ nondecreasing on } [0, \infty))$$

when

$$W_a = \left(\sum_{i=1}^k a_i \int_{Z(i-1)}^{Z(i)} n(u)\, du \Big/ \int_0^{Z(k)} n(u)\, du \right) \geqq c^*_{\alpha,a,k} .$$

Note that $\int_{Z(i-1)}^{Z(i)} n(u)\, du$ represents the total time on test between the $(i - 1)$st and ith observed failures. The distribution of W_a can be computed under H_0 using the fact that $Y_i = \int_{Z(i-1)}^{Z(i)} n(u)\, du$ $(i = 1, 2, \cdots, k)$ are distributed as independent exponential random variables under H_0 conditioned on the value of k. We show that ϕ_a^* is an unbiased test for IFRA alternatives for weights $a = (a_1, a_2, \cdots, a_n)$ for which $a_1 \geqq a_2 \geqq \cdots \geqq a_n$.

2. Distribution of W_a under H_0. Let $r(t) = f(t)/[1 - F(t)]$ be the failure rate function for F. We will need the following lemma, stated without proof in Bray, Crawford, and Proschan (1967).

LEMMA 1. *For any distribution $F(F(0) = 0)$ with failure rate $r(t)$, $Y_i = \int_{Z(i-1)}^{Z(i)} r(u)n(u)\, du$, $i = 1, 2, \cdots, k$ are independently distributed with density e^{-y}.*

PROOF. Let $Y_1 = \int_0^{Z(1)} r(u)n(u)\, du$ and $S_0(t) = \int_0^t r(u)n(u)\, du$. Note that $S_0(t)$ is well defined up to the time of the first observed failure since $n(u)$ depends only on the specified truncation times L_i $(i = 1, 2, \cdots, n)$ up until $Z_{(1)}$. Then

$$P[Y_1 > y_1] = P[S_0(Z_{(1)}) > y_1] = P[Z_{(1)} > S_0^{-1}(y_1)] = \exp[-S_0(S_0^{-1}(y_1))],$$

i.e., the probability of no failure in $[0, S_0^{-1}(y_1)]$. Hence

$$P[Y_1 > y_1] = e^{-y_1}.$$

Thus Y_1 has density e^{-y_1}.

Now let $Y_2 = \int_{Z(1)}^{Z(2)} r(u)n(u)\, du$ and $S_{x_1}(t) = \int_{x_1}^t r(u)n(u)\, du$. Note that conditionally on $Z_{(1)} = x_1$, S_{x_1} is well defined for $x_1 \leqq t < Z_{(2)}$. Hence

$$\begin{aligned} P[Y_2 > y_2 \mid Z_{(1)} = x_1] &= P[S_{x_1}(Z_{(2)}) > y_2 \mid Z_{(1)} = x_1] \\ &= P[Z_{(2)} > S_{x_1}^{-1}(y_2) \mid Z_{(1)} = x_1] = \exp[-S_{x_1}(S_{x_1}^{-1}(y_2))] = e^{-y_2}. \end{aligned}$$

Thus Y_2 is independent of Y_1 and also exponentially distributed with mean 1. If we continue in this manner, conditioning on previous events, we establish the lemma. □

Under H_0, $r(t) \equiv \lambda$ and we see from the lemma that, given k observed failures,

$$W_a =_{\text{st}} \left(\sum_{i=1}^k a_i Y_i \Big/ \sum_{j=1}^k Y_j \right),$$

where $=_{\text{st}}$ denotes stochastic equality and $Y_1, Y_2, \cdots, Y_k$ are independent, exponentially distributed random variables with unit mean.

3. Unbiasedness under IFRA alternatives. We need the following lemma to establish unbiasedness. Define $R(t) = \int_0^t r(u)\,du$ and $T(t) = \int_0^t n(u)\,du$.

LEMMA 2. *If $R(t)/t$ is nondecreasing in $t \geqq 0$, $n(t) \geqq 0$, and $T(t)/t$ is nonincreasing in $t \geqq 0$, then*

(i) $r(t) \geqq \int_0^t r(u)\,du/t \geqq \int_0^t r(u)\,dT(u)/T(t)$

(ii) $\int_0^t r(u)\,dT(u)/T(t)$ *is nondecreasing in* $t \geqq 0$,

when the indicated integrals exist.

PROOF. To show (i). The first inequality follows from differentiating $R(t)/t$. Since $R(t)/t \geqq 0$ is nondecreasing in $t \geqq 0$ we can approximate $R(t)$ arbitrarily closely from below by a positive linear combination of functions of the form

$$\begin{aligned} R(t) &= 0, \quad 0 \leqq t < x, \\ &= t, \quad t \geqq x, \end{aligned}$$

[cf. Barlow, Marshall, and Proschan (1967)]. If we can establish the second inequality in (i) for functions $R(t)$ of this type, then the second inequality in (i) will hold in general, by the Lebesgue monotone convergence theorem. For $t < x$, both sides of the second inequality of (i) are zero. For $t \geqq x$,

$$\begin{aligned} \int_0^t n(u)\,dR(u)/T(t) &= [n(x)x + \textstyle\int_x^t n(u)\,du]/T(t) \\ &= 1 + [xn(x) - T(x)]/T(t). \end{aligned}$$

Thus the left side of the second inequality of (i) equals one while the right side is less than one since $xn(x) - T(x) \leqq 0$, a consequence of $T(x)/x$ being nonincreasing in $x \geqq 0$.

To show (ii). Clearly

$$(d/dt)[\textstyle\int_0^t r(u)n(u)\,du/\int_0^t n(u)\,du] \geqq 0$$

if and only if

$$r(t)n(t)\textstyle\int_0^t n(u)\,du \geqq n(t)\int_0^t r(u)n(u)\,du,$$

which follows from (i). □

Note that if $r(t)$ is nondecreasing in $t \geqq 0$, then (ii) follows for all $n(t) \geqq 0$; i.e., the assumption that $T(t)/t$ is nondecreasing may be dropped.

Lemma 2 may be used in testing for IFRA in models other than the one described in the introduction; see for example the model of Bray, Crawford, and Proschan (1967).

THEOREM 1. *If F is* IFRA *with failure rate $r(t)$ and $Z_{(1)} \leqq Z_{(2)} \leqq \cdots \leqq Z_{(k)}$ are the observed failure times, $n(t) \geqq 0$ for $t \geqq 0$, and $T(t)/t \geqq 0$ is nonincreasing in $t \geqq 0$, then (conditional on k),*

$$W_a = \textstyle\sum_{i=1}^k a_i \int_{Z_{(i-1)}}^{Z_{(i)}} n(u)\,du/\int_0^{Z_{(k)}} n(u)\,du \geqq_{\text{st}} \sum_{i=1}^k a_i Y_i/\sum_{i=1}^k Y_i,$$

where $a_1 \geqq a_2 \geqq \cdots \geqq a_n$ and $Y_1, Y_2, \cdots, Y_k$ are independently distributed as exponential random variables with unit mean.

PROOF. Since $n(u) \geqq 0$ and $T(t)/t$ is nonincreasing, Lemma 2 applies, yielding

$$\beta_i/\alpha_i = \int_0^{Z(i)} r(u)n(u)\,du \Big/ \int_0^{Z(i)} n(u)\,du$$

nondecreasing in $i = 1, 2, \cdots, k$. By lemma 1 we need only show that

$$(1)\quad \sum_{i=1}^{k} a_i \int_{Z(i-1)}^{Z(i)} n(u)\,du \Big/ \int_0^{Z(k)} n(u)\,du \geqq \sum_{i=1}^{k} a_i \int_{Z(i-1)}^{Z(i)} r(u)n(u)\,du \Big/ \int_0^{Z(k)} r(u)n(u)\,du$$

i.e.,

$$\sum_{i=1}^{k} a_i(\alpha_i - \alpha_{i-1})\alpha_k^{-1} \geqq \sum_{i=1}^{k} a_i(\beta_i - \beta_{i-1})\beta_k^{-1}$$

where $\alpha_0 = \beta_0 \equiv 0$. Note that

$$\sum_{i=1}^{k} a_i(\alpha_i - \alpha_{i-1}) = (a_1 - a_2)\alpha_1 + (a_2 - a_3)\alpha_2 + \cdots + a_k\alpha_k = \sum_{i=1}^{k} \Delta_i\alpha_i$$

where $\Delta_i = a_i - a_{i-1} \geqq 0$ for $i = 1, 2, \cdots, k-1$ and $\Delta_k = a_k$. Hence $\beta_i/\alpha_i \leqq \beta_k/\alpha_k$ implies $\sum_{i=1}^{k} \Delta_i\alpha_i/\alpha_k \geqq \sum_{i=1}^{k} \Delta_i\beta_i/\beta_k$, which proves (1). □

4. Application of total time on test. Assuming an exponential distribution, the results of Bickel and Doksum (1968) may be used to establish the asymptotic normality of W_a in the incomplete data case for selected vectors $a = (a_1, \cdots, a_k)$. Perhaps the most useful test is the total time on test statistic. In the case of a complete sample of size n, this is S_1^* in the Bickel-Doksum paper, obtained by choosing $a_i = -i/(n+1)$, after algebraic manipulation. Epstein (1960) adapted this test to the life testing problem and called it test 3. In the case of incomplete data as described in the introduction, with k failures observed; the total time on test statistic is

$$W_{a^0} = \sum_{i=1}^{k-1} (k - i) \int_{Z(i-1)}^{Z(i)} n(u)\,du \Big/ \int_0^{Z(k)} n(u)\,du,$$

obtained by choosing $a^0 = (k-1, k-2, \cdots, 1, 0)$.

The exact distribution conditioned on the number of observed failures $k \geqq 2$ is easily computed in this case. Table 1 is a short table of percentage points. Note that, under H_0

$$W_{a^0} =_{\text{st}} U_1 + U_2 + \cdots + U_{k-1}$$

where U_i $(i = 1, 2, \cdots, k-1)$ are independent uniform random variables on [0, 1]. Since the distribution of W_{a^0} is symmetric about $\frac{1}{2}(k-1)$ we tabulate upper percentiles only.

5. Monotone tests under IFR alternatives. Bickel and Doksum (1968) define a test ϕ to be monotone in the normalized spacings $D_1, \cdots, D_n$ if $\phi(D_1', \cdots, D_n') \leqq \phi(D_1, \cdots, D_n)$ for all $(D_1, \cdots, D_n)$ and $(D_1', \cdots, D_n')$ such that for $i < j$, $D_i' \geqq D_j'$ implies $D_i \geqq D_j$. We show that if D_i is replaced by $\int_{Z(i-1)}^{Z(i)} n(u)\,du$

TABLE 1

Percentiles χ_α of total time on test statistic, W_{a^0}

$k-1$	α				
	.900	.950	.975	.990	.995
2	1.553	1.684	1.776	1.859	1.900
3	2.157	2.331	2.469	2.609	2.689
4	2.753	2.953	3.120	3.300	3.411
5	3.339	3.565	3.754	3.963	4.097
6	3.917	4.166	4.376	4.610	4.762
7	4.489	4.759	4.988	5.244	5.413
8	5.056	5.346	5.592	5.869	6.053
9	5.619	5.927	6.189	6.487	6.683
10	6.178	6.504	6.781	7.097	7.307
11	6.735	7.077	7.369	7.702	7.924
12	7.289	7.647	7.953	8.302	8.535

k = number of failures observed in incomplete sample

$P[W_{a^0} \leqq \chi_\alpha] = \alpha$

in the incomplete data case, then a monotone test is unbiased for testing H_0 versus H_1 when $n(u) \geqq 0$ for $u \geqq 0$. The test rejects H_0 for large values of ϕ.

We need

LEMMA 3. *Let $r(u)\uparrow$ and $n(u) \geqq 0$ for $u \geqq 0$. Then for $0 \leqq a < b \leqq c < d$,*

$$\int_a^b n(u)r(u)\,du/\int_a^b n(u)\,du \leqq \int_c^d n(u)r(u)\,du/\int_c^d n(u)\,du.$$

PROOF.

$$\int_a^b n(u)r(u)\,du/\int_a^b n(u)\,du \leqq r(b)\int_a^b n(u)\,du/\int_a^b n(u)\,du$$
$$\leqq r(c)\int_c^d n(u)\,du/\int_c^d n(u)\,du \leqq \int_c^d n(u)r(u)\,du/\int_c^d n(u)\,du. \quad \square$$

From Lemma 3, we immediately obtain

THEOREM 2. *Let ϕ be a monotone test of H_0 versus H_1 based on a sample of incomplete data as described in the introduction. Then*

$$E[\phi(\int_0^{Z(1)} n(u)\,du, \cdots, \int_{Z(k-1)}^{Z(k)} n(u)\,du) \mid F \text{ IFR}]$$
$$\geqq E[\phi(\int_0^{Z(1)} n(u)\,du, \cdots, \int_{Z(k-1)}^{Z(k)} n(u)\,du) \mid F \textit{ exponential}]$$
$$= E[\phi(Y_1, \cdots, Y_k)],$$

where $Y_1, \cdots, Y_k$ are independent exponentially distributed random variables.

PROOF. For $i < j$,

$$\int_{Z(j-1)}^{Z(j)} n(u)\,du/\int_{Z(i-1)}^{Z(i)} n(u)\,du$$
$$\leqq \int_{Z(j-1)}^{Z(j)} r(u)n(u)\,du/\int_{Z(i-1)}^{Z(i)} r(u)n(u)\,du =_{st} Y_j/Y_i.$$

The inequality follows from Lemma 3; the stochastic equality follows from Lemma 1.

Thus $\phi(Y_1, \cdots, Y_k) \leqq_{st} \phi(\int_0^{Z(1)} n(u)\, du, \cdots, \int_{Z(k-1)}^{Z(k)} n(u)\, du)$. The conclusion follows by taking expectations. □

Acknowledgment. We would like to acknowledge the help of T. A. Bray, R. Pyke, and R. Wolff.

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