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DIVERGENCE PROPERTIES OF SOME MARTINGALE TRANSFORMS

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A martingale difference sequence $d = (d_n, n \geqq 1)$ relative to the sequence $(\mathfrak{F}_k, k \geqq 0)$ of σ-fields is said to satisfy condition MZ if

(i) $E(d_n^2 \mid \mathfrak{F}_{n-1}) = 1$ a.e. for all n
(ii) $E(|d_n| \mid \mathfrak{F}_{n-1}) \geqq K$ a.e. for some $K > 0$ and all n.

This definition is due to Richard F. Gundy, who in [3] studies transforms of martingales with difference sequences d satisfying condition MZ, that is, processes of the form $(\sum_1^n v_i d_i, n = 1, 2, \cdots)$ where $v = (v_i, i \geqq 1)$ is a sequence of functions such that v_n is $\mathfrak{F}_{n-1}$ measurable. v is called a multiplier sequence. Gundy proves that for such a transform the three sets

$$A = \{\lim \textstyle\sum_1^n v_i d_i \text{ exists and is finite}\}$$

$$B = \{\Sigma v_1^2 < \infty\}$$

$$C = \{\Sigma v_i^2 d_i^2 < \infty\}$$

are equal with probability 1. Here we prove that, as is known in some special cases, A, B, and C are equivalent to the set

$$D^+ = \{\sup \textstyle\sum_1^n v_i d_i < \infty\}$$

(and thus also to $D^- = \{\inf \sum_1^n v_i d_i > -\infty\}$). Thus the sample functions of the process $(\sum_1^n v_i d_i, n \geqq 1)$ almost surely either converge or oscillate between $-\infty$ and $+\infty$. This was conjectured by Gundy in [3]. Y. S. Chow has already given a partial answer in [2], showing the equivalence of A, B, C, and $|D| = \{\sup |\sum_1^n v_i d_i| < \infty\}$.

The definition of property MZ is extended to finite sequences of martingale differences in the obvious way. Whenever a statement like $(d_i, \mathfrak{F}_i, i \geqq 1)$ *satisfies condition* MZ is made the 0th σ-field required by the definition is to be taken as $\{\Omega, \varnothing\}$. In addition we can and do always assume without loss of generality that $E(d_1 \mid \mathfrak{F}_0) = 0$ a.e..

Lemma 1. *Let* $\epsilon > 0$. *There is a number* $\delta_\epsilon = \delta_\epsilon(K)$ *such that if* $d = (d_1, \cdots, d_n)$ *satisfies* (i) *and* (ii) *and* $(v_1, \cdots, v_n) = v$ *is a multiplier sequence such that* $P(\sum_1^n v_i^2 > \epsilon) > \epsilon$ *then*

$$P(\sup_{1 \leqq k \leqq n} |\textstyle\sum_1^k v_i d_i| > \delta_\epsilon) > \delta_\epsilon .$$

Proof. Suppose this is not true. For $i \geqq 1$ let $d_i = (d_{ij}, \mathfrak{F}_{ij}, 1 \leqq j \leqq n_i)$ be independent martingale difference sequences, that is let $\mathfrak{F}_{1n_1}, \mathfrak{F}_{2n_2}, \cdots$ be independent σ-fields, and let $(v_{ij}, 1 \leqq j \leqq n_i)$ be associated multiplier sequences such that for each i the conditions of the lemma are satisfied and such that

$$P(\sup_{1 \leqq k \leqq n_i} |\textstyle\sum_{j=1}^k v_{ij} d_{ij}| > 1/2^i) \leqq 1/2^i.$$

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Let $x_s = \sum_{i=1}^{s} n_i$, $x_0 = 0$. Consider the martingale difference sequence $d' = (d_n', \mathfrak{F}_n', n \geqq 1)$ and associated multiplier sequence $v' = (v_i', i \geqq 1)$ where $d_n' = d_{ij}$, $\mathfrak{F}_n' = \sigma(\mathfrak{F}_{1n_1}, \mathfrak{F}_{2n_2}, \cdots, \mathfrak{F}_{in_i}, \mathfrak{F}_{(i+1)j})$, and $v_n' = v_{ij}$, i and j chosen by the rules $x_{i-1} < n \leqq x_i$, $j = n - x_{i-1}$. Then $P(\sum_1^n v_i'd_i'$ converges to a finite limit$) = 1$, since

$$P(\sup_{k>0} |\textstyle\sum_1^{x_j+k} v_i'd_i' - \sum_1^{x_j} v_i'd_i'| > 1/2^j)$$
$$\leqq \textstyle\sum_{z=j}^{\infty} P(\sup_{x_z \leqq u \leqq x_{z+1}} |\sum_{i=1}^{u} v_i'd_i' - \sum_{i=1}^{x_z} v_i'd_i'| > 1/2^{z+1})$$
$$= \textstyle\sum_{z=j+1}^{\infty} P(\sup_{1 \leqq k \leqq n_z} |\sum_1^k v_{zi}\, d_{zi}| > 1/2^z) \leqq \sum_{z=j+1}^{\infty} 1/2^z = 1/2^j.$$

However $P(\sum v_i'^2 = \infty) = 1$, since the sets $A_j = \{\sum_{x_j+1}^{x_{j+1}} v_i'^2 > \epsilon\}$ are independent and all have probability exceeding ϵ. This contradicts Gundy's theorem, proving Lemma 1.

In Lemma 1 we add independent martingales and use a divergence result to show a certain class of martingales cannot be too small. This is in a sense a mirror of an argument used by Burkholder in [1], where he adds independent martingales in a slightly different way and uses a convergence result to show another class of martingales cannot be too large.

Remark. Marcinkiewicz and Zygmund show in [5] that there is a constant $\Delta(K)$ such that if $d = (d_1, \cdots, d_n)$ is a sequence of independent random variables of expectation 0 which satisfy (i) and (ii) and if $(v_1, \cdots, v_n)$ is a sequence of constants, then $E|\sum_1^n d_i v_i| \geqq \Delta(K)E((\sum_1^n d_i v_i)^2)$, implying Lemma 1 in this special case. However this does not hold in the situation under consideration here, since random walk R, stopped the first time t it hits -1 or at time n, whichever comes first, is the transform of the martingale with the first n Rademacher functions as differences by the multiplier sequences $(I_{\{k<t\}}, k = 1, \cdots, n)$ and as $n \to \infty$, $E(R^2_{\min(n,t)}) \to \infty$, $E(|R_{\min(n,t)}|) \to 2$.

Lemma 2. *There is a positive number $\rho_\epsilon = \rho_\epsilon(K)$ such that if d, v satisfy the conditions of Lemma 1 and in addition $\sum_1^n v_i^2 \leqq 2$ then $P(\sum_1^n v_i\, d_i > \rho_\epsilon) > \rho_\epsilon$.*

Proof. Let $b_k = \sum_1^k v_i\, d_i$. Then $(b_k, 1 \leqq k \leqq n)$ is a martingale. Let t be the first k such that $b_k \geqq \delta_\epsilon$. Then $E|b_n| \geqq E(|b_{\min(n,t)}|) \geqq \delta_\epsilon P(\sup |b_i| > \delta_\epsilon) \geqq \delta_\epsilon^2$ by Lemma 1. Thus $E(b_n^+) = \frac{1}{2}E(|b_n|) \geqq \frac{1}{2}\delta_\epsilon^2$. Now $E((b_n^+)^2) \leqq E(b_n^2) = E(\sum_1^n v_i^2) \leqq 2$. Thus if $P(b_n^+ > x) \leqq x$, we have $\frac{1}{2}\delta_\epsilon^2 \leqq E(b_n^+) = E(b_n^+ I_{\{b_n^+ > x\}}) + E(b_n^+ I_{\{b_n^+ \leqq x\}}) \leqq x + E((b_n^+)^2)^{\frac{1}{2}} E(I_{\{b_n^+ > x\}})^{\frac{1}{2}} \leqq x + (2x)^{\frac{1}{2}}$. This cannot happen for $x \leqq (\delta_\epsilon^2/2)^6$, and thus we can take $\rho_\epsilon = (\delta_\epsilon^2/2)^6$.

Lemma 3. *There is a positive number $\lambda_\epsilon = \lambda_\epsilon(K)$ such that if $d = (d_i, \mathfrak{F}_i, 1 \leqq i \leqq n)$ satisfies* (i) *and* (ii) *and if v is an associated multiplier sequence with $P(\sum_1^n v_i^2 > 1) > \epsilon$, then*

$$P(\sup_{1 \leqq k \leqq n} \textstyle\sum_1^k v_i\, d_i > \lambda_\epsilon) > \lambda_\epsilon.$$

Proof. Let $s_k = \sum_1^k v_i\, d_i$, and for $0 < x < \epsilon/2$ let $t(x) = t$ be the first time k that $\sum_1^k v_i^2 > x$. If $P(|v_t| \leqq 1) \geqq \epsilon/2 \geqq x$, the multiplier sequence $u = (u_k = v_k I_{\{k \leqq t \text{ and } |v_k| \leqq 1\}}, k = 1, \cdots, n)$ satisfies $\sum u_i^2 \leqq 2$, $P(\sum u_i^2 > x) \geqq \epsilon/2 > x$, and thus by Lemma 2, $P(\sup s_k > \rho_x) \geqq P(\sum_1^n u_k\, d_k > x) > \rho_x$.

On the other hand, let $T = T_x = \{|v_{t(x)}| > 1\}$ and suppose $P(T) \geqq \epsilon/2$. Then since $v_t I_T$ is $\mathfrak{F}_{t-1}$ measurable,

$$E(I_T\, d_t \operatorname{sgn} v_t) = E(I_T \operatorname{sgn} v_t E(d_t \mid \mathfrak{F}_{t-1})) = 0$$

$$E((I_T\, d_t \operatorname{sgn} v_t)^2) = E(I_T\, d_t^2) = P(T) \leqq 1$$

$$E(|I_T\, d_t \operatorname{sgn} v_t|) = E(I_T\, |d_t|) \geqq KP(T) \geqq K\epsilon/2.$$

This is the same set of conditions we faced in the proof of Lemma 2, and as in that proof we can show, if $\beta = (K\epsilon/4)^6$, that $P(I_T\, d_t \operatorname{sgn} v_t > \beta) > \beta$. Thus, since $|v_t| > 1$ on T, $P(d_t v_t > \beta) \geqq P(|v_t|\, d_t \operatorname{sgn} v_t I_T > \beta) > \beta$.

Now let $x_0 = \beta^3/8$. Since $E(s^2_{t(x)-1}) = E(\sum_1^{t(x)-1} v_i^2) \leqq x$, we have using Chebyshev's inequality that $P(|s_{t(x_0)-1}| > \beta/2) \leqq \beta/2$, so if $P(T_{x_0}) \geqq \epsilon/2$, $P(\sup s_i > \beta/2) \geqq \beta/2) \geqq P(s_{t(x_0)} > \beta/2) \geqq P(v_{t(x_0)} d_{t(x_0)} > \beta) - P(|s_{t(x_0)-1}| > \beta/2) > \beta - \beta/2 = \beta/2$. Thus we can take $\lambda_\epsilon = \min(\rho_{x_0}, \beta/2)$.

THEOREM. *If d is a martingale difference sequence which satisfies* MZ *and v is an associated multiplier sequence the sets A, B, C, D^+ are equivalent.*

PROOF. Clearly $A \subset D^+$. We will show that $B \supset D^+$. Suppose d satisfies (i) and (ii). Pick $\theta > 0$, and let $E = \{\sup \sum_1^k v_i d_i < \theta, \sum_1^\infty v_i^2 = \infty\}$. Assume $P(E) > 0$, and let $\beta = \lambda_{P(E)/2}$. Pick n so large that there is a set $R \varepsilon \mathfrak{F}_n$ such that $P(R \Delta E) \leqq \beta/3$, $P(R \cap E) > P(E)/2$. Pick γ so small that $P(\sum_1^n v_i d_i > \gamma, R) > P(R) - \beta/3$. Pick j so large that $P(\sum_{n+1}^{n+j} v_i^2 > [(\theta - \gamma)/\beta]^2, R) > P(E)/2$. Then $P(\sup_{1 \leqq x \leqq j} \sum_{n+1}^{n+x} v_i d_i > \theta - \gamma, R) \geqq \beta$ as can be seen by applying Lemma 3 to the martingale difference sequence $(d_{n+i}, \mathfrak{F}_{n+i}, 1 \leqq i \leqq j)$ and associated multiplier sequence $(I_R(\beta[\theta - \gamma]^{-1})v_{n+i}, 1 \leqq i \leqq j)$. Hence $P(\sup_k \sum_1^k v_i d_i > \theta, E) \geqq P(\sup_{1 \leqq x \leqq j} \sum_{n+1}^{n+x} v_i d_i > \theta - \gamma, R) - P(\sum_1^n v_i d_i < \gamma, R) - P(R \Delta E) \geqq \beta - \beta/3 - \beta/3 > 0$, contradicting the definition of E. Thus $P(E) = 0$, and the theorem is proved.

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