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ON BARTLETT'S TEST AND LEHMANN'S TEST FOR HOMOGENEITY OF VARIANCES[1]

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1. Introduction and summary. The purpose of this paper is to compare a modified likelihood ratio test (Bartlett [2]) with the asymptotically UMP invariant test (Lehmann [8]) for testing homogeneity of variances of k normal populations. We denote these tests by the "M test" and "L test," respectively. The M test has been investigated by many authors, whereas the L test has not. Fitting beta type distributions, Mahalanobis [9] and Nayer [11] computed the percentage points of M, when the numbers of observations in each sample are the same. Nayer's results were confirmed by Bishop and Nair [3], using the exact null distribution of M in a form of infinite series derived by Nair [10]. Asymptotic series expansion of the null distribution of M was obtained by Hartley [6], using Mellin inversion formula, from which tables for percentage points were calculated by Thompson and Merrington [16], without assuming the equality of k-sample sizes. Later in a more general formulation, Box [4] derived the asymptotic series expansions of the null distributions of many test statistics, including that of the M test, by using the characteristic function. Recently Pearson [12] obtained some approximate powers of the M test both by fitting a gamma type distribution to the inverse of the modified likelihood ratio statistic and by using the Monte Carlo method. No attempt was made, however, to investigate the asymptotic non-null distribution of M. Sugiura [18] has shown the limiting distribution of M in multivariate case under fixed alternative hypothesis to be normal.

In Section 2 of this paper we shall show that the L test is not unbiased, though the M test is known to be unbiased (Pitman [14], Sugiura and Nagao [19]). In Section 3, we shall derive the limiting distributions of L and M under sequences of alternative hypothesis with arbitrary rate of convergence to the null hypothesis as sample sizes tend to infinity. Limiting distributions are characterized by χ^2, noncentral χ^2, and normal distributions, according to the rate of convergence of the sequence. In Section 4, asymptotic expansion of the null distribution of L is given in terms of χ^2-distributions, and asymptotic formulas for the percentage points of L and M are obtained by using the general inverse expansion formula of Hill and Davis [7], with some numerical examples. In Section 5, asymptotic expansions of the non-null distributions of L and M under a fixed alternative hypothesis are derived in terms of normal distribution func-

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tion and its derivatives, from which approximate powers are computed. It may be remarked that the limiting non-null distributions of L and M degenerate at the null hypothesis, by which asymptotic null distributions cannot be derived.

2. Biasedness of the L test. Let $X_{i1}, X_{i2}, \cdots, X_{iN_i}$ be a random sample from a normal distribution with mean μ_i and variance σ_i^2 $(i = 1, 2, \cdots, k)$. For testing the hypothesis $H: \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2$ against all alternatives $K: \sigma_i^2 \neq \sigma_j^2$ for some i and j $(i \neq j)$ with unspecified μ_i, the L test criterion due to Lehmann ([8], page 274–275) is given by

$$(2.1) \qquad L = \tfrac{1}{2} \textstyle\sum_{\alpha=1}^{k} n_\alpha \{\log (S_\alpha/n_\alpha) - n^{-1} \sum_{\beta=1}^{k} n_\beta \log (S_\beta/n_\beta)\}^2,$$

where $S_j = \sum_{\alpha=1}^{N_j} (X_{j\alpha} - \bar{X}_j)^2$ with $\bar{X}_j = N_j^{-1} \sum_{\alpha=1}^{N_j} X_{j\alpha}$, and $n_j = N_j - 1$ with $n = \sum_{\alpha=1}^{k} n_\alpha$. The M test criterion due to Bartlett [2], without correction factor, is given by

$$(2.2) \qquad M = n \log (\textstyle\sum_{\alpha=1}^{k} S_\alpha/n) - \sum_{\alpha=1}^{k} n_\alpha \log (S_\alpha/n_\alpha)$$

with the same notation as above. The L (or M) test rejects the hypothesis H, when the observed value of L (or M) is larger than a preassigned constant. The M test is equivalent to the modified liklihood ratio test known to be unbiased (Pitman [14]). The modification consists of replacing sample size N_α by degrees of freedom n_α. The following theorem shows that the L test is not always unbiased.

THEOREM 2.1. *In the two-sample problem* $(k = 2)$, *the L test is unbiased if and only if the two sample sizes are equal. In this case* $(n_1 = n_2)$, *the L test is equivalent to the M test.*

PROOF. If $k = 2$, $L = (n_1 n_2/2n)\{\log (n_2 S_1/n_1 S_2)\}^2$ and the acceptance region of the L test is simply $c < n_2 S_1/(n_1 S_2) < 1/c$ for some constant c $(0 < c < 1)$. Putting the derivative of the power function at the null hypothesis to zero, Ramachandran [15] showed that an acceptance region, $c_1 < n_2 S_1/(n_1 S_2) < c_2$ for any constants c_1 and c_2 $(0 < c_1 < c_2)$, gives an unbiased test if and only if the condition

$$(2.3) \qquad c_2^{n_1}(1 + n_1 c_2/n_2)^{-n} = c_1^{n_1}(1 + n_1 c_1/n_2)^{-n}$$

is satisfied. In our case $c_1 = c$, $c_2 = 1/c$ and (2.3) becomes

$$(2.4) \qquad c^{n_1 - n_2} = [(n_2 + n_1 c)/(n_1 + n_2 c)]^n$$

for $0 < c < 1$. To show that (2.4) has a solution if and only if $n_1 = n_2$, take logarithms of both sides and define

$$(2.5) \qquad g(\alpha) = (\alpha - \bar{\alpha}) \log c - \log [(\bar{\alpha} + \alpha c)/(\alpha + \bar{\alpha} c)],$$

where $\alpha = n_1/n$, $\bar{\alpha} = 1 - \alpha$. It is easily verified that $g(0) = g(\frac{1}{2}) = g(1) = 0$ and $g'' < 0$ for $0 < \alpha < \frac{1}{2}$, $g'' > 0$ for $\frac{1}{2} < \alpha < 1$. Hence the only solution is $n_1 = n_2$ (excluding the case n_1 or n_2 is 0). When $n_1 = n_2$, $M = \frac{1}{2} n \log [(\frac{1}{4})\{1 + (S_1/S_2)\}^2 (S_2/S_1)]$, the acceptance region of which is equivalent to $c < S_1/S_2 < 1/c$ for some c. Hence our proof is completed.

3. Limiting distributions under sequences of alternative hypotheses. Since the statistic

$$S_\alpha/\sigma_\alpha^2 = \sum_{\beta=1}^{N_\alpha} (X_{\alpha\beta} - \bar{X}_\alpha)^2/\sigma_\alpha^2$$

has a χ^2-distribution with n_α degrees of freedom under the alternative K, the statistic $T_\alpha = [(S_\alpha/\sigma_\alpha^2) - n_\alpha]/(2n_\alpha)^{\frac{1}{2}}$ has asymptotically the standard normal distribution as n_α tends to infinity. To express the statistics L and M in terms of $T_1, \cdots, T_k$, put $n_\alpha = \rho_\alpha n$ with $\sum_{\alpha=1}^k \rho_\alpha = 1$. Clearly

$$\log(S_\alpha/n_\alpha) = \log \sigma_\alpha^2 + \log(1 + (2/n_\alpha)^{\frac{1}{2}} T_\alpha), \tag{3.1}$$

which implies, for large n with fixed $\rho_\alpha(> 0)$,

$$\begin{aligned} L &= \tfrac{1}{2}n \sum_{\alpha=1}^k \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2 + n \sum_{\alpha=1}^k \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma}) \log(1 + (2/n_\alpha)^{\frac{1}{2}} T_\alpha) \\ &\quad + \tfrac{1}{2}n \sum_{\alpha=1}^k \rho_\alpha\{\log(1 + (2/n_\alpha)^{\frac{1}{2}} T_\alpha) \\ &\quad - \sum_{\beta=1}^k \rho_\beta \log(1 + (2/n_\beta)^{\frac{1}{2}} T_\beta)\}^2 \\ &= (n/2) \sum_{\alpha=1}^k \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2 + (2n)^{\frac{1}{2}} \sum_{\alpha=1}^k \rho_\alpha^{\frac{1}{2}}(\tilde{\sigma}_\alpha - \tilde{\sigma}) T_\alpha \\ &\quad + \sum_{\alpha=1}^k (\tilde{\sigma} - \tilde{\sigma}_\alpha + 1) T_\alpha^2 - (\sum_{\alpha=1}^k \rho_\alpha^{\frac{1}{2}} T_\alpha)^2 + O_p(n^{-\frac{1}{2}}) \end{aligned} \tag{3.2}$$

where $\tilde{\sigma}_\alpha = \log \sigma_\alpha^2$ and $\tilde{\sigma} = \sum_{\alpha=1}^k \rho_\alpha \log \sigma_\alpha^2$. Similarly

$$\begin{aligned} M = n(\log \bar{\sigma} - \tilde{\sigma}) + (2n)^{\frac{1}{2}} \sum_{\alpha=1}^k (\nu_\alpha - 1)\rho_\alpha^{\frac{1}{2}} T_\alpha + \sum_{\alpha=1}^k T_\alpha^2 \\ - (\sum_{\alpha=1}^k \rho_\alpha^{\frac{1}{2}} \nu_\alpha T_\alpha)^2 + O_p(n^{-\frac{1}{2}}), \end{aligned} \tag{3.3}$$

with $\bar{\sigma} = \sum_{\alpha=1}^k \rho_\alpha \sigma_\alpha^2$ and $\nu_\alpha = \sigma_\alpha^2/\bar{\sigma}$.

Now we shall specify the sequence of alternatives K_δ as $\sigma_\alpha = \sigma + \theta_\alpha n^{-\delta}$ for $\alpha = 1, 2, \cdots, k$ and $\delta > 0$, where not all θ's are assumed to be equal. If $0 < \delta < \frac{1}{2}$, we can rewrite the expression of L in (3.2) as

$$\begin{aligned} L = (n/2) \sum_{\alpha=1}^k \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2 \\ + 2\sigma^{-1} 2^{\frac{1}{2}} n^{\frac{1}{2}-\delta} \sum_{\alpha=1}^k (\theta_\alpha - \bar{\theta}) \rho_\alpha^{\frac{1}{2}} T_\alpha + O_p(n^{\frac{1}{2}-2\delta}), \end{aligned} \tag{3.4}$$

where $\bar{\theta} = \sum_{\alpha=1}^k \rho_\alpha \theta_\alpha$ and $(n/2) \sum_{\alpha=1}^k \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2 = O(n^{1-2\delta})$. This means that the statistic $n^{\delta-\frac{1}{2}}[L - \frac{1}{2}n \sum_{\alpha=1}^k \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2]$ has asymptotically a normal distribution with mean zero and variance $\tau_L^2 = (8/\sigma^2) \sum_{\alpha=1}^k \rho_\alpha(\theta_\alpha - \bar{\theta})^2$. If $\delta > \frac{1}{2}$ we can write the statistic L from (3.2) as

$$L = \sum_{\alpha=1}^k T_\alpha^2 - (\sum_{\alpha=1}^k \rho_\alpha^{\frac{1}{2}} T_\alpha)^2 + O_p(n^{\frac{1}{2}-\delta}), \tag{3.5}$$

which shows that L has asymptotically a χ^2-distribution with $k - 1$ degrees of freedom. In this case, the sequence of alternatives K_δ converges so fast to the null hypothesis that the limiting distribution is the same as under the null hypothesis. On the boundary $\delta = \frac{1}{2}$, we can write

$$L = \sum_{\alpha=1}^k \{T_\alpha + (2\rho_\alpha)^{\frac{1}{2}}(\theta_\alpha - \bar{\theta})/\sigma\}^2 - (\sum_{\alpha=1}^k \rho_\alpha^{\frac{1}{2}} T_\alpha)^2 + O_p(n^{-\frac{1}{2}}). \tag{3.6}$$

Thus the statistic L has asymptotically a noncentral χ^2-distribution with $k - 1$ degrees of freedom and noncentrality parameter $\delta_L^2 = (2/\sigma^2) \sum_{\alpha=1}^k \rho_\alpha(\theta_\alpha - \bar{\theta})^2$. Summarizing the above results, we have the following theorem.

THEOREM 3.1. *Under the sequence of alternatives* $K_\delta: \sigma_\alpha = \sigma + \theta_\alpha n^{-\delta}$ *for* $\alpha = 1, 2, \cdots, k$ *and* $\delta > 0$, *where not all* θ's *are equal, the limiting distributions of the test statistic L given by* (2.1) *for large n with fixed* $\rho_\alpha = n_\alpha/n > 0$ *are the following*:

(1) *When* $0 < \delta < \frac{1}{2}$, $n^{\delta - \frac{1}{2}}\{L - (n/2) \sum_{\alpha=1}^{k} \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2\}$ *has asymptotically a normal distribution with mean zero and variance* $\tau_L^2 = (8/\sigma^2) \sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})^2$, *where* $\tilde{\theta} = \sum_{\alpha=1}^{k} \rho_\alpha\theta_\alpha$, $\tilde{\sigma}_\alpha = \log \sigma_\alpha^2$ *and* $\tilde{\sigma} = \sum_{\alpha=1}^{k} \rho_\alpha\tilde{\sigma}_\alpha$.

(2) *When* $\delta > \frac{1}{2}$, *L has asymptotically a* χ^2*-distribution with* $k - 1$ *degrees of freedom.*

(3) *When* $\delta = \frac{1}{2}$, *L has asymptotically a noncentral* χ^2*-distribution with* $k - 1$ *degrees of freedom and noncentrality parameter* $\delta_L^2 = (2/\sigma^2) \sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})^2$.

The result (3) in the above theorem has been used in discussing the asymptotic relative efficiency of nonparametric tests for scale parameters by Deshpande [5] and Sugiura [17]. However, for completeness, we have included it in the statement of the theorem. Similarly we have the following results for the modified likelihood ratio statistic M given by (2.2) from the expression (3.3) of M.

THEOREM 3.2. *Under the same assumptions as in Theorem* 3.1, *the limiting distributions of the test statistic M under* K_δ *are the following*:

(1) *When* $0 < \delta < \frac{1}{2}$, $n^{\delta - \frac{1}{2}}\{M - n \log (\sum_{\alpha=1}^{k} \rho_\alpha\sigma_\alpha^2 / \prod_{\alpha=1}^{k} \sigma_\alpha^{2\rho_\alpha})\}$ *has asymptotically a normal distribution with mean zero and variance*

$$\tau_M^2 = (8/\sigma^2) \sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})^2.$$

(2) *When* $\delta > \frac{1}{2}$, *M has asymptotically a* χ^2*-distribution with* $k - 1$ *degrees of freedom.*

(3) *When* $\delta = \frac{1}{2}$, *M has asymptotically a noncentral* χ^2*-distribution with* $k - 1$ *degrees of freedom and noncentrality parameter* $\delta_M^2 = (2/\sigma^2) \sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})^2$.

Noting that the two noncentrality parameters δ_L^2 and δ_M^2 in Theorem 3.1 and Theorem 3.2 are the same, we immediately have the following corollary.

COROLLARY. *Pitman's asymptotic relative efficiency of the L test with respect to the M test is equal to* 1.

When $\delta \geqq \frac{1}{2}$, the limiting distributions of L and M are the same. Even when $0 < \delta < \frac{1}{2}$, the asymptotic variances τ_L^2 and τ_M^2 are equal. Thus we are interested in the asymptotic means of L and M, namely, in cases (1), $E_L = (n/2) \sum_{\alpha=1}^{k} \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2$ and $E_M = n\{\log (\sum_{\alpha=1}^{k} \rho_\alpha\sigma_\alpha^2) - \sum_{\alpha=1}^{k} \rho_\alpha \log \sigma_\alpha^2\}$. We can expect the L test to have the larger asymptotic power when $E_L > E_M$, and the smaller asymptotic power, when $E_L < E_M$. We can easily see that

$$\begin{aligned} E_L &= \sigma^{-2} 2n^{1-2\delta} \textstyle\sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})^2 \\ &\qquad - \sigma^{-3} 2n^{1-3\delta} \textstyle\sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})\theta_\alpha^2 + O(n^{1-4\delta}) \\ E_M &= \sigma^{-2} 2n^{1-2\delta} \textstyle\sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})^2 \\ &\qquad - \tfrac{1}{3}\sigma^{-3} 2n^{1-3\delta} \textstyle\sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})(\theta_\alpha + 2\tilde{\theta})^2 + O(n^{1-4\delta}). \end{aligned} \tag{3.7}$$

Hence the first main terms in the above expansions of E_L and E_M are equal. Putting $\delta = \frac{1}{4}$ and $\theta_1 = \theta_2 = \cdots = \theta_{k-1} = \theta$ (equality of first $k - 1$ variances), we have

$$E_L - E_M = \tfrac{1}{3}\sigma^{-3} 4n^{\frac{1}{4}}\rho_k(1 - \rho_k)(1 - 2\rho_k)(\theta - \theta_k)^3 + O(1). \tag{3.8}$$

Hence for large n, $E_L > E_M$, when $\rho_k < \frac{1}{2}$ and $\theta > \theta_k$, whereas the reverse inequality holds when $\rho_k < \frac{1}{2}$ and $\theta < \theta_k$. We cannot make a unique choice from the two tests L and M which will be better against all alternatives from the asymptotic powers near hypothesis.

4. Expansions of the null distributions of L and M. We shall first derive an asymptotic expansion of the null distribution of L given by (2.1). The statistic L is rewritten as

$$(4.1)\quad L = \tfrac{1}{2}[\textstyle\sum_{\alpha=1}^k n_\alpha\{\log(S_\alpha/n_\alpha)\}^2 - n^{-1}\{\sum_{\beta=1}^k n_\beta \log(S_\beta/n_\beta)\}^2].$$

Under the null hypothesis H, we may assume that S_α has a χ^2-distribution with n_α degrees of freedom. Thus the statistic $(n_\alpha/2)^{\frac{1}{2}}\log(S_\alpha/n_\alpha)$ has the density function

$$(4.2)\qquad c_{n_\alpha}\exp\{(\tfrac{1}{2}n_\alpha)^{\frac{1}{2}}y - \tfrac{1}{2}n_\alpha \exp[(2/n_\alpha)^{\frac{1}{2}}y]\},\quad -\infty < y < +\infty,$$

where $c_n = (n/2)^{\frac{1}{2}(n-1)}\{\Gamma(\frac{1}{2}n)\}^{-1}$. We can express the characteristic function of L as

$$(4.3)\quad C(t) = (\textstyle\prod_{\alpha=1}^k c_{n_\alpha})\int \exp[it\sum_{\alpha=1}^k y_\alpha^2 - it(\sum_{\alpha=1}^k \rho_\alpha^{\frac{1}{2}}y_\alpha)^2]$$
$$\cdot\exp\{\textstyle\sum_{\alpha=1}^k((\frac{1}{2}n_\alpha)^{\frac{1}{2}}y_\alpha - \frac{1}{2}n_\alpha\exp[(2/n_\alpha)^{\frac{1}{2}}y_\alpha])\}\,dy_1\cdots dy_k$$

The second exponential part in the above integrand is expanded asymptotically for large n using the formula

$$(4.4)\quad \textstyle\sum_{\alpha=1}^k n_\alpha\exp[(2/n_\alpha)^{\frac{1}{2}}y_\alpha]$$
$$= n + \textstyle\sum_{\alpha=1}^k((2n_\alpha)^{\frac{1}{2}}y_\alpha + y_\alpha^2 + \frac{1}{3}n_\alpha^{-\frac{1}{2}}2^{\frac{1}{2}}y_\alpha^3 + \frac{1}{6}n_\alpha^{-1}y_\alpha^4) + O(n^{-3/2}).$$

We find

$$C(t) = (\textstyle\prod_{\alpha=1}^k c_{n_\alpha}e^{-\frac{1}{2}n_\alpha})\{\int\exp[(it-\frac{1}{2})\sum_{\alpha=1}^k y_\alpha^2 - it(\sum_{\alpha=1}^k\rho_\alpha^{\frac{1}{2}}y_\alpha)^2]$$
$$(4.5)\qquad \cdot\{1 - \tfrac{1}{3}2^{-\frac{1}{2}}\textstyle\sum_{\alpha=1}^k n_\alpha^{-\frac{1}{2}}y_\alpha^3 - (1/12)\sum_{\alpha=1}^k n_\alpha^{-1}y_\alpha^4$$
$$+ (1/36)(\textstyle\sum_{\alpha=1}^k n_\alpha^{-\frac{1}{2}}y_\alpha^3)^2\}\,dy_1\cdots dy_k + O(n^{-3/2})\}.$$

The quadratic form $(it - \frac{1}{2})\sum_{\alpha=1}^k y_\alpha^2 - it(\sum_{\alpha=1}^k \rho_\alpha^{\frac{1}{2}}y_\alpha)^2$ can be written as $-\frac{1}{2}y'\Sigma^{-1}y$, where $y' = (y_1, y_2, \cdots, y_k)$ and $\Sigma = (\sigma_{\alpha\beta})_{\alpha,\beta=1\ldots k}$ with

$$(4.6)\qquad \sigma_{\alpha\beta} = (\delta_{\alpha\beta} - 2it(\rho_\alpha\rho_\beta)^{\frac{1}{2}})/(1-2it).$$

The symmetric matrix Σ has a simple characteristic root equal to 1 and $(k-1)$-ple root equal to $1/(1-2it)$. Noting that all characteristic roots of Σ have positive real parts, we can use the following well-known formulas based on moments of the k-variate normal distribution with mean zero and covariance matrix $\Sigma = (\sigma_{\alpha\beta})$, to get the integral in (4.5).

$$(4.7)\qquad E[Y_\alpha^{2l+1}] = 0,\quad E[Y_\alpha^4] = 3\sigma_{\alpha\alpha}^2,\quad E[Y_\alpha^6] = 15\sigma_{\alpha\alpha}^3,$$
$$E[Y_\alpha^3Y_\beta^3] = 9\sigma_{\alpha\alpha}\sigma_{\beta\beta}\sigma_{\alpha\beta} + 6\sigma_{\alpha\beta}^3.$$

Since all product moments of odd degree from a normal population with mean zero are zero, we can see that the term of order $n^{-3/2}$ in (4.5) vanishes, giving

$$\begin{aligned} C(t) = {} & (\textstyle\prod_{\alpha=1}^{k} c_{n_\alpha} e^{-n_\alpha/2})(2\pi)^{\frac{1}{2}k}(1 - 2it)^{-(k-1)/2} \\ & \cdot[1 - \tfrac{1}{4}n^{-1} \textstyle\sum_{\alpha=1}^{k} \rho_\alpha^{-1}((1 - 2it\rho_\alpha)/(1 - 2it))^2 \\ & + (5/12)n^{-1} \textstyle\sum_{\alpha=1}^{k} \rho_\alpha^{-1}((1 - 2it\rho_\alpha)/(1 - 2it))^3 \\ & + \tfrac{1}{4}n^{-1} \textstyle\sum_{\alpha\neq\beta} (1 - 2it\rho_\alpha)(1 - 2it\rho_\beta)(-2it)/(1 - 2it)^3 \\ & + (\tfrac{1}{6})n^{-1}((-2it)/(1 - 2it))^3 \textstyle\sum_{\alpha\neq\beta} \rho_\alpha\rho_\beta + O(n^{-2})]. \end{aligned} \tag{4.8}$$

Applying Stirling's formula $\log \Gamma(x) = \log (2\pi)^{\frac{1}{2}} + (x - \frac{1}{2}) \log x - x - (1/12)x^{-1} + O(x^{-2})$, to the coefficient c_{n_α} in (4.8), we get

$$\textstyle\prod_{\alpha=1}^{k} (c_{n_\alpha} e^{-\frac{1}{2}n_\alpha}(2\pi)^{\frac{1}{2}}) = 1 - (\frac{1}{6})n^{-1} \sum_{\alpha=1}^{k} \rho_\alpha^{-1} + O(n^{-2}). \tag{4.9}$$

Arranging the second factor of the characteristic function (4.8) according to the magnitude of negative powers of $(1 - 2it)$, and using (4.9), we obtain the following asymptotic formula.

$$\begin{aligned} C(t) = {} & (1 - 2it)^{-(k-1)/2}[1 + (1/12)n^{-1}\{2(1 - \tilde{\rho}) \\ & + (3k^2 + 6k - 6 - 3\tilde{\rho})/(1 - 2it)^2 \\ & - (3k^2 + 6k - 4 - 5\tilde{\rho})/(1 - 2it)^3\}] + O(n^{-2}), \end{aligned} \tag{4.10}$$

where $\tilde{\rho} = \sum_{\alpha=1}^{k} \rho_\alpha^{-1}$. Inversion of this characteristic function yields the following theorem.

THEOREM 4.1. *The null distribution of Lehmann's test statistic L given by (2.1), expanded asymptotically in terms of the χ^2-distributions for large n with fixed $\rho_\alpha = n_\alpha/n$ (positive), is*

$$\begin{aligned} P(L < z) = {} & P_{k-1} + (1/12)n^{-1}\{a_1P_{k-1} + a_2P_{k+3} + a_3P_{k+5}\} + O(n^{-2}), \\ & a_1 = 2(1 - \tilde{\rho}), \qquad a_2 = 3k^2 + 6k - 6 - 3\tilde{\rho}, \\ & a_3 = -3k^2 - 6k + 4 + 5\tilde{\rho}, \end{aligned} \tag{4.11}$$

where $P_f = P(\chi_f^2 < z)$ and $\tilde{\rho} = \sum_{\alpha=1}^{k} \rho_\alpha^{-1}$.

From this theorem, we can easily get the asymptotic mean of the statistic L under the null hypothesis H,

$$E[L \mid H] = k - 1 + (3/2) \textstyle\sum_{\alpha=1}^{k} n_\alpha^{-1} - \frac{1}{2}k(k + 2)/n + O(n^{-2}). \tag{4.12}$$

This can also be obtained by computing directly the asymptotic means of $\log (S_\alpha/n_\alpha)$ and $\{\log (S_\alpha/n_\alpha)\}^2$ in (2.1). A correction factor d can be determined such that $E[dL \mid H] = k - 1 + O(n^{-2})$, that is the expectation of dL is equal to the mean of the limiting distribution up to the order n^{-2}. We have

$$d = [1 + \tfrac{1}{2}(k - 1)^{-1}\{3 \textstyle\sum_{\alpha=1}^{k} n_\alpha^{-1} - k(k + 2)/n\}]^{-1}. \tag{4.13}$$

Then the statistic dL is expected to show better approximation by χ^2-variate with $k - 1$ degrees of freedom, for large n. We could not choose, however, a correction factor such that the term of order n^{-1} in the asymptotic expansion of the null distribution vanishes, as is the case with Bartlett's test M.

We shall take a correction factor c for the M test, because of Box [4], as

$$c = 1 - \tfrac{1}{3}(k-1)^{-1}\left(\textstyle\sum_{\alpha=1}^{k} n_\alpha^{-1} - n^{-1}\right), \tag{4.14}$$

which is asymptotically equivalent to Bartlett's correction up to the order n^{-1}, Bartlett [2]. Then (Box [4], Anderson [1], page 255) we can get the asymptotic expansion of the null distribution of cM as

$$\begin{aligned} P(cM < z) = {} & P_f + m^{-2}\omega_2(P_{f+4} - P_f) + m^{-3}\omega_3(P_{f+6} - P_f) \\ & + m^{-4}\{\omega_4(P_{f+8} - P_f) - \omega_2^2(P_{f+4} - P_f)\} + O(m^{-5}), \end{aligned} \tag{4.15}$$

where $m = cn$, $P_f = P(\chi_f^2 < z)$ with $f = k - 1$ and

$$\begin{aligned} \omega_2 &= -(\tilde{\rho} - 1)^2/36(k-1) \\ \omega_3 &= (\tilde{\rho} - 1)^3/81(k-1)^2 - (\tilde{\rho}_3 - 1)/45 \\ \omega_4 &= \omega_2^2/2 - (\tilde{\rho} - 1)^4/216(k-1)^3 + (\tilde{\rho} - 1)(\tilde{\rho}_3 - 1)/45(k-1), \end{aligned} \tag{4.16}$$

with $\tilde{\rho} = \sum_{\alpha=1}^{k} \rho_\alpha^{-1}$ and $\tilde{\rho}_3 = \sum_{\alpha=1}^{k} \rho_\alpha^{-3}$. Applying the general inverse expansion formula of Hill and Davis [7] to (4.11) and (4.15), we can get the asymptotic formulas for percentage points of L and cM in terms of the percentage point u of the χ^2-distribution with f degrees of freedom as

$$\begin{aligned} u + [u/(6nf_{(3)})][(\tilde{\rho} - 1)\{2(f+2)(f+4) + 2u(f+4) + 5u^2\} \\ - 3f(f+4)u^2] + O(n^{-2}), \end{aligned} \tag{4.17}$$

$$\begin{aligned} & u + m^{-2}2\omega_2 \textstyle\sum_{\alpha=1}^{2} u^\alpha/f_{(\alpha)} + m^{-3}2\omega_3 \textstyle\sum_{\alpha=1}^{3} u^\alpha/f_{(\alpha)} \\ & + m^{-4}\{\omega_4 \textstyle\sum_{\alpha=1}^{4} 2u^\alpha/f_{(\alpha)} - [\omega_2^2 u/f_{(2)}^2](u+f+2)(u^2 - 4u + f^2 - 4)\} \\ & + O(m^{-5}), \end{aligned} \tag{4.18}$$

where $f_{(\alpha)} = f(f+2)\cdots(f+2\alpha-2)$. We shall examine the effectiveness of these formulas in the following examples.

EXAMPLE 4.1. Using the 5% points of the χ^2-distribution in Pearson and Hartley ([13], page 136), we get the following approximate 5% point of the L test from (4.17).

	Case 1	Case 2
	$n_1 = n_2 = 50$	$n_1 = 50, n_2 = 100, n_3 = 150$
first term	3.841	5.991
term of order n^{-1}	0.088	0.118
approximate value	3.929	6.109

The improvement of the approximations to the 5% points of L compared with the formula (4.11) and that using the correction factor d in (4.13) are shown below:

Case 1	Case 2
$P(L > 3.841) = 0.0526 + O(n^{-2})$	$P(L > 5.991) = 0.0529 + O(n^{-2})$
$P(dL > 3.841) = 0.0503 + O(n^{-2})$	$P(dL > 5.991) = 0.0507 + O(n^{-2})$
$P(L > 3.929) = 0.0500 + O(n^{-2})$	$P(L > 6.109) = 0.0500 + O(n^{-2})$.

EXAMPLE 4.2. The asymptotic formula (4.18) for the percentage point of the cM test gives the following results.

	Case 1	Case 2	Case 3	Case 4
	$n_1 = n_2 = 50$	$n_1 = 50$, $n_2 = 100$, $n_3 = 150$	$n_1 = 4$, $n_2 = 20$	$n_1 = \cdots = n_5 = 4$
first term	3.84146	5.99147	3.8415	9.4877
term of $O(m^{-2})$	−0.00045	−0.00023	−0.0389	−0.1512
term of $O(m^{-3})$	0	0.00000	−0.0045	−0.0116
term of $O(m^{-4})$	0.00000	0.00000	0.0030	0.0165
approximate value	3.84101	5.99124	3.801	9.341

When $k = 2$, both the L test and the cM test are equivalent to F tests, based on the different acceptance regions, except when $n_1 = n_2$, and the exact 5% points of cM can be computed by Table 743 in Ramachandran [15], giving 3.80 in Case 3 ($n_1 = 4$, $n_2 = 20$). Thus our approximate 5% point 3.801 is accurate, at least to two decimal places. For $k > 2$, Thompson and Merrington [16] gave tables for 5% and 1% points of M, based on the asymptotic formula of the distribution of M by Hartley [6], which were reproduced in Pearson and Hartley [13]. They showed 10.37 as a 5% point of M in Case 4 ($n_1 = n_2 = \cdots = n_5 = 4$), that is, 9.333 as 5% point of cM, the exact value of which, due to Bishop and Nair [3], is 10.38 for M (9.342 for cM) (see Thompson and Merrington [16]). Hence our approximate value 9.341 is reasonable. It should be noted that in Case 4, the term of order m^{-4} is larger than of order m^{-3} in absolute values, which shows irregularity of the asymptotic expansion for small n_α.

5. Expansions of the non-null distributions.

5.1 *Expansion of the non-null distribution of L.* We shall consider the asymptotic expansion of the non-null distribution of L (Lehmann's test) under a fixed alternative. Putting

$$L' = L - \tfrac{1}{2}n \sum_{\alpha=1}^{k} \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2 \tag{5.1}$$

in (3.2), we can easily see that $(L'/n^{\frac{1}{2}}) - \sum_{\alpha=1}^{k} (2\rho_\alpha)^{\frac{1}{2}}(\tilde{\sigma}_\alpha - \tilde{\sigma})T_\alpha = O_p(n^{-\frac{1}{2}})$. Hence the statistic $L'/n^{\frac{1}{2}}$ converges in law to the normal distribution with mean zero and variance $\tau_L^2 = \sum_{\alpha=1}^{k} 2\rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2$. More precisely we have

$$n^{-\frac{1}{2}}L' = l_0(T) + n^{-\frac{1}{2}}l_1(T) + n^{-1}l_2(T) + O_p(n^{-3/2}), \tag{5.2}$$

where

$$
\begin{aligned}
l_0(T) &= \textstyle\sum_{\alpha=1}^{k} (2\rho_\alpha)^{\frac{1}{2}}(\tilde{\sigma}_\alpha - \tilde{\sigma})T_\alpha ; \\
l_1(T) &= \textstyle\sum_{\alpha=1}^{k} a_\alpha T_\alpha^{\,2} - (\sum_{\alpha=1}^{k} \rho_\alpha^{\frac{1}{2}} T_\alpha)^2 ; \\
l_2(T) &= \textstyle\sum_{\alpha=1}^{k} a_\alpha' T_\alpha^{\,3} + (\sum_{\alpha=1}^{k} (2\rho_\alpha)^{\frac{1}{2}} T_\alpha)(\sum_{\alpha=1}^{k} T_\alpha^{\,2}) ;
\end{aligned} \tag{5.3}
$$

with $a_\alpha = \tilde{\sigma} - \tilde{\sigma}_\alpha + 1$ and $a_\alpha' = 2^{\frac{1}{2}}\rho_\alpha^{-\frac{1}{2}}\{(\frac{2}{3})(\tilde{\sigma}_\alpha - \tilde{\sigma}) - 1\}$. Hence the characteristic function of $L'/(n^{\frac{1}{2}}\tau_L)(\tau_L > 0)$ is expressed as

$$
\begin{aligned}
C_L(t) &= E[\exp(itl_0(T)/\tau_L)\{1 + n^{-\frac{1}{2}} itl_1(T)/\tau_L \\
&\quad + n^{-1}[itl_2(T)/\tau_L + \tfrac{1}{2}(it)^2 l_1(T)^2/\tau_L^{\,2}]\}] + O(n^{-3/2}).
\end{aligned} \tag{5.4}
$$

Since $T_\alpha = (\chi^2_{n_\alpha} - n_\alpha)/(2n_\alpha)^{\frac{1}{2}}$, we easily obtain

$$
\begin{aligned}
E[e^{tT_\alpha}] &= (1 - (2/n_\alpha)^{\frac{1}{2}}t)^{-n_\alpha/2} \exp(-(n_\alpha/2)^{\frac{1}{2}}t), \\
&= \{1 + \tfrac{1}{3}(2/n_\alpha)^{\frac{1}{2}}t^3 + n_\alpha^{-1}(\tfrac{1}{2}t^4 + t^6/9)\}e^{t^2/2} + O(n^{-3/2}).
\end{aligned} \tag{5.5}
$$

Similarly $E[T_\alpha^{\,l} e^{tT_\alpha}]$ $(l = 1, 2, 3, 4)$ are given by substituting m for n and putting $\Delta = 0$ in formulas (5.16.2)–(5.16.5), respectively.

Applying formula (5.5) to the first term in (5.4), with the abbreviated notation $b_\alpha = (2\rho_\alpha)^{\frac{1}{2}}(it/\tau_L)(\tilde{\sigma}_\alpha - \tilde{\sigma})$ in $l_0(T)$, we have

$$
\begin{aligned}
E[\exp(itl_0(T)/\tau_L)] &= e^{-t^2/2}[1 + \tfrac{1}{3}(2/n)^{\frac{1}{2}} \textstyle\sum_{\alpha=1}^{k} b_\alpha^{\,3}/\rho_\alpha^{\frac{1}{2}} \\
&\quad + n^{-1}\{(1/9)(\textstyle\sum_{\alpha=1}^{k} b_\alpha^{\,3}/\rho_\alpha^{\frac{1}{2}})^2 + \tfrac{1}{2}\sum_{\alpha=1}^{k} b_\alpha^{\,4}/\rho_\alpha\}] + O(n^{-3/2}).
\end{aligned} \tag{5.6}
$$

Noting that $\sum_{\alpha=1}^{k} \rho_\alpha^{\frac{1}{2}} b_\alpha = 0$, we can write each expectation in (5.4) as

$$
\begin{aligned}
E[l_1(T)\exp(itl_0(T)/\tau_L)] &= e^{-t^2/2}[\textstyle\sum a_\alpha b_\alpha^{\,2} + \sum a_\alpha - 1 \\
&\quad + 2^{\frac{1}{2}}n^{-\frac{1}{2}}\{\tfrac{1}{3}(\textstyle\sum b_\alpha^{\,3}/\rho_\alpha^{\frac{1}{2}})(\sum a_\alpha b_\alpha^{\,2}) + \tfrac{1}{3}(\sum b_\alpha^{\,3}/\rho_\alpha^{\frac{1}{2}})(\sum a_\alpha - 1) \\
&\quad + 2\textstyle\sum a_\alpha b_\alpha^{\,3}/\rho_\alpha^{\frac{1}{2}} + 2\sum a_\alpha b_\alpha/\rho_\alpha^{\frac{1}{2}}\}] + O(n^{-1})
\end{aligned} \tag{5.7}
$$

$$
E[l_2(T)\exp(itl_0(T)/\tau_L)] = e^{-t^2/2}[\textstyle\sum a_\alpha' b_\alpha^{\,3} + 3\sum a_\alpha' b_\alpha] + O(n^{-\frac{1}{2}}) \tag{5.8}
$$

$$
\begin{aligned}
E[l_1(T)^2\exp(itl_0(T)/\tau_L)] &= e^{-t^2/2}[(\textstyle\sum a_\alpha b_\alpha^{\,2})^2 + 4\sum a_\alpha^{\,2} b_\alpha^{\,2} \\
&\quad + 2\textstyle\sum a_\alpha b_\alpha^{\,2}(\sum a_\alpha - 1) + 2\sum a_\alpha^{\,2} \\
&\quad + (\textstyle\sum a_\alpha)^2 - 4\sum \rho_\alpha a_\alpha - 2\sum a_\alpha \\
&\quad + 3] + O(n^{-\frac{1}{2}}),
\end{aligned} \tag{5.9}
$$

where the symbol $\sum$ means the summation $\sum_{\alpha=1}^{k}$. It follows that the characteristic function of $L'/(n^{\frac{1}{2}}\tau_L)$ can be expanded asymptotically as

$$
\begin{aligned}
C_L(t) &= e^{-t^2/2}[1 + n^{-\frac{1}{2}}\{(it/\tau_L)(\textstyle\sum_{\alpha=1}^{k}[\tilde{\sigma} - \tilde{\sigma}_\alpha] + k - 1) \\
&\quad + (it/\tau_L)^3(\tau_L^{\,2} - \tfrac{2}{3}\textstyle\sum_{\alpha=1}^{k}\rho_\alpha[\tilde{\sigma}_\alpha - \tilde{\sigma}]^3)\} + n^{-1}\sum_{\alpha=1}^{3}(it/\tau_L)^{2\alpha} g_{2\alpha}],
\end{aligned} \tag{5.10}
$$

where the coefficients g_2, g_4, g_6, are given by

$$
\begin{aligned}
g_2 &= \sum (\tilde{\sigma}_\alpha - \tilde{\sigma})^2 + \tfrac{1}{2}\{\sum (\tilde{\sigma}_\alpha - \tilde{\sigma})\}^2 - (k+3) \sum (\tilde{\sigma}_\alpha - \tilde{\sigma}) \\
&\quad + \tfrac{1}{2}(k^2 - 1) \\
g_4 &= \tfrac{2}{3} \sum \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^4 + \tfrac{2}{3} \sum (\tilde{\sigma}_\alpha - \tilde{\sigma}) \sum \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^3 \\
&\quad - \tfrac{2}{3}(k+5) \sum \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^3 - \sum (\tilde{\sigma}_\alpha - \tilde{\sigma})\tau_L^2 + (k+1)\tau_L^2 \\
g_6 &= (2/9)\{\sum \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^3\}^2 - \tfrac{2}{3}\tau_L^2 \sum \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^3 + \tfrac{1}{2}\tau_L^4.
\end{aligned} \tag{5.11}
$$

Inverting this characteristic function, we have the following theorem.

THEOREM 5.1. *Under the fixed alternative* $K: \sigma_i^2 \neq \sigma_j^2$ *for at least some* i, j $(i \neq j)$, *the distribution of the statistic* $L' = L - (n/2) \sum_{\alpha=1}^k \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2$, *where* L *is given by* (2.1) *with* $\tilde{\sigma}_\alpha = \log \sigma_\alpha^2$ *and* $\tilde{\sigma} = \sum_{\alpha=1}^k \rho_\alpha \tilde{\sigma}_\alpha$, *is expanded asymptotically for large* n *as*

$$
\begin{aligned}
P(L'/(n^{\frac{1}{2}}\tau_L) < z) &= \Phi(z) - n^{-\frac{1}{2}}[\Phi^{(1)}(z)\tau_L^{-1}\{\textstyle\sum_{\alpha=1}^k (\tilde{\sigma} - \tilde{\sigma}_\alpha) + k - 1\} \\
&\quad + \Phi^{(3)}(z)\tau_L^{-3}\{\tau_L^2 - \tfrac{2}{3}\textstyle\sum_{\alpha=1}^k \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^3\}] \\
&\quad + n^{-1}\textstyle\sum_{\alpha=1}^3 \Phi^{(2\alpha)}(z) g_{2\alpha}/\tau_L^{2\alpha} + O(n^{-3/2}),
\end{aligned} \tag{5.12}
$$

where $\tau_L^2 = 2 \sum_{\alpha=1}^k \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2$ and $\Phi^{(j)}(z)$ means the jth derivative of the standard normal distribution function $\Phi(z)$. The coefficients $g_{2\alpha}$ are given by (5.11).

5.2 *Expansion of the non-null distribution of cM*. We shall derive the asymptotic expansion of the distribution of cM (Bartlett's test) under a fixed alternative. Putting $cn_\alpha = m_\alpha$ $(\alpha = 1, 2, \cdots, k)$ with the correction factor c in (4.14), we can write from (2.2)

$$cM = m \log \left(\textstyle\sum_{\alpha=1}^k S_\alpha/m\right) - \textstyle\sum_{\alpha=1}^k m_\alpha \log (S_\alpha/m_\alpha),$$

where $m = \sum_{\alpha=1}^k m_\alpha$. Let $U_\alpha = [(S_\alpha/\sigma_\alpha^2) - m_\alpha]/(2m_\alpha)^{\frac{1}{2}}$, then cM is expressed by U_α as

$$cM = m(\log \bar{\sigma} - \tilde{\sigma}) + m^{\frac{1}{2}} q_0(U) + q_1(U) + m^{-\frac{1}{2}} q_2(U) + O_p(m^{-1}), \tag{5.13}$$

where $\bar{\sigma} = \sum_{\alpha=1}^k \rho_\alpha \sigma_\alpha^2$, $\tilde{\sigma} = \sum_{\alpha=1}^k \rho_\alpha \log \sigma_\alpha^2$ and

$$
\begin{aligned}
q_0(U) &= \textstyle\sum_{\alpha=1}^k (2\rho_\alpha)^{\frac{1}{2}}(\nu_\alpha - 1) U_\alpha \\
q_1(U) &= \textstyle\sum_{\alpha=1}^k U_\alpha^2 - (\textstyle\sum_{\alpha=1}^k \rho_\alpha^{\frac{1}{2}} \nu_\alpha U_\alpha)^2 \\
q_2(U) &= \tfrac{2}{3} 2^{\frac{1}{2}} \{(\textstyle\sum_{\alpha=1}^k \rho_\alpha^{\frac{1}{2}} \nu_\alpha U_\alpha)^3 - \textstyle\sum_{\alpha=1}^k U_\alpha^3/\rho_\alpha^{\frac{1}{2}}\}
\end{aligned} \tag{5.14}
$$

with $\nu_\alpha = \sigma_\alpha^2/\bar{\sigma}$ for abbreviation. Note that since the random variables $U_1, U_2, \cdots, U_k$ are independent and each of them has asymptotically the standard normal distribution as $m \to \infty$, the statistic $M'/m^{\frac{1}{2}} = \{cM - m(\log \bar{\sigma} - \tilde{\sigma})\}/m^{\frac{1}{2}}$ is distributed asymptotically according to the normal distribution with mean zero and variance $\tau_M^2 = 2\sum_{\alpha=1}^k \rho_\alpha(\nu_\alpha - 1)^2$. Further, the characteristic function of $M'/(m^{\frac{1}{2}}\tau_M)$ $(\tau_M > 0)$ can be expressed as

$$(5.15)\quad C_M(t) = E[\exp(itq_0(U)/\tau_M)\{1 + m^{-\frac{1}{2}}itq_1(U)/\tau_M + m^{-1}[itq_2(U)/\tau_M + \tfrac{1}{2}(it)^2q_1(U)^2/\tau_M^2]\}] + O(m^{-3/2}).$$

By the definition of U_α,

$$(5.16.1)\quad E[e^{tU_\alpha}] = e^{t^2/2}[1 + m_\alpha^{-\frac{1}{2}}\{\tfrac{1}{2}\,2^{\frac{1}{2}}\Delta\rho_\alpha t + \tfrac{1}{3}\,2^{\frac{1}{2}}t^3\} + m_\alpha^{-1}\{\tfrac{1}{2}t^4 + \tfrac{1}{2}\Delta\rho_\alpha t^2 + (\tfrac{1}{3}t^3 + \tfrac{1}{2}\Delta\rho_\alpha t)^2\}] + O(m^{-3/2})$$

$$(5.16.2)\quad E[U_\alpha e^{tU_\alpha}] = e^{t^2/2}[t + m_\alpha^{-\frac{1}{2}}\,2^{\frac{1}{2}}\{\tfrac{1}{3}t^4 + t^2(1 + \tfrac{1}{2}\Delta\rho_\alpha) + \tfrac{1}{2}\Delta\rho_\alpha\}] + O(m^{-1})$$

$$(5.16.3)\quad E[U_\alpha^2 e^{tU_\alpha}] = e^{t^2/2}[t^2 + 1 + m_\alpha^{-\frac{1}{2}}\,2^{\frac{1}{2}}\{\tfrac{1}{3}t^5 + (7/3 + \tfrac{1}{2}\Delta\rho_\alpha)t^3 + ((3/2)\Delta\rho_\alpha + 2)t\}] + O(m^{-1})$$

$$(5.16.4)\quad E[U_\alpha^3 e^{tU_\alpha}] = e^{t^2/2}(t^3 + 3t) + O(m^{-\frac{1}{2}})$$

$$(5.16.5)\quad E[U_\alpha^4 e^{tU_\alpha}] = e^{t^2/2}(t^4 + 6t^2 + 3) + O(m^{-\frac{1}{2}}),$$

where $\Delta = n(1 - c) = O(1)$. If we set $\Delta = 0$ and change m_α to n_α in (5.16.1), we have the same result as in (5.5). After some computation with the abbreviated notations $b_\alpha = (2\rho_\alpha)^{\frac{1}{2}}(\nu_\alpha - 1)it/\tau_M$ in $q_0(U)$ and $\sum a_\alpha = \sum_{\alpha=1}^k a_\alpha$, we have

$$(5.17)\quad E[\exp(itq_0(U)/\tau_M)] = e^{-\frac{1}{2}t^2}[1 + \tfrac{1}{3}(2/m)^{\frac{1}{2}}\textstyle\sum b_\alpha^3/\rho_\alpha^{\frac{1}{2}} + m^{-1}\{(1/9)(\sum b_\alpha^3/\rho_\alpha^{\frac{1}{2}})^2 + \tfrac{1}{2}\sum b_\alpha^4/\rho_\alpha + \tfrac{1}{2}\Delta\sum b_\alpha^2\}] + O(m^{-3/2}).$$

Putting $a_\alpha = \rho_\alpha^{\frac{1}{2}}\nu_\alpha$ in $q_1(U)$ and $q_2(U)$ in (5.14), we have

$$(5.18)\quad E[q_1(U)\exp(itq_0(U)/\tau_M)] = e^{-\frac{1}{2}t^2}[\textstyle\sum b_\alpha^2 - (\sum a_\alpha b_\alpha)^2 + k - \sum a_\alpha^2 + m^{-\frac{1}{2}}\,2^{\frac{1}{2}}\{\tfrac{1}{3}(\sum b_\alpha^3/\rho_\alpha^{\frac{1}{2}})(\sum b_\alpha^2 - [\sum a_\alpha b_\alpha]^2) + (\sum b_\alpha^3/\rho_\alpha^{\frac{1}{2}})(\tfrac{1}{3}k + 2 - \tfrac{1}{3}\sum a_\alpha^2) - 2(\sum a_\alpha b_\alpha^2/\rho_\alpha^{\frac{1}{2}})(\sum a_\alpha b_\alpha) + 2\sum b_\alpha(1 - a_\alpha^2)/\rho_\alpha^{\frac{1}{2}} - \Delta\sum a_\alpha\rho_\alpha^{\frac{1}{2}}\sum a_\alpha b_\alpha\}] + O(m^{-1})$$

$$(5.19)\quad E[q_2(U)\exp(itq_0(U)/\tau_M)] = \tfrac{1}{3}e^{-\frac{1}{2}t^2}2^{3/2}\{(\textstyle\sum a_\alpha b_\alpha)^3 - \sum b_\alpha^3/\rho_\alpha^{\frac{1}{2}} + 3\sum a_\alpha^2\sum a_\alpha b_\alpha - 3\sum b_\alpha/\rho_\alpha^{\frac{1}{2}}\} + O(m^{-\frac{1}{2}})$$

$$(5.20)\quad E[q_1(U)^2\exp(itq_0(U)/\tau_M)] = e^{-\frac{1}{2}t^2}[\{\textstyle\sum b_\alpha^2 - (\sum a_\alpha b_\alpha)^2\}^2 + 2\sum b_\alpha^2(k + 2 - \sum a_\alpha^2) + 2(\sum a_\alpha b_\alpha)^2(3\sum a_\alpha^2 - k - 4) + 3(\sum a_\alpha^2)^2 - 2(k + 2)\sum a_\alpha^2 + k(k + 2)] + O(m^{-\frac{1}{2}}),$$

which implies the following asymptotic formula of the characteristic function of $M'/(m^{\frac{1}{2}}\tau_M)$:

$$(5.21)\quad \begin{aligned} C_M(t) = e^{-t^2/2}[1 &+ m^{-\frac{1}{2}}\{(it/\tau_M)(k - \textstyle\sum_{\alpha=1}^k \rho_\alpha \nu_\alpha^2) \\ &+ (it/\tau_M)^3((4/3)\textstyle\sum_{\alpha=1}^k \rho_\alpha(\nu_\alpha - 1)^3 + \tau_M^2 - \frac{1}{2}\tau_M^4)\} \\ &+ m^{-1}\textstyle\sum_{\alpha=1}^3 (it/\tau_M)^{2\alpha} h_{2\alpha}] + O(m^{-3/2}), \end{aligned}$$

where the coefficients h_2, h_4 and h_6 are given by

$$(5.22)\quad \begin{aligned} h_2 &= (11/2)(\textstyle\sum \rho_\alpha \nu_\alpha^2)^2 - 4\textstyle\sum \rho_\alpha \nu_\alpha^3 - (k+2)\textstyle\sum \rho_\alpha \nu_\alpha^2 + k(k+2)/2 \\ &\quad - \tfrac{1}{2}\Delta \tau_M^2 \\ h_4 &= 2\textstyle\sum \rho_\alpha(\nu_\alpha - 1)^4 + (4/3)\textstyle\sum \rho_\alpha(\nu_\alpha - 1)^3(k + 4 - \textstyle\sum \rho_\alpha \nu_\alpha^2) \\ &\quad + \tau_M^2\{k + 1 - 4\textstyle\sum \rho_\alpha \nu_\alpha(\nu_\alpha - 1)^2\} + \tfrac{1}{2}\tau_M^4(3\textstyle\sum \rho_\alpha \nu_\alpha^2 - k - 5) \\ &\quad + \tfrac{1}{3}\tau_M^6 \\ h_6 &= (8/9)\{\textstyle\sum \rho_\alpha(\nu_\alpha - 1)^3\}^2 + \tfrac{2}{3}\textstyle\sum \rho_\alpha(\nu_\alpha - 1)^3(2 - \tau_M^2)\tau_M^2 \\ &\quad + \tfrac{1}{8}\tau_M^4(2 - \tau_M^2)^2. \end{aligned}$$

Inverting this characteristic function, we have the following theorem.

THEOREM 5.2. *Under the fixed alternative K, the distribution of the statistic* $M' = cM - m(\log \bar{\sigma} - \tilde{\sigma})$, *where cM is the modified likelihood ratio statistic given by* (2.2) *and* (4.14) *with* $\bar{\sigma} = \sum_{\alpha=1}^k \rho_\alpha \sigma_\alpha^2$ *and* $\tilde{\sigma} = \sum_{\alpha=1}^k \rho_\alpha \log \sigma_\alpha^2$, *can be expanded asymptotically for large* $m(= nc)$ *as*

$$(5.23)\quad \begin{aligned} P(M'/(m^{\frac{1}{2}}\tau_M) < z) = \Phi(z) &- m^{-\frac{1}{2}}[\Phi^{(1)}(z)(k - \textstyle\sum_{\alpha=1}^k \rho_\alpha \nu_\alpha^2)/\tau_M \\ &+ \Phi^{(3)}(z)\tau_M^{-3}\{(4/3)\textstyle\sum_{\alpha=1}^k \rho_\alpha(\nu_\alpha - 1)^3 + \tau_M^2 - \frac{1}{2}\tau_M^4\}] \\ &+ m^{-1}\textstyle\sum_{\alpha=1}^3 \Phi^{(2\alpha)}(z) h_{2\alpha} \tau_M^{-2\alpha} + O(m^{-3/2}), \end{aligned}$$

where $\tau_M^2 = 2\sum_{\alpha=1}^k \rho_\alpha(\nu_\alpha - 1)^2$ *with* $\nu_\alpha = \sigma_\alpha^2/\bar{\sigma}$ *and* $h_{2\alpha}$ $(\alpha = 1, 2, 3)$ *are given by* (5.22) *with* $\Delta = n(1 - c)$.

The limiting distribution of the statistic M in multivariate models has been obtained by Sugiura [18] and coincides with the first term of the formula (5.23) in Theorem 5.2. Since the asymptotic variances τ_L^2 and τ_M^2 vanish when the hypothesis is true, the asymptotic non-null distributions of L and cM have singularities at the null hypothesis, so that our formulas in Theorem 5.1 and Theorem 5.2 do not give good approximation, when the alternative hypothesis K is near to the null hypothesis. Also it means that asymptotic expansions of the non-null distributions of L and M do not cover the expansions of the null distributions.

5.3. *Numerical examples.* We shall finally obtain some numerical values of the asymptotic power of Lehmann's test $(= L)$ and Bartlett's test $(= cM)$ in the following special cases.

EXAMPLE 5.1. When $k = 2$ and $n_1 = n_2$, the L test is equivalent to the M test by Theorem 2.1. Hence the two powers computed by the formulas (5.12) and (5.23) should be equal within the accuracy of the percentage points. Using the 5 % points of L and cM obtained in Example 4.1. and Example 4.2. for $n_1 = n_2 = 50$, we have the following approximate powers for the alternative $K: \sigma_1^2 = 2\sigma_2^2$.

	$P_K(L > 3.929)$ Formula (5.12)	$P_K(cM > 3.841)$ Formula (5.23)
first term	0.6641	0.6643
second term	0.0134	0.0124
third term	0.0014	0.0020
approximate power	0.6789	0.6787

Thus our formulas give a reasonable approximation in this case.

EXAMPLE 5.2. When $k = 2$ and $n_1 = 4$, $n_2 = 20$, the exact values of the power of the M test for some alternatives have been given by Ramachandran [15] in his Table 744a. Using the 5 % point of cM obtained by Example 4.2. and specifying the alternatives $K: \sigma_2^2 = \delta\sigma_1^2$, we have the following approximate powers of cM test from the formula (5.23).

	$P_\delta(cM > 3.801)$		
	$\delta = 10$	$\delta = 5$	$\delta = 10/3$
first term	0.6563	0.3215	0.1391
second term	0.0748	0.0804	0.1146
third term	0.0000	−0.0013	−0.0171
approximate power	0.731	0.401	0.237
exact power	0.729	0.397	0.230

For the smaller values of δ, it happens that the first term is smaller than the second term. Thus we cannot apply our asymptotic formula effectively for alternatives near the null hypothesis.

EXAMPLE 5.3. When $k = 5$ and $n_1 = n_2 = \cdots = n_5 = 4$, Pearson [12] obtained some approximate powers of the M test both by the Monte Carlo method and by fitting a gamma–type distribution to the inverse of the modified likelihood ratio statistic. For the alternative $K: \sigma_1^2 = \frac{1}{4}$, $\sigma_2^2 = \sigma_3^2 = \sigma_4^2 = 1$, $\sigma_5^2 = 4$ (alternative VI in Pearson [12]), our asymptotic formula (5.23) gives the following approximate power of the M test, based on the 5 % point obtained in Example 4.2.

$P_K(cM > 9.341)$

first term	second term	third term	approximate power
0.3119	0.1601	−0.0116	0.460

Pearson's approximate powers are 0.440 (by Monte Carlo method) and 0.493 (by fitting a gamma-type distribution).

EXAMPLE 5.4. When $k = 3$ and $n_1 = 50$, $n_2 = 100$, $n_3 = 150$, the formulas (5.12) and (5.23), together with the 5% points in Example 4.1 and Example 4.2, give the following approximate powers for alternatives $K: \sigma_2^2 = \delta\sigma_1^2$ and $\sigma_3^2 = \delta^2\sigma_1^2$.

	$P_\delta(L > 6.109)$		$P_\delta(cM > 5.991)$	
	$\delta = 1.5$	$\delta = 0.7$	$\delta = 1.5$	$\delta = 0.7$
first term	0.8474	0.7549	0.8430	0.7658
second term	0.0784	0.0555	0.0700	0.0615
third term	−0.0014	0.0069	0.0028	0.0077
approximate power	0.924	0.817	0.916	0.835

This example seems to show that for $\delta = 1.5$, the power of Lehmann's test is larger than that of Bartlett's test and for $\delta = 0.7$ the reverse inequality holds, though the differences are small.

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