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THE REPRESENTATION OF FUNCTIONALS OF BROWNIAN MOTION BY STOCHASTIC INTEGRALS[1]

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0. Abstract. It is known that any functional of Brownian motion with finite second moment can be expressed as the sum of a constant and an Itô stochastic integral. It is also known that homogeneous additive functionals of Brownian motion with finite expectations have a similar representation.

This paper extends these results in several ways. It is shown that any finite functional of Brownian motion can be represented as a stochastic integral. This representation is not unique, but if the functional has a finite expectation it does have a unique representation as a constant plus a stochastic integral in which the process of indefinite integrals is a martingale. A corollary of this result is that any martingale (on a closed interval) that is measurable with respect to the increasing family of σ-fields generated by a Brownian motion is equal to a constant plus an indefinite stochastic integral. Sufficiently well-behaved Fréchet-differentiable functionals have an explicit representation as a stochastic integral in which the integrand has the form of conditional expectations of the differential.

1. Introduction. It is known that any functional of Brownian motion with finite second moment can be expressed as the sum of a constant and an Itô stochastic integral. This result,[2] which was pointed out to me by T. Duncan and T. Kailath, is a direct consequence of a modification by Itô [4] of the expansion of Cameron and Martin [1] for square-integrable functionals. The argument goes as follows: If $\{x(t):0 \leqq t \leqq 1\}$ is a Brownian motion and ξ is a functional of $x(t)$ with finite second moment, Itô's expansion for ξ is of the form:

$$\xi = E\xi + \sum_{n=1}^{\infty} I_n$$

where the I_n are iterated stochastic integrals. The infinite sum converges in quadratic mean and can be condensed by a completion argument into a single stochastic integral, so that

$$\xi = E\xi + \int_0^1 \phi(t)\,dx(t)$$

for some random function $\phi(t)$.

It is also known that all homogeneous additive functionals of Brownian motion with zero expectations are of the form

$$\int_s^t f(x(u))\,dx(u).$$

This is due to Ventsel [8].

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[1] This work was carried out while the author was at the Mathematics Center, Bell Telephone Laboratories, Murray Hill, New Jersey.

[2] This result has also been proved by Kunita and Watanabe [6].

These results raise several questions on the nature of the relation between functionals of Brownian motion and stochastic integrals, and some of these are considered in this paper. The results are as follows. Sufficiently well-behaved Fréchet-differentiable functionals have an explicit expression as a stochastic integral in which the integrand is made up from conditional expectations of the Fréchet differential. Functionals with finite expectations have a unique representation as a constant plus a stochastic integral, the process of indefinite integrals being a martingale. The corollary of this result is that any martingale (on a closed interval) that is measurable with respect to the increasing family of σ-fields generated by a Brownian motion is equal to a constant plus an indefinite stochastic integral. Finally, we show that for any finite functional of Brownian motion there exists a stochastic integral equal to it.

2. Itô integrals. We briefly review some of the properties of Itô integrals. Let $\{x(t): 0 \leqq t < \infty\}$ be a Brownian motion on a probability space $(\Omega, \mathscr{B}, P)$. Let $\{\mathscr{H}_t: 0 \leqq t < \infty\}$ be an increasing family of σ-fields in $\mathscr{B}$, each $\mathscr{H}_t$ containing the negligible sets of $\mathscr{B}$, such that (i) $x(t)$ is $\mathscr{H}_t$-measurable and (ii) $x(t') - x(t)$ is independent of $\mathscr{H}_t$ for all $0 \leqq t < t'$. Then the following conditions are sufficient for the existence of the stochastic integral $\int_0^\infty \phi(t)\, dx(t)$ [2], [3], [5]:

CONDITION A. $\phi(\cdot, \cdot)$ is an $\mathscr{A} \times \mathscr{B}$-measurable function, where $\mathscr{A}$ is the σ-field of Lebesgue-measurable sets on $[0, \infty)$, and for fixed t, $\phi(t, \cdot)$ is $\mathscr{H}_t$-measurable.

CONDITION B. $\int_0^\infty \phi(t)^2\, dt < \infty$ a.s.
Condition B can be replaced by the stronger version:

CONDITION B′. $\int_0^\infty E[\phi(t)^2]\, dt < \infty$.
If B′ holds, then for $s \geqq 0$

$$E[\textstyle\int_s^\infty \phi(t)\, dx(t) \,|\, \mathscr{H}_s] = 0 \quad \text{a.s.}$$

so that the family of indefinite integrals form a martingale, and

$$E[(\textstyle\int_s^\infty \phi(t)\, dx(t))^2 \,|\, \mathscr{H}_s] = E[\textstyle\int_s^\infty \phi(t)^2\, dt \,|\, \mathscr{H}_s] \quad \text{a.s.}$$

The following lemma is an almost immediate consequence of this property.

LEMMA 1. *If a sequence of stochastic integrals* $\xi_n = \int_0^1 \varphi_n(t)\, dx(t)$ $n = 1, 2, \cdots$, *converges in quadratic mean to* ξ, *and if the integrands* $\varphi_n(t)$ *satisfy* B′, *then there is a function* $\varphi(t,\omega)$ *satisfying* A *and* B *such that*

$$\xi = \textstyle\int_0^1 \varphi(t)\, dx(t) \quad \text{a.s.} \tag{1}$$

PROOF. Since $\lim_{n\to\infty} E[(\xi_n - \xi)^2] = 0$, $\{\xi_n\}$ is a Cauchy sequence in $L_2(\Omega)$. But

$$E[(\xi_n - \xi_m)^2] = \textstyle\int_0^1 E[(\varphi_n(t) - \varphi_m(t))^2]\, dt.$$

Therefore $\{\varphi_n(t, \omega)\}$ is a Cauchy sequence in $L_2([0, 1] \times \Omega)$ and by completeness there exists a function $\varphi(t, \omega)$ to which $\{\varphi_n(t, \omega)\}$ converges in quadratic mean. So except perhaps on a (t, ω)-negligible set, on which we can take $\varphi(t, \omega)$ to be zero, there exists a subsequence $\{\varphi_{n_i}(t,\omega)\}$ converging to $\varphi(t,\omega)$ for each t and ω. $\varphi(t,\omega)$ is therefore $\mathscr{H}_t$-measurable and $\varphi(\cdot,\cdot)$ satisfies A and B′. Moreover

$$E[(\xi_n-\xi)^2] = \int_0^1 E[(\phi_n(t)-\phi(t))^2]\,dt$$

and (1) follows.

3. Fréchet-differentiable functionals. Let $x = \{x(t): 0 \leqq t \leqq 1\}$ be a Brownian motion on $(\Omega, \mathscr{B}, P)$ with continuous sample paths, and let $\mathscr{F}_t$ be the σ-field generated by $\{x(s); 0 \leqq s \leqq t\}$ together with the negligible sets of $\mathscr{B}$.

Let ξ be a functional on the Banach space of $C[0,1]$, the space of continuous functions, together with the norm $||z|| = \max_{0\leqq t\leqq 1}|z(t)|$. Let $\mathscr{B}_c$ denote the Borel field generated by the norm topology on $C[0,1]$. Suppose ξ is $\mathscr{B}_c$-measurable and satisfies the following condition:

CONDITION C. For all $z, z' \in C[0,1]$,

$$\xi(z+z')-\xi(z) = F(z,z')+R(z,z')$$

where $F(z,z')$ is a continuous linear functional in z' for each z and $R(z,z')$ satisfies

$$R(z,z') \leqq K\,||z'||^{1+\delta}(1+||z||^{\alpha})(1+||z'||^{\alpha})$$

for some positive constants K, α and δ. $F(z,z')$ is the Fréchet differential of ξ and, being a continuous linear functional on $C[0,1]$, has the Riesz representation

$$F(z,z') = \int_0^1 z'(s)\lambda(ds,z)$$

where $\lambda(s,z)$ is a function of bounded variation in s, which we shall take to be right-continuous in s.

Now consider the functional of Brownian motion $\xi(x)$. Our first result is the following.

THEOREM 1. *If ξ satisfies condition* C *and $E[\xi(x)] = 0$, then*

$$\xi(x) = \int_0^1 E[\lambda(1,x)-\lambda(t,x)\,|\,\mathscr{F}_t]\,dx(t) \quad \text{a.s.} \tag{2}$$

The following estimate of the modulus of continuity of Brownian motion will be needed in the proof of this theorem.

LEMMA 2. *Let $v_x(h) = \max_{|t-s|\leqq h, 0\leqq s\leqq t\leqq 1}|x(t)-x(s)|$. Then for fixed $\delta > 0$ there exists a constant K_δ such that for any h, $0 \leqq h \leqq 1$,*

$$E[v_x(h)^{4+4\delta}] \leqq K_\delta h^{2+\delta}. \tag{3}$$

PROOF. Since $v_x(h)$ is monotone and $nh \to 1$ as $h \to 0$ where n is the integral part of $1/h$, we need only prove the assertion for h of the form $1/n$. Let

$$\eta_r^{\,+} = \max_{rh\leqq t\leqq (r+1)h}[x(t)-x(rh)],$$
$$\eta_r^{\,-} = -\min_{rh\leqq t\leqq (r+1)h}[x(t)-x(rh)]$$

and let $\eta_r = \max\{\eta_r^{\,+}, \eta_r^{\,-}\}$. It can be easily verified that for any r, $1 \leqq r < n$, and for $(r-1)h \leqq s \leqq t \leqq (r+1)h$, $\max_{|t-s|\leqq h}|x(t)-x(s)| \leqq 3\max\{\eta_{r-1}, \eta_r\}$, and so

$v_x(h) \leqq 3\max_{0\leqq r\leqq n-1}\{\eta_r\}$. Now η_r, $r = 0, 1, \cdots, n-1$, are independent random variables with a common distribution; say F. Moreover, the variables η_r^+ and η_r^- all have the same distribution as $|x(h)|$ (see [2] page 392, Theorem 2.1) which is the modulus of a normal random variable with variance h. Consequently,

$$P\{\max_{0\leqq r\leqq n-1}\eta_r < a\} = \prod_{r=0}^{n-1} P\{\eta_r < a\}$$
$$= F(a)^n$$

and noting that for all integral $m, \eta_0^{2m} \leqq (\eta_0^+)^{2m} + (\eta_0^-)^{2m}$, we have

$$E(\max_{0\leqq r\leqq n-1}\eta_r)^{2m} = n\int_{-\infty}^{\infty} a^{2m}F(a)^{n-1}\,dF(a)$$
$$\leqq n\int_{-\infty}^{\infty} a^{2m}\,dF(a)$$
$$= nE[\eta_0^{2m}]$$
$$\leqq n(E[(\eta_0^+)^{2m}] + E[(\eta_0^-)^{2m}])$$
$$= 2(2m-1)(2m-3)\cdots 1\,.\,h^{m-1}.$$

Now choose m sufficiently large so that $m^{-1}(m-1)(2+2\delta) > 2+\delta$ and $m > 2+2\delta$. Then applying Jensen's inequality in the form

$$E[X^{4+4\delta}] \leqq E[X^{2m}]^{(2+2\delta)/m}$$

we have

$$E[v_x(h)^{4+4\delta}] \leqq 3^{4+4\delta}E[\max_{0\leqq r\leqq n-1}\eta_r^{4+4\delta}]$$
$$\leqq 3^{4+4\delta}K^{(2+2\delta)/m}h^{2+\delta}$$

where $K = 2(2m-1)(2m-3)\cdots 1$, and the lemma follows.

Proof of the Theorem. Let $\psi(t,\omega)$ denote $E[\lambda(1,x)-\lambda(t,x)\,|\,\mathscr{F}_t](\omega)$. We begin by verifying the following inequalities.

(i) $E[\xi(x)^2] < \infty$.

(ii) $\int_0^1 E[\lambda(t,x)^2]\,dt < \infty$.

(iii) $\int_0^1 E[\psi(t)^2]\,dt < \infty$.

In Condition C set z' to x and z to zero. Then

$$|\xi(x)| \leqq |\xi(0)| + \|x\|\int_0^1|\lambda(ds,0)| + K\|x\|^{1+\delta}(1+\|x\|^{\alpha}).$$

As we have already noted, $\|x\| = \max_{0\leqq t\leqq 1}|x(t)|$ is the sum of two variables with the same distribution as $|x(1)|$, the moments of which are finite. So (i) follows. Let $\{z_n^t\}$ be a sequence of continuous functions for which

$$z_n^t(s) = 1, \qquad 0 \leqq s \leqq t$$
$$1 > z_n^t(s) \to 0 \qquad \text{as } n\to\infty \text{ for } t < s \leqq 1.$$

Then

$$\lambda(t, z) = \lim_{n\to\infty} \int_0^1 z_n{}^t(s)\lambda(ds, z)$$

since $\lambda(t, z) = \lambda(t+, z)$. But by C

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\{\xi(z+\varepsilon z_n{}^t) - \xi(z)\} = \int_0^1 z_n{}^t(s)\lambda(ds, z).$$

Consequently $\int_0^1 z_n{}^t(s)\lambda(ds, \cdot)$, and therefore $\lambda(t, \cdot)$, is $\mathscr{B}_c$-measurable.

Using Condition C for the two cases where $(z, z') = (0, x)$ and $(0, x+z_n{}^t)$ and the inequality

$$\|x+z_n{}^t\|^\alpha \leqq (\|x\| + \|z_n{}^t\|)^\alpha \leqq 2^{\alpha-1}(1+\|x\|^\alpha)$$

which holds if α is taken to be greater than one, we obtain the estimate

$$\begin{aligned}|\textstyle\int_0^1 z_n{}^t(s)\lambda(ds, x)| &\leqq |\xi(x+z_n{}^t)| + |\xi(x)| + 2K(1+\|x\|^\alpha)\\ &\leqq 2\xi(0) + (1+2\|x\|)\textstyle\int_0^1 |\lambda(ds, 0)| + K(2+\|x\|^{1+\delta})(1+\|x\|^\alpha)\\ &\quad + 2^\delta K(1+\|x\|^{1+\delta})(1+2^{\alpha-1}(1+\|x\|^\alpha)),\end{aligned}$$

(ii) follows by dominated convergence. $\int_0^1 E[(\lambda(1, x) - \lambda(t, x))^2]\,dt$ is also finite and (iii) follows from Jensen's inequality for conditional expectations.

Let $y = \{y(t): 0 \leqq t \leqq 1,\ y(0) = 0\}$, be a Brownian motion with continuous sample paths that is independent of x and therefore of $\mathscr{F}_1$. Let $0 = t_0 < t_1 < \cdots < t_n = 1$. Define the family of processes $x_i, i = 0, \cdots, n$, by

$$\begin{aligned} x_i(t) &= x(t) &&\text{if} \quad 0 \leqq t \leqq t_i;\\ &= x(t_i) + y(t-t_i) &&\text{if} \quad t_i < t \leqq 1.\end{aligned}$$

Then each x_i-process is also a Brownian motion. Moreover, we observe that the joint distributions of $\{x(t): 0 \leqq t \leqq 1\}$ are identical with those of $\{x(t): 0 \leqq t \leqq t_i,\ x_i(t): t_i \leqq t \leqq 1\}$. Consequently,

$$\begin{aligned} E[\xi(x) \mid \mathscr{F}_{t_i}] &= E[\xi(x_i) \mid \mathscr{F}_{t_i}]\\ &= E[\xi(x_i) \mid \mathscr{F}_1], \quad \text{a.s.}\end{aligned} \tag{4}$$

We now use this expression to obtain an expansion of $\xi(x)$. Summing over i from 0 to $n-1$ we have

$$\begin{aligned}\xi(x) &= \textstyle\sum_i \{E[\xi(x) \mid \mathscr{F}_{t_{i+1}}] - E[\xi(x) \mid \mathscr{F}_{t_i}]\}\\ &= \textstyle\sum_i E[\xi(x_{i+1}) - \xi(x_i) \mid \mathscr{F}_1] \quad \text{a.s.}\end{aligned} \tag{5}$$

Now

$$\begin{aligned} x_{i+1}(t) - x_i(t) &= 0 &&\text{if} \quad t \leqq t_i;\\ &= x(t) - x(t_i) - y(t-t_i)\\ &= x(t_{i+1}) - x(t_i) + (x(t) - x(t_{i+1}) - y(t-t_i)),\\ & &&\text{if} \quad t_i < t \leqq t_{i+1};\\ &= x(t_{i+1}) - x(t_i) + y(t-t_{i+1}) - y(t-t_i),\\ & &&\text{if} \quad t_{i+1} < t;\end{aligned} \tag{6}$$

and so by the differentiability of ξ

$$\begin{aligned} E[\xi(x_{i+1})-\xi(x_i)\,|\,\mathscr{F}_1] &= E[\textstyle\int_0^1 (x_{i+1}(s)-x_i(s))\lambda(ds,x_i)+R(x_i,x_{i+1}-x_i)\,|\,\mathscr{F}_1] \\ &= E[\lambda(1,x_i)-\lambda(t_i,x_i)\,|\,\mathscr{F}_1](x(t_{i+1})-x(t_i)) \\ &\quad + E[\textstyle\int_{t_{i+1}}^1 \{y(s-t_{i+1})-y(s-t_i)\}\lambda(ds,x_i)\,|\,\mathscr{F}_1] \\ &\quad + E[\textstyle\int_{t_i}^{t_{i+1}} \{x(s)-x(t_{i+1})-y(s-t_i)\}\lambda(ds,x_i)\,|\,\mathscr{F}_1] \\ &\quad + E[R(x_i,x_{i+1}-x_i)\,|\,\mathscr{F}_1] \quad \text{a.s.} \end{aligned}$$

where the integrations are over $(t_i, t_{i+1}]$ and $(t_{i+1}, 1]$. Observe that the conditional expectation with respect to $\mathscr{F}_{t_i}$ of the left side of this equation is zero. So we can subtract from the right side its conditional expectation with respect to $\mathscr{F}_{t_i}$ and leave the equation unchanged. The result is

$$E[\xi(x_{i+1})-\xi(x_i)\,|\,\mathscr{F}_1] = E[\lambda(1,x_i)-\lambda(t_i,x_i)\,|\,\mathscr{F}_1](x(t_{i+1})-x(t_i))+a_i+b_i, \quad \text{a.s.,}$$

where

$$\begin{aligned} a_i &= E[\textstyle\int_{t_i}^{t_{i+1}} \{x(s)-x(t_{i+1})\}\lambda(ds,x_i)\,|\,\mathscr{F}_1], \\ b_i &= E[R(x_i,x_{i+1}-x_i)\,|\,\mathscr{F}_1]-E[R(x_i,x_{i+1}-x_i)\,|\,\mathscr{F}_{t_i}]. \end{aligned}$$

Noting that, like (4),

$$E[\lambda(t,x_i)\,|\,\mathscr{F}_1] = E[\lambda(t,x)\,|\,\mathscr{F}_{t_i}]$$

we have from (5) that

$$\xi(x) = \textstyle\sum_i \psi(t_i)(x(t_{i+1})-x(t_i))+\sum_i a_i+\sum_i b_i. \quad \text{a.s.} \tag{7}$$

Now it is possible to choose a sequence $\{t_i^m\}m = 1, 2, \cdots$, of partitions of the unit interval, with $h_m = \max_i |t_{i+1}^m - t_i^m| \to 0$, such that ([2] pages 440–441)

$$\textstyle\sum_i \int_{t_i^m}^{t_{i+1}^m} E[\{\psi(t)-\psi(t_i^m)\}^2]\,dt \to 0 \quad \text{and} \tag{8}$$

$$\textstyle\sum_i \int_{t_i^m}^{t_{i+1}^m} E[\{\lambda(t,x)-\lambda(t_i^m,x)\}^2]\,dt \to 0. \tag{9}$$

Let the partition $\{t_i\}$ be a member of such a sequence. The relation (8) implies that, as $h = \max_i |t_{i+1}-t_i| \to 0$, the Riemann–Stieltjes sum in (7) converges in quadratic mean to the stochastic integral $\int_0^1 \psi(t)\,dx(t)$. So to prove the theorem it only remains to show that the sums $\sum_i a_i$ and $\sum_i b_i$ vanish. Let $\mathscr{H}_i$ be the σ-field generated by x_i and let $Q_i(w, dz)$ be a regular conditional distribution of x on $\mathscr{B}_c$ given $\mathscr{H}_i$. This exists since $(C[0,1], \mathscr{B}_c)$ is a Borel space. Then $\lambda(s, x_i)$ is $\mathscr{H}_i$-measurable and $x(s)-x(t_{i+1})$, $s \geqq t_i$, is independent of $\mathscr{H}_i$, By Jensen's inequality for conditional expectations

$$\begin{aligned} E[a_i^2] &\leqq E[\{\textstyle\int_{t_i}^{t_{i+1}} (x(s)-x(t_{i+1}))\lambda(ds,x_i)\}^2] \\ &= E[E[\{\cdot\}^2\,|\,\mathscr{H}_i]], \quad \text{a.s.} \end{aligned}$$

the term in the parentheses $\{\cdot\}$ being repeated. The conditional expectation in this last expression is equal to

$$\int_{C[0,1]}\{\int_{t_i}^{t_{i+1}} z(s)-z(t_{i+1})\lambda(ds,x_i)\}^2 Q_i(\cdot,dz)$$

which is a triple integral with respect to the measures $\lambda(ds,x_i)$, $\lambda(ds',x_i)$ and $Q_i(\cdot,dz)$. Applying Fubini's theorem, we can change the order of integration and obtain

$$E[a_i^2] \leqq E[\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\{t_{i+1}-\max(s,s')\}\lambda(ds,x_i)\lambda(ds',x_i)].$$

The double integral within the expectation is over the rectangle $t_i < s \leqq t_{i+1}$, $t_i < s' \leqq t_{i+1}$. If this is split into the two triangles $t_i < s \leqq t_{i+1}$, $t_i < s' \leqq s$ and $t_i < s < s'$, $t_i < s' < t_iP_{+1}$ and Fubini's theorem is again used, the double integral becomes

$$\begin{aligned}
&\int_{t_i+0}^{t_{i+1}+0}\int_{t_i+0}^{s+0}(t_{i+1}-s)\lambda(ds')\lambda(ds)+\int_{t_i+0}^{t_{i+1}+0}\int_{t_i+0}^{s'-0}(t_{i+1}-s')\lambda(ds)\lambda(ds')\\
&\quad=\int_{t_i+0}^{t_{i+1}+0}(t_{i+1}-s)[\lambda(s+0)+\lambda(s-0)-2\lambda(t_i+0)]\lambda(ds)\\
&\quad=\int_{t_i+0}^{t_{i+1}+0}(t_{i+1}-s)d[(\lambda(s)-\lambda(t_i+0))^2]\\
&\quad=\int_{t_i}^{t_{i+1}}(\lambda(s)-\lambda(t_i+0))^2\,ds
\end{aligned}$$

the last step following by integration by parts. Consequently, with the convention that $\lambda(t,x)$ is right-continuous, the inequality of expectations reduces to

$$E[a_i^2] \leqq E[\int_{t_i}^{t_{i+1}}(\lambda(s)-\lambda(t_i))^2\,ds].$$

It can be easily verified that

$$E[(\textstyle\sum_i a_i)^2] = \sum_i E[a_i^2].$$

If these last two relations are combined, it then follows from (9) that $E[(\sum_i a_i)^2] \to 0$ as $h \to 0$.

By Jensen's inequality

$$\begin{aligned}
E[b_i^2] &\leqq 2E[R(x_i,x_{i+1}-x_i)^2]\\
&\leqq E[2K^2\|x_{i+1}-x_i\|^{2+2\delta}(1+\|x_i\|^{\alpha})^2(1+\|x_{i+1}-x_i\|^{\alpha})^2],
\end{aligned}$$

which, by Schwartz's inequality,

$$\leqq 2K^2E[\|x_{i+1}-x_i\|^{4+4\delta}]^{\frac{1}{2}}E[(1+\|x_i\|^{\alpha})^4(1+\|x_{i+1}-x_i\|^{\alpha})^4]^{\frac{1}{2}}.$$

Now the distribution of $\|x_i\|$ is, for each i, the same as η_0(with $h = 1$), the random variable occurring in the proof of Lemma 2, the moments of which are all finite. Therefore the second expectation in the above expression is uniformly bounded in i. From (6) we see that

$$\|x_{i+1}-x_i\| \leqq \nu_x(|t_{i+1}-t_i|)+\nu_y(|t_{i+1}-t_i|).$$

Hence by Lemma 2, for some constant K' independent of i,

$$E[b_i^2] \leqq K'|t_{i+1}-t_i|^{1+\frac{1}{2}\delta} \leqq K'(t_{i+1}-t_i)h^{\frac{1}{2}\delta}.$$

Consequently,

$$E[(\sum_i b_i)^2] = \sum_i E[(b_i)^2] \leqq K'h^{\frac{1}{2}\delta},$$

which vanishes with h, and the theorem is proved.

4. Functionals with finite expectations. Suppose now ξ is a general $\mathscr{B}_c$-measurable functional. First we restate and confirm the representation result for functionals with finite second moments mentioned in the introduction:

THEOREM 2. *If* $E[\xi(x)] = 0$ *and* $E[\xi(x)^2] < \infty$, *then*

$$\xi(x) = \int_0^1 \varphi(t)\, dx(t) \quad \text{a.s.} \tag{10}$$

where the integrand $\varphi(t, \omega)$ *is unique in* $L_2\{[0, 1] \times \Omega\}$.

PROOF. Lemma 1 shows that if there is a set of stochastic integrals dense (in quadratic mean) in the set of functionals of x with zero mean and finite second moment then a representation of $\xi(x)$ of the form (10) certainly exists. The partial sums of Itô's orthogonal expansion is such a set. Another is the dense set of functions of the form $g(x(t_1), \cdots, x(t_n))$, where g is continuously differentiable and has bounded derivatives; for by Theorem 1 such functions certainly have integral representations. $\varphi(t, \omega)$ is unique in $L_2\{[0, 1] \times \Omega\}$ because if $\varphi'(t, \omega)$ is an alternative integrand

$$\int_0^1 E[(\varphi(t) - \varphi'(t))^2]\, dt = E[(\xi(x) - \xi(x))^2] = 0.$$

The extension of this theorem to all functionals with finite expectation is as follows.

THEOREM 3. *If* $E[|\xi(x)|] < \infty$, *there exists a function* $\psi(t, \omega)$ *such that for all* $s, t, 0 \leqq s \leqq t \leqq 1$,

$$E[\xi(x) \mid \mathscr{F}_t] - E[\xi(x) \mid \mathscr{F}_s] = \int_s^t \psi(u)\, dx(u) \quad \text{a.s.} \tag{11}$$

and ψ *is unique except perhaps on sets of* (t, ω)*-measure zero.*

An immediate corollary of this, obtained by simple rescaling of time, is that any martingale on a closed interval that is measurable with respect to the increasing family of σ-fields generated by a Brownian motion has a unique representation as the sum of a constant and an indefinite stochastic integral. This is a partial converse to the result that stochastic integrals with square-integrable integrands are martingales.

PROOF. We can choose $E[\xi(x)]$ to be zero as this does not alter (11). Let $z(t)$, $0 \leqq t \leqq 1$, be a separable version of the martingale $E[\xi(x) \mid \mathscr{F}_t]$. We begin by showing that $z(t)$ has continuous sample paths with probability 1. Choose a sequence $\{b_n, n = 1, 2, \cdots\}$ of $\mathscr{F}_1$-measurable random variables with finite second moments in such a way that $E|b_n - \xi(x)| < 1/n^2$ and let $b_n(t)$, $0 \leqq t \leqq 1$, be a separable version of $E[b_n \mid \mathscr{F}_t]$. Theorem 2 then tells us that $b_n = \int_0^1 \varphi_n(t)\, dx(t)$ where $\varphi_n(t, \omega)$ is some square-integrable function. The indefinite integral $\int_0^t \varphi_n(s)\, dx(s)$ has the property of being a martingale with respect to $\{\mathscr{F}_t\}$([2] page 444) and therefore

$$b_n(t) = \int_0^t \varphi_n(t)\, dx(t) \quad \text{a.s.}$$

But separable versions of $\{\int_0^t \varphi_n(t)\,dx(t)\}$ are continuous with probability 1, and so $b_n(t)$ is also continuous with probability 1. Since $z(t)-b_n(t)$ is a martingale

$$P\{\sup_{0 \leqq t \leqq 1} |z(t)-b_n(t)| > a\} \leqq a^{-1}E|z(1)-b_n(1)| \leqq (an)^{-2}.$$

Consequently, by the Borel–Cantelli lemma,

$$\sup_{0 \leqq t \leqq 1} |z(t)-b_n(t)| \to 0 \quad \text{a.s.}$$

$b_n(t)$ is therefore a sequence of continuous functions converging uniformly to $z(t)$ with probability 1, and this implies that $z(t)$ is also continuous with probability 1.

Now let τ_n be the last time that $\sup_{0 \leqq s \leqq t}|z(s)| \leqq n$. It is clear that $\{\tau_n \leqq t\} \in \mathscr{F}_t$. τ_n is therefore a stopping time ([2] page 366) and Doob's optional sampling theorem ([2] page 376, Theorem 11.8) tells us that the stopped process $z_n(t)$, where $z_n(t) = z(\tau_n \wedge t)$, is also a martingale. Moreover,

$$\sup_{0 \leqq t \leqq 1} |z_n(t)| \leqq n \quad \text{a.s.,}$$

and so, by Theorem 2,

$$z_n(t) = \int_0^t \psi_n(t)\,dx(t)$$

where $\psi_n(t,\omega)$ is some square-integrable function.

Now consider the process

$$\int_0^t \psi_m(s)\chi\{\sup_{0 \leqq r \leqq s} |z(r)| \leqq n\}\,dx(s)$$

where $\chi\{\cdot\}$ is the indicator function of the set $\{\cdot\}$. This is clearly $z_m(t)$ stopped at τ_n, which is the same as $z_n(t)$ for $m \geqq n$. Therefore for $m \geqq n$

$$\int_0^1 E[(\psi_m(s)\chi\{\sup_{0 \leqq s \leqq t} |z(r)| \leqq n\} - \psi_n(s))^2]\,ds = 0$$

and on the set $\{\sup_{0 \leqq s \leqq t}|z(s)| \leqq n\}$

$$\psi_n(t) = \psi_{n+1}(t) = \psi_{n+2}(t) = \cdots$$

for almost all (t,ω), Now define $\psi(t,\omega)$ to be $\psi_1(t,\omega)$ on $\{\sup_{0 \leqq s \leqq t}|z(s)| \leqq 1\}$, $\psi_2(t,\omega)$ on $\{1 < \sup_{0 \leqq s \leqq t}|z(s)| \leqq 2\}$ and so on. ψ is clearly a (t,ω)-measurable function and for fixed t is $\mathscr{F}_t$-measurable. Moreover for arbitrary n

$$\{\int_0^1 \psi(t)^2\,dt = \infty\} \subset \{\int_0^1 (\psi(t)-\psi_n(t))^2\,dt > 0\}$$
$$\subset \{\sup_{0 \leqq s \leqq 1} |z(s)| > n\}$$

the probability of which vanishes as $n \to \infty$.

So $\int_0^t \psi(s)^2\,ds < \infty$ a.s. and the integral $\int_0^t \psi(s)\,dx(s)$ is well defined for each t. Now Itô's property G.5 for stochastic integrals ([5] page 15) tells us that

$$P\{|\int_0^t (\psi_n(s)-\psi(s))\,dx(s)| > 0\} \leqq P\{\int_0^t (\psi_n(s)-\psi(s))^2\,ds > 0\}$$

which, as before, vanishes as $n \to \infty$. $z_n(t)$ therefore converges in probability both to $\int_0^t \psi(s)\,dx(s)$ and $z(t)$. So

$$z(t) = \int_0^t \psi(s)\,dx(s) \quad \text{a.s.}$$

The uniqueness for almost all (t,ω) of $\psi(t)$ follows from Itô's Theorem 1.2 [3], but its proof is simple enough: denoting the difference between two possible integrands by $\psi'(t)$, and $\int_0^t \psi'(s)\,dx(s)$ by $z'(t)$, Itô's rule ([3] Theorem 1.1) gives us

$$z'(t)^2 = \int_0^t \psi'(s)^2\,ds + 2\int_0^t z'(s)\psi'(s)\,dx(s) \quad \text{a.s.}$$

and since $z'(t) = 0$ for $0 \leqq t \leqq 1$,

$$\int_0^t \psi'(s)^2\,ds = 0 \quad \text{a.s.}$$

This completes the proof of Theorem 3.

5. The general case. First we note that Theorem 2 and Theorem 3 can be rephrased with the time-domain of the Brownian motion and the stochastic integrals taken as $[0,\infty)$ rather than $[0,1]$. This follows because any functional of the Brownian motion $\{x(t): 0 \leqq t < \infty\}$ is also a functional of the Brownian motion $\{x'(t'): 0 \leqq t' \leqq 1\}$, where $x'(t') = \int_0^t e^{-\frac{1}{2}s}\,dx(s)$, $t' = 1-e^{-t}$, and a stochastic integral on $x'(t)$ can be transformed into one on $x(t)$ with the formula

$$\int_0^\infty \psi'(t)\,dx'(t) = \int_0^\infty \psi'(1-e^{-t})\,e^{-\frac{1}{2}t}\,dx(t) \quad \text{a.s.}$$

In this section we shall work in terms of the Brownian motion $x = \{x(t): 0 \leqq t < \infty\}$. ξ will be taken to be a $\mathscr{F}_\infty$-measurable random variable, $\mathscr{F}_\infty$ being the σ-field generated by $\{x(t): 0 \leqq t < \infty\}$, rather than an explicit functional on the value space of x.

Consider the following identities (shown me by L. A. Shepp; see Lemma 3 for their proof): for any constant K

$$K = \int_0^\infty \chi\{\sup_{0\leqq s\leqq t} x(s) \leqq K\}\,dx(t),$$

and

$$0 = \int_0^\infty (\chi\{\sup_{0\leqq s\leqq t} x(s) \leqq 1\} + \chi\{\inf_{0\leqq s\leqq t} x(s) \geqq -1\})\,dx(t).$$

It follows from the first identity that any function with finite expectation, which by Theorem 3 is equal to the sum of a constant and a stochastic integral, is also equal to a single stochastic integral, though the resulting indefinite integral is no longer generally a martingale and no longer represents the conditional expectations of the functional. And it follows from the second identity that such a representation of a functional is not unique. It turns out that this simpler type of representation is possible for all functionals:

THEOREM 4. *For any finite, $\mathscr{F}_\infty$-measurable random variable ξ there exists an integrand $\psi(t,\omega)$, jointly measurable and $\mathscr{F}_t$-measurable for fixed t, such that* $\xi = \int_0^\infty \psi(t)\,dx(t)$ a.s.

From our earlier remarks it then follows that:

COROLLARY. *If ξ is in addition $\mathscr{F}_t$-measurable, the $\psi(t)$ can be chosen so that*

$$\xi = \int_0^t \psi(s)\,dx(s). \quad \text{a.s.}$$

As we shall see in Lemma 3, it is straightforward to represent any $\mathscr{F}_t$-measurable variable as a stochastic integral on $[0, \infty)$; what difficulty there is in the proof of Theorem 4 lies in showing this is true for all $\mathscr{F}_\infty$-measurable variables. The proof is based on a generalization of Theorem 2: we show there is a finite stopping time T such that ξ has a finite *conditional* second moment given $\mathscr{F}_T$, and then represent ξ as the sum of its conditional mean and a stochastic integral of the Brownian motion beyond T,

LEMMA 3. *Let $\mathscr{H}_0$ be a σ-field independent of $\mathscr{F}_\infty$, and let $\mathscr{H}_t$ be the σ-field generated by $\mathscr{H}_0$ and $\mathscr{F}_t$. Then any $\mathscr{H}_0$-measurable random variable η has the representation*

$$\eta = \int_0^\infty \phi(t)\, dx(t) \quad \text{a.s.}$$

where $\phi(t)$ is $\mathscr{H}_t$-measurable for fixed t.

PROOF. Let T_n be the stopping time $\inf\{r/n : x(r/n) \geqq 1,\ r = 0, 1, \cdots\}$. Then by the continuity of the sample paths of x, as $n \to \infty$, $T_n \to T$, the first passage time by x of the level 1, and $x(T_n) \to x(T) = 1$. But

$$\begin{aligned} x(T_n)\eta &= \textstyle\sum_{r=0}^{\infty} \eta\chi\{r/n < T_n\}(x((r+1)/n) - x(r/n)) \\ &= \textstyle\int_0^\infty \eta\chi\{t < T_n\}\, dx(t), \end{aligned}$$

and since

$$\textstyle\int_0^\infty \eta^2(\chi\{t < T_n\} - \chi\{t < T\})^2\, dt \leqq n^{-1}\eta^2 \to 0,$$

the last expression converges in probability to $\int_0^\infty \eta\chi\{t < T\}\, dx(t)$ and the lemma is proved.

LEMMA 4. *With $\{\mathscr{H}_t\}$ defined as in Lemma 3, if η is $\mathscr{H}_\infty$-measurable and $E[\eta^2 \mid \mathscr{H}_0] < \infty$ a.s., there exists an integrand $\phi(t)$, $\mathscr{H}_t$-measurable for fixed t, such that*

$$\eta = E[\eta \mid \mathscr{H}_0] + \textstyle\int_0^\infty \phi(t)\, dx(t) \quad \text{a.s.} \tag{12}$$

In the above formulation $E[\eta^2 \mid \mathscr{H}_0]$ is a generalized conditional expectation in the sense that we are not assuming that $E\eta^2 < \infty$ (see, for instance, [7] page 121).

PROOF. First suppose $E\eta^2 < \infty$. Then we can approximate η in quadratic mean by a sequence of step functions η_n, each taking only a finite number of values and measurable on the field $\mathscr{H}_0 \cup \mathscr{F}_\infty$:

$$\eta_n = \textstyle\sum_i \chi_{H_i}\chi_{F_i}, \qquad H_i \in \mathscr{H}_0, \qquad F_i \in \mathscr{F}_\infty.$$

Theorem 2 gives us that

$$\chi_{F_i} = E\chi_{F_i} + \textstyle\int_0^\infty \psi_i(t)\, dx(t)$$

for some appropriate integrand $\psi_i(t)$, and approximating the stochastic integral by a sequence of integrals with time-step function integrands then shows that

$$\begin{aligned} \eta_n &= \textstyle\sum_i \chi_{H_i} E\chi_{F_i} + \int_0^\infty \sum \chi_{H_i}\psi_i(t)\, dx(t) \\ &= E[\eta_n \mid \mathscr{H}_0] + \textstyle\int_0^\infty \phi_n(t)\, dx(t) \end{aligned}$$

where $\phi_n(t)$ is $\mathscr{H}_t$-measurable and $\int_0^\infty E\phi_n^2(t)\,dt < \infty$. So by Lemma 1, η also has the representation (12). In the general case, where $E[\eta^2 \mid \mathscr{H}_0] < \infty$, set

$$\begin{aligned} \eta_n' &= \eta \qquad \text{if} \qquad E[\eta^2 \mid \mathscr{H}_0] \leqq n; \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then $\eta_n' \to \eta$. Moreover $E[\eta_n'^2] \leqq n$ and so by our previous argument, for some integrand $\phi_n'(t)$

$$\eta_n' = E[\eta_n' \mid \mathscr{H}_0] + \int_0^\infty \phi_n'(t)\,dx(t).$$

But

$$\begin{aligned} \int_0^\infty E[(\phi_n'(t)-\phi_m'(t))^2 \mid \mathscr{H}_0]\,dt &= E[(\eta_n'-\eta_m)^2 \mid \mathscr{H}_0] - E[\eta_n'-\eta_m' \mid \mathscr{H}_0]^2 \\ &\to 0 \quad \text{a.s.} \end{aligned}$$

as $m, n \to \infty$. Therefore $\int_0^\infty (\phi_n'(t)-\phi_m'(t))^2\,dt \to 0$ in probability and, by the argument of the latter part of the proof of Lemma 1, there is an integrand $\phi(t)$ such that $\int_0^\infty (\phi_n'(t)-\phi(t))^2\,dt \to 0$. The corresponding integrals thus converge and, since by Jensen's inequality and dominated convergence $E[\eta_n' \mid \mathscr{H}_0] \to E[\eta \mid \mathscr{H}_0]$ in probability, the lemma follows.

Proof of Theorem 4. We begin by proving that there is a finite stopping time T, taking integral values, such that $E[\xi^2 \mid \mathscr{F}_T] < \infty$ a.s. Consider the sequence $Z_n = E[\xi^2 \mid \mathscr{F}_n]$, $n = 0, 1, \cdots$, where the conditional expectations may take the value $+\infty$. Though Z_n is not a martingale in the normal sense since we are not assuming that $E|Z_n| < \infty$, it is uniformly bounded below and so can at least be regarded as a generalized supermartingale as defined by Neveu ([7] page 131). So by martingale convergence $Z_\infty = \lim_{n\to\infty} Z_n$ a.s. exists and

$$E[Z_\infty \mid \mathscr{F}_n] \leqq Z_n = E[\xi^2 \mid \mathscr{F}_n] \quad \text{a.s.}$$

Since this is true for all n and Z_∞ and ξ^2 are measurable over $\mathscr{F}_\infty$, which is generated by $\bigcup_{n=1}^\infty \mathscr{F}_n$, it establishes that $Z_\infty \leqq \xi^2$ a.s. However for $N = 1, 2, \cdots$

$$\begin{aligned} Z_\infty = \lim_n Z_n &\geqq \lim_n E[\xi^2 \wedge N \mid \mathscr{F}_n] \\ &= \xi^2 \wedge N \quad \text{a.s.} \end{aligned}$$

and, since ξ^2 is finite, $Z_\infty = \xi^2$ a.s. As its limit is finite the sequence $\{Z_n\}$ is finite a.s. for all n sufficiently large. Now let $T = \min\{n : Z_n < \infty\}$. Then $T < \infty$ a.s. It is easy to verify T is a stopping time, since it is countably valued. Moreover Z_T is finite by the definition of T, and since $\{Z_n\}$ is a positive supermartingale the optional sampling theorem implies

$$E[\xi^2 \mid \mathscr{F}_T] \leqq Z_T < \infty \quad \text{a.s.}$$

which is what we wanted to show.

Now by the strong Markov property of Brownian motions, $x'(t) = x(T+t) - x(T)$ is also a Brownian motion and is independent of $\mathscr{F}_T$. Now if in Lemma 3 and

Lemma 4 we take the Brownian motion to be $x'(t)$, and $\mathscr{H}_0$ to be $\mathscr{F}_T$, then $\mathscr{H}_t$ becomes $\mathscr{F}_{T+t}$ and we have

$$\xi = E[\xi \mid \mathscr{F}_T] + \int_0^\infty \phi'(t)\,dx'(t)$$
$$= \int_0^\infty \psi'(t)\,dx'(t)$$

for some $\psi'(t,\omega)$, which is jointly measurable, $\mathscr{F}_{T+t}$-measurable for fixed t, and which satisfies $\int_0^\infty \psi'(t)^2\,dt < \infty$ a.s. It remains to show that this stochastic integral in $x'(t)$ can be reinterpreted as an integral in $x(t)$. Let

$$\psi(t) = \psi'(t-T)\chi\{T \leqq t\} = \sum_{m=0}^{[t]} \psi'(t-m)\chi\{T=m\},$$

where $[\cdot]$ denotes integral part. Then we can verify that $\psi(t,\omega)$ is jointly measurable, $\mathscr{F}_t$-measurable for fixed t and that $\int_0^\infty \psi(t)^2\,dt < \infty$ a.s.; therefore $\psi(t)$ is integrable with respect to $x(t)$. Let the sequence $\{t_i\}$, depending on an integer n, be an ordered version of the nonnegative elements of $\{r/n-\alpha : r = 1, 2, \cdots\}$ for some α, $0 \leqq \alpha \leqq 1$. Let

$$\psi_n'(t) = \psi'(t_i) \qquad \text{for} \quad t_i \leqq t < t_{i+1}$$
$$\psi_n(t) = \sum_{m=0}^{[t_i]} \psi'(t_i - m)\chi\{T=m\} \qquad \text{for} \quad t_i \leqq t < t_{i+1}.$$

Then it is possible to choose α and a sequence of integers $\{n_j\}$ so that as $n = n_j \to \infty$,

$$\int_0^\infty (\psi_n'(t) - \psi'(t))^2\,dt \to 0$$

and

$$\int_0^\infty (\psi_n(t) - \psi(t))^2\,dt \to 0,$$

both in probability ([2] page 440). Then

$$\int_0^\infty \psi_n'(t)\,dx'(t) \to \int_0^\infty \psi'(t)\,dx'(t)$$

and

$$\int_0^\infty \psi_n(t)\,dx(t) \to \int_0^\infty \psi(t)\,dx(t)$$

in probability. But

$$\int_0^\infty \psi_n'(t)\,dx'(t) = \sum_i \psi'(t_i)(x(T+t_{i+1}) - x(T+t_i))$$
$$= \sum_j \sum_{m=0}^{[t_i]} \psi'(t_j - m)\chi\{T=m\}(x(t_{j+1}) - x(t_j))$$
$$= \int_0^\infty \psi_n(t)\,dx(t).$$

Consequently

$$\xi = \int_0^\infty \psi'(t)\,dx'(t) = \int_0^\infty \psi(t)\,dx(t) \quad \text{a.s.}$$

and the theorem is proved.

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