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ALMOST CERTAIN SUMMABILITY OF INDEPENDENT, IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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1. Summary. The Strong Law of Large Numbers, valid for independent, identically distributed (i.i.d.) random variables $\{X_n, n \geqq 1\}$ with finite first moment, may be regarded as merely one of a host of summability methods applicable to the divergent[3] sequence $\{X_n\}$. Here, a subclass of regular (Toeplitz) summability methods will be considered and concern will focus on the almost certain (a.c.) convergence to zero of the transformed sequence

(1) $$T_n = A_n^{-1} \sum_{j=1}^n a_j X_j$$

when centered where

(i) $$a_n \geqq 0, \qquad A_n = \sum_{j=1}^n a_j \to \infty,$$

thereby ensuring regularity.

If $T_n - C_n \to_{\text{a.c.}} 0$ for some choice of centering constants C_n, the i.i.d. random variables $\{X_n\}$ will be called a_n-summable with probability one or simply a_n-summable. The Strong Law is the special case ($a_n \equiv 1$) of Cesaro-one summability with $C_n \equiv EX$.

Of course, if $X_n^* = X_n - X_n'$, $n \geqq 1$ are the symmetrized X_n (i.e., $\{X_n'\}$ is i.i.d., independent of $\{X_n\}$ with the same distribution), then a_n-summability of $\{X_n\}$ implies a_n-summability of $\{X_n^*\}$ with vanishing centering constants, i.e.

(2) $$T_n^* = A_n^{-1} \sum_{j=1}^n a_j X_j^* \to_{\text{a.c.}} 0.$$

It will be shown, on the one hand, that no such choice of $\{a_n\}$ and $\{C_n\}$ will render i.i.d. $\{X_n\}$ with the St. Petersburg (mass 2^{-n} at the point 2^n, $n \geqq 1$) or Cauchy distribution a_n-summable. On the other hand, necessary and sufficient conditions for certain types of a_n-summability more refined than (implied by) Cesaro-one will be proffered. The prototype of these appears in Corollary 1 and Corollary 2.

2. Results. Criteria, in the case of a numerical sequence x_n, for a comparison of a_n-summability and a_n'-summability are given in [1]. For example, if a_n, a_n' are strictly positive and $a_{n+1}'/a_n' \leqq a_{n+1}/a_n$, then a_n-summability implies a_n'-summability.

If $\{X_n^*, n \geqq 1\}$ is i.i.d. and a_n-summable with $C_n^* = 0$, then necessarily

(*) $$\lim_{n\to\infty} \frac{A_{n+1}}{A_n} = 1.$$

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[3] Only in the degenerate case X_n a.c. constant is a sequence of i.i.d. random variables a.c. convergent to a finite limit.

For if $A_{n_i}/A_{n_i-1} > 1+\delta > 1$ for some subsequence n_i, $i \geqq 1$ of the positive integers, then (2) would entail

$$X_{n_i}^* = \frac{A_{n_i-1}}{a_{n_i}}(T_{n_i}^* - T_{n_i-1}^*) + T_{n_i}^* \to_{\text{a.c.}} 0$$

since $A_{n_i-1}/a_{n_i} < 1/\delta$. But a sequence of non-degenerate i.i.d. random variables cannot converge a.c. to a finite constant so that (*) follows. (The argument is a minor adaptation of Theorem 15 of ([1] page 59)].)

THEOREM 1. *Independent, identically distributed random variables* $\{X_n\}$ *with the St. Petersburg or Cauchy distribution* (*or merely obeying* $\liminf_{x\to\infty} xP\{|X_1| > x\} > 0$) *are not* a_n*-summable for any* $\{a_n\}$ *satisfying* (i).

PROOF. If $\{X_n\}$ is a_n-summable, the symmetrized sequence $\{X_n^*\}$ is a_n-summable with vanishing centering constants whence by a prior remark (*) holds or equivalently $A_n/a_n \to \infty$. Choosing $x > 0$ so that $P\{|X_1| < x\} \geqq \frac{1}{2}$,

$$P\{|X_n^*| > x\} = P\{|X_n - X_n'| > x\} \geqq P\{|X_n| > 2x, |X_n'| < x\}$$
$$\geqq \tfrac{1}{2}P\{|X_1| > 2x\}$$

whence there exist positive constants c, x_0 with $P\{|X_n^*| > x\} \geqq c/x$ for $x \geqq x_0$. Consequently, if n_0 is a positive integer ensuring $A_n/a_n \geqq x_0$ for $n \geqq n_0$,

$$\sum_{n=1}^{\infty} P\left\{|X_n^*| > \frac{A_n}{a_n}\right\} \geqq c \sum_{n=n_0}^{\infty} \frac{a_n}{A_n} = \infty \tag{3}$$

by the Abel–Dini theorem.

However, (2) and (*) entail $(a_n/A_n)X_n^* \to_{\text{a.c.}} 0$ which is incompatible with (3) in view of the Borel–Cantelli lemma.

The next theorem subsumes the classical Strong Law as the special case $a(x) \equiv 1$.

THEOREM 2. *If* $a(x)$, $x > 0$ *is a positive non-increasing function and* $a_n = a(n)$, $A_n = \sum_{i=1}^{n} a_i$, $b_n = A_n/a_n$ *where*

$$A_n \to \infty \tag{i}$$

$$0 < \liminf_{n\to\infty} \frac{b_n}{n} a(\log b_n) \leqq \limsup_{n\to\infty} \frac{b_n}{n} a(\log b_n) < \infty \tag{ii}$$

$$xa(\log^+ x) \quad \textit{is non-decreasing for} \quad x > 0 \tag{iii}$$

then i.i.d. $\{X_n\}$ *are* a_n*-summable if and only if*

$$E|X|\,a(\log^+ |X|) < \infty. \tag{4}$$

PROOF. Sufficiency: Since $0 < a(x)\downarrow$, (i) guarantees that $b_n \uparrow \infty$.

Choose m_0 such that $n \geqq m_0$ implies

$$\alpha n \leqq b_n a(\log b_n) \leqq \beta n \tag{ii'}$$

whence $b_n \geqq \alpha n[a(\log b_m)]^{-1}$ for $n \geqq m \geqq m_0$ entailing

$$\sum_{j=m}^{\infty} b_j^{-2} \leqq \frac{a^2(\log b_m)}{\alpha^2 m}, \qquad m \geqq m_0. \tag{5}$$

Consequently, defining

$$Y_j = X_j I_{[|X_j| \leqq b_j]}, \qquad j \geqq 1 \tag{6}$$

it follows from (5) and (ii′) that for $m \geqq m_0$,

$$\begin{aligned}\textstyle\sum_{j=m}^{\infty} EY_j^2/b_j^2 &= \textstyle\sum_{j=m}^{\infty} b_j^{-2}\left(\int_{[|X_1| \leqq b_{m-1}]} X_1^2 + \sum_{i=m}^{j} \int_{[b_{i-1} < |X_1| \leqq b_i]} X_1^2\right) \\ &\leqq O(1) + \textstyle\sum_{i=m}^{\infty} \sum_{j=i}^{\infty} b_j^{-2} \int_{[b_{i-1} < |X_1| \leqq b_i]} X_1^2 \\ &\leqq O(1) + \alpha^{-2} \textstyle\sum_{i=m}^{\infty} i^{-1} a^2(\log b_i) \int_{[b_{i-1} < |X_1| \leqq b_i]} X_1^2 \\ &\leqq O(1) + \beta\alpha^{-2} \textstyle\sum_{i=m}^{\infty} a(\log b_i) \int_{[b_{i-1} < |X_1| \leqq b_i]} |X_1| \\ &\leqq O(1) + \beta \cdot \alpha^{-2} \textstyle\sum_{i=m}^{\infty} \int_{[b_{i-1} < |X_1| \leqq b_i]} |X_1|\, a(\log |X_1|) < \infty\end{aligned}$$

by (4). Thus, $\sum_{j=1}^{\infty} b_j^{-1}(Y_j - EY_j)$ converges a.c. and so by Kronecker's lemma

$$A_n^{-1} \textstyle\sum_{j=1}^{n} a_j(Y_j - EY_j) \rightarrow_{\text{a.c.}} 0. \tag{7}$$

Via (iii) and (ii′), for $m \geqq m_0$

$$\begin{aligned}\textstyle\sum_m^{\infty} P\{|X_n| \geqq b_n\} &\leqq \textstyle\sum_m^{\infty} P\{|X_n|\, a(\log |X_n|) \geqq b_n a(\log b_n)\} \\ &\leqq \textstyle\sum_m^{\infty} P\{|X_1|\, a(\log |X_1|) \geqq \alpha n\} < \infty,\end{aligned}$$

whence by the Borel–Cantelli lemma

$$P\{X_n \neq Y_n, \text{i.o.}\} = 0. \tag{8}$$

Combining (7) and (8), $\{X_n\}$ is a_n-summable with centering constants

$$C_n = A_n^{-1} \textstyle\sum_{j=1}^{n} a_j EY_j.$$

Conversely, if $\{X_n\}$ is a_n-summable, then $b_n^{-1}X_n^* = (a_n/A_n)X_n^* \rightarrow_{\text{a.c.}} 0$ and so, once more invoking the Borel–Cantelli lemma $\sum_{n=1}^{\infty} P\{|X_n^*| > b_n\} < \infty$. By (iii) and (ii′) for $n > m \geqq m_0$

$$\begin{aligned}&\textstyle\sum_m^n \int_{[b_{j-1} < |X_1^*| \leqq b_j]} |X_1^*|\, a(\log^+ |X_1^*|) \\ &\quad\leqq \textstyle\sum_m^n b_j a(\log b_j) P\{b_{j-1} < |X_1^*| \leqq b_j\} \\ &\quad\leqq \beta \textstyle\sum_m^n j P\{b_{j-1} < |X_1^*| \leqq b_j\} \\ &\quad= \beta[\textstyle\sum_m^{n-1} P\{|X_1^*| > b_j\} + mP\{|X_1^*| > b_{m-1}\} - nP\{|X_1^*| > b_n\}] \\ &\quad\leqq O(1) + \beta \textstyle\sum_m^{\infty} P\{|X_1^*| > b_j\} < \infty\end{aligned}$$

whence

$$\begin{aligned}\infty > E|X_1^*|\, a(\log |X_1^*|) &\geqq \textstyle\int_{[|X_1'| < C]} (|X_1| - C)\, a(\log(|X| + C)) \\ &= P\{|X_1'| < C\} \cdot E[|X_1| - C] \cdot a(\log(|X_1| + C))\end{aligned}$$

which readily implies (4).

COROLLARY 1. *If* X, X_n, $n \geqq 1$ *are* i.i.d., *then*

$(\log n)^{-1}\sum_{j=1}^{n}(X_j/j)-C_n \to_{\text{a.c.}} 0$ *if and only if* $E(|X|/\log|X|)\,I_{[|X|>e]} < \infty$.

Moreover, C_n *may be taken to be* $(\log n)^{-1}\sum_{j=1}^{n} j^{-1}EX_j\,I_{[|X_j|\leqq j\log j]}$.

COROLLARY 2. *If* X, X_n, $n \geqq 1$ *are* i.i.d. *and for some* $k \geqq 2$

$$a_n = [n(\log n)\cdots(\log_{k-1} n)]^{-1}$$

where $\log_1 n = \log n$, $\log_k n = \log(\log_{k-1} n)$, $k \geqq 2$, *then* $\{X_n\}$ *is* a_n*-summable if and only if for all large* $C > 0$,

$$E\frac{|X|\,I_{[|X|>C]}}{(\log|X|)\cdots(\log_k|X|)} < \infty.$$

REFERENCE

[1] HARDY, G. H. (1949). *Divergent Series*. Clarendon Press, Oxford.