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ON THE L_p-CONVERGENCE FOR $n^{-1/p}S_n$, $0 < p < 2$[1]

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Let $(X_n, n \geqq 1)$ be a sequence of random variables and $S_n = X_1 + \cdots + X_n$. By an ingenious method, Pyke and Root [4] prove that if $X_1, X_2, \cdots$ are i.i.d. random variables with $E|X_1|^p < \infty$ for some $0 < p < 2$, then $E|S_n - a_n|^p = o(n)$ as $n \to \infty$, where $a_n = 0$ if $0 < p < 1$ and $a_n = nEX_1$ if $1 \leqq p < 2$. By using an inequality due to Essen and Von Bahr [3], Chatterji [2] extends the result to the following form: If $X_1, X_2, \cdots$ are dominated in distribution by a random variable X with $E|X|^p < \infty$ for some $0 < p < 2$, then $E|S_n - a_n|^p = o(n)$ as $n \to \infty$, where $a_n = 0$ if $0 < p < 1$ and $a_n = \sum_1^n E(X_k \mid X_1, \cdots, X_{k-1})$ if $1 \leqq p < 2$. In this note, by applying an inequality due to Burkholder [1], we will prove the following result, which relaxes the domination condition of [2] to uniform integrability.

THEOREM. *Let $(|X_n|^p, n \geqq 1)$ be uniformly integrable for some $0 < p < 2$. Then as $n \to \infty$*

$$E|S_n - a_n|^p = o(n),$$

where $a_n = 0$ if $0 < p < 1$, and $a_n = \sum_1^n E(X_k \mid X_1, \cdots, X_{k-1})$ if $1 \leqq p < 2$.

PROOF. Define $Y_k = X_k$ if $0 < p < 1$ and $Y_k = X_k - E(X_k \mid X_1, \cdots, X_{k-1})$ for $1 \leqq p < 2$. It is easy to see that if $(|X_n|^p, n \geqq 1)$ is uniformly integrable, so is $(E(|X_n|^p \mid X_1, \cdots, X_{k-1}), n \geqq 1)$. Hence $(|Y_n|^p, n \geqq 1)$ is uniformly integrable. For $\varepsilon > 0$, choose $M > 0$ so that $\int_{[|Y_n| > M]} |Y_n|^p < \varepsilon$ for all $n \geqq 1$. Put

$$Y_n' = Y_n I_{[|Y_n| \leqq M]}, \qquad Y_n'' = Y_n - Y_n'.$$

(a) If $0 < p < 1$,

$$E|S_n|^p = E|\textstyle\sum_1^n (Y_k' + Y_k'')|^p \leqq E(\sum_1^n |Y_k'|)^p + E(\sum_1^n |Y_k''|)^p$$
$$\leqq E(\textstyle\sum_1^n |Y_k'|)^p + \sum_1^n E|Y_k''|^p \leqq (nM)^p + n\varepsilon,$$

and hence $E|S_n|^p = o(n)$ as $n \to \infty$.

(b) If $I < p < 2$, by Burkholder inequality [1] there exists a constant $A_p > 0$ satisfying

$$A_p E|\textstyle\sum_1^n Y_k|^p \leqq E(\sum_1^n Y_k^2)^{p/2}, \qquad n \geqq 1.$$

Hence

$$A_p E|S_n - a_n|^p = A_p E|\textstyle\sum_1^n Y_k|^p \leqq E(\sum_1^n Y_k^2)^{p/2}$$
$$= E\{\textstyle\sum_1^n (Y_k'^2 + Y_k''^2)\}^{p/2} \leqq E(\sum_1^n Y_k'^2)^{p/2} + E(\sum_1^n Y_k''^2)^{p/2}$$
$$\leqq E(\textstyle\sum_1^n Y_k'^2)^{p/2} + E\sum_1^n |Y_k''|^p \leqq (nM^2)^{p/2} + n\varepsilon.$$

Therefore $E|S_n - a_n|^p = o(n)$ as $n \to \infty$.

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(c) Let $p = 1$. Then $\sum_1^\infty n^{-1}\{Y_n' - E(Y_n' \mid X_1, \cdots, X_{n-1})\}$ converges a.s., since $|Y_n'| \leqq M$. Therefore $\sum_1^n \{Y_k' - E(Y_k' \mid X_1, \cdots, X_{k-1})\} = o(n)$ a.s. by Kronecker lemma; $E|\sum_1^n \{Y_k' - E(Y_k' \mid X_1, \cdots, X_{k-1})\}| = o(n)$ by Lebesgue dominated convergence theorem. Since

$$E\left|\textstyle\sum_1^n \{Y_k'' - E(Y_k'' \mid X_1, \cdots, X_{k-1})\}\right| \leqq 2E \textstyle\sum_1^n |Y_k''| \leqq 2n\varepsilon,$$

we obtain that

$$\begin{aligned} E\left|\textstyle\sum_1^n Y_k\right| &= E\left|\textstyle\sum_1^n \{Y_k' - E(Y_k' \mid X_1, \cdots, X_{k-1}) + Y_k'' - E(Y_k'' \mid X_1, \cdots, X_{k-1})\}\right| \\ &\leqq o(n) + 2n\varepsilon, \end{aligned}$$

and therefore $E|S_n - a_n| = E|\sum_1^n Y_k| = o(n)$ as $n \to \infty$.

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