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THE DISCOUNTED CENTRAL LIMIT THEOREM AND ITS BERRY–ESSÉEN ANALOGUE[1]

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1. Introduction and summary. Let $X_0, X_1, X_2, \cdots$ be a sequence of independent random variables with common distribution function $F(x)$. Let v be a discount factor $(0 < v < 1)$. Then we define

$$X_v = \textstyle\sum_{k=0}^{\infty} v^k X_k, \tag{1.1}$$

which may be interpreted as the *present value* of a sum of certain periodic and identically distributed payments X_k.

We assume that the first three moments of X_k are finite:

$$\mu = \textstyle\int_{-\infty}^{+\infty} x\, dF(x) < \infty, \qquad \sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2\, dF(x) < \infty, \tag{1.2}$$
$$\rho = \textstyle\int_{-\infty}^{+\infty} |x-\mu|^3\, dF(x) < \infty.$$

It will be shown that the normalized random variable

$$Z_v = \frac{(1-v)^{\frac{1}{2}}}{\sigma}\left(X_v - \frac{\mu}{1-v}\right) \tag{1.3}$$

is asymptotically normal for $v \to 1$. The analogue of the Berry–Esséen theorem (see [1], [2], [3], [4]) will be established for the difference $F_v(x) - \mathcal{N}_v(x)$, $F_v(x)$ being the distribution of Z_v, whereas $\mathcal{N}_v(x)$ is the normal distribution with zero mean and variance $(1+v)^{-1}$:

$$\mathcal{N}_v(x) = \left(\frac{1+v}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{x} \exp\left(-\frac{1+v}{2}t^2\right) dt. \tag{1.4}$$

The proof uses those Fourier techniques which are masterfully presented in Feller's book [3] for the proof of the "ordinary" Berry–Esséen theorem.

In the last section the corresponding estimate is established for the compound Poisson process.

Other aspects of the Discounted Central Limit Theorem are treated in a paper by Whitt [5].

2. The discounted version of the Berry–Esséen Theorem. We do not attempt to prove asymptotic normality of Z_v under the most general conditions; let us proceed directly to the analogue of the Berry–Esséen theorem (from which asymptotic normality naturally follows).

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THEOREM 1. *If* (1.2) *holds, then for all* x

$$|F_v(x)-\mathcal{N}_v(x)| \leqq C(\rho/\sigma^3)(1-v)^{\frac{1}{2}}. \tag{2.1}$$

In the proof we shall see that this is true for $C = 5.4$.

PROOF. Of course we may assume $\mu = 0$. Let us denote by ϕ and Φ the characteristic functions of X_k and Z_v respectively. We need the following basic

LEMMA. *For all* x *and* $T > 0$

$$|F_v(x)-\mathcal{N}_v(x)| \leqq \frac{1}{\pi}\int_{-T}^{+T}\frac{1}{|\zeta|}\left|\Phi_v(\zeta)-\exp\left(-\frac{1}{2}\frac{\zeta^2}{1+v}\right)\right|d\zeta+\frac{24m}{\pi T} \tag{2.2}$$

with

$$m = \max \mathcal{N}_v'(x) = \mathcal{N}_v'(0) = \left(\frac{1+v}{2\pi}\right)^{\frac{1}{2}}.$$

This lemma is a special case of Feller's "Lemma 2" ([3] page 512). Furthermore we need Feller's (5.6) of page 516: For $|\zeta| \leqq \sigma^2/\rho$ we have

$$\left|\log\phi(\zeta)+\tfrac{1}{2}\sigma^2\zeta^2\right| \leqq \tfrac{5}{12}\rho\,|\zeta|^3. \tag{2.3}$$

The formulas (1.1) and (1.3) mean that

$$\Phi_v(\zeta) = \prod_{k=0}^{\infty}\phi\left(\frac{(1-v)^{\frac{1}{2}}}{\sigma}v^k\zeta\right), \tag{2.4}$$

such that

$$\log\Phi_v(\zeta) = \sum_{k=0}^{\infty}\log\phi\left(\frac{(1-v)^{\frac{1}{2}}}{\sigma}v^k\zeta\right). \tag{2.5}$$

From this and (2.3) we obtain for $|\zeta| \leqq (1-v)^{-\frac{1}{2}}\sigma^3/\rho$

$$\left|\log\Phi_v(\zeta)+\frac{1}{2}\frac{\zeta^2}{1+v}\right| \leqq \frac{5}{12}\frac{\rho}{\sigma^3}\frac{(1-v)^{\frac{1}{2}}}{1+v+v^2}|\zeta|^3. \tag{2.6}$$

Let $T = (1-v)^{-\frac{1}{2}}\sigma^3/\rho$. Thus the integrand in (2.2) is dominated by

$$\frac{1}{|\zeta|}\exp\left(-\frac{1}{2}\frac{\zeta^2}{1+v}\right)\left[\exp\left(\frac{5}{12}\frac{\rho}{\sigma^3}\frac{(1-v)^{\frac{1}{2}}}{1+v+v^2}|\zeta|^3\right)-1\right]. \tag{2.7}$$

But the latter is, because of $|e^t-1| \leqq |t|\cdot e^{|t|}$, dominated by

$$\begin{aligned}&\frac{5}{12}\frac{\rho}{\sigma^3}\frac{(1-v)^{\frac{1}{2}}}{1+v+v^2}\zeta^2\exp\left(-\frac{1}{2}\frac{\zeta^2}{1+v}+\frac{5}{12}\frac{\rho}{\sigma^3}\frac{(1-v)^{\frac{1}{2}}}{1+v+v^2}|\zeta|^3\right)\\ &\leqq \frac{5}{12}\frac{\rho}{\sigma^3}\frac{(1-v)^{\frac{1}{2}}}{1+v+v^2}\zeta^2\exp\left(-\frac{1}{2}\frac{\zeta^2}{1+v}+\frac{5}{12}\frac{\rho}{\sigma^3}\frac{(1-v)^{\frac{1}{2}}}{1+v+v^2}\zeta^2 T\right).\end{aligned} \tag{2.8}$$

Extending the integral in (2.2) from $-\infty$ to $+\infty$, using the above estimate, and remembering that $\int_{-\infty}^{+\infty}t^2\exp(-t^2)\,dt = (\pi^{\frac{1}{2}}/2)$, lead to

$$|F_v(x)-\mathcal{N}_v(x)| \leqq \rho/\sigma^3\,C(1-v)^{\frac{1}{2}}, \tag{2.9}$$

where

$$C = \frac{1}{\pi^{\frac{1}{2}}}\frac{5}{24}\frac{1}{1+v+v^2}\left(\frac{1}{2}\frac{1}{1+v} - \frac{5}{12}\frac{1}{1+v+v^2}\right)^{-\frac{3}{2}} + \frac{24(1+v)^{\frac{1}{2}}}{\pi(2\pi)^{\frac{1}{2}}}. \tag{2.10}$$

One may take $C = 5.4$, because for $0.96 \leqq v \leqq 1$ the above formula leads to a smaller constant, whereas for $0 \leqq v \leqq 0.96$ we find that $5.4\,(1-v)^{\frac{1}{2}}$ is greater than one, such that (2.1) is true anyway.

3. Further discussion. The factor $(1-v)^{\frac{1}{2}}$ in (2.1) cannot be improved for $v \to 1$. Indeed, for any alternative inequality to (2.1), with $(1-v)^{\frac{1}{2}}$ replaced by some function $f(v)$, the passage to the limit ($v \to 1$, $t \to 0$, respectively) in the next section shows that for some $\lambda > 0$

$$\frac{f(e^{-\beta t})}{t^{\frac{1}{2}}} = \frac{f(v)}{\left(\frac{1}{\beta}\log\frac{1}{v}\right)^{\frac{1}{2}}} \geqq \lambda. \tag{3.1}$$

The statement now follows from $(\log 1/v)^{\frac{1}{2}} = (1-v)^{\frac{1}{2}}\{1+O(1-v)\}$.

Obviously the constant C can be improved, and it would be desirable to find better estimates.

4. Formulation for the compound Poisson process. Let $\{X_t\}_{t \geqq 0}$ denote a compound Poisson process with $X_0 = 0$. Thus

$$P[X_t \leqq x] = \sum_{k=0}^{\infty} e^{-\alpha t}\frac{(\alpha t)^k}{k!}F^{*k}(x). \tag{4.1}$$

Here α is the Poisson parameter, $F(x)$ the distribution of the magnitude of the individual jumps (which in our context, should be interpreted as payments). We assume that the first three moments of the latter exist. It is easily verified that

$$E[X_t] = \alpha t m, \qquad \sigma_t^2 = \mathrm{Var}\,[X_t] = \alpha t \textstyle\int_{-\infty}^{+\infty}|x|^2\,dF(x), \tag{4.2}$$

$$\lim_{t\to 0}\rho_t/t = \lim_{t\to 0}E[|X_t-\alpha t m|^3]/t = \alpha\textstyle\int_{-\infty}^{+\infty}|x|^3\,dF(x),$$

with $m = \int_{-\infty}^{+\infty} x\,dF(x)$.

Let $\beta > 0$ be a *rate of interest*. Thus the random variable

$$X_\beta = \textstyle\int_{t=0}^{\infty} e^{-\beta t}\,dX_t \tag{4.3}$$

represents the sum of the *discounted payments*. We wish to establish the Central Limit Theorem for the corresponding normalized random variable

$$Z_\beta = \left(\frac{\beta}{\alpha\cdot\int x^2\,dF(x)}\right)^{\frac{1}{2}}\left[X_\beta - \frac{\alpha m}{\beta}\right]. \tag{4.4}$$

Let $F_\beta(x)$ be the distribution of Z_β.

The problem is easily solved by reduction to the discrete case, followed by an obvious passage to the limit. From Theorem 1 and (4.2) one gets

THEOREM 2. *If* $\int_{-\infty}^{+\infty} |x|^3 \, dF(x) < \infty$, *then for all* x

$$|F_\beta(x) - \mathcal{N}_1(x)| \leqq C\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}} \frac{\int |x|^3 \, dF(x)}{[\int x^2 \, dF(x)]^{\frac{3}{2}}}, \tag{4.5}$$

where we know that $C = 5.4$ *does the job.*

Clearly this estimate can be established for a larger class of homogeneous processes with independent increments. One needs essentially that

$$\lim_{t \to 0} \frac{\rho_t}{\sigma_t^3} (1 - e^{-\beta t})^{\frac{1}{2}} \tag{4.6}$$

exists.

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