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NOTE ON THE CHARACTERIZATION OF CERTAIN ASSOCIATION SCHEMES

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1. Introduction. Much attention has been paid in recent years to characterizing association schemes by means of their parameters (see e.g. [2]). There have been essentially two different approaches, one using the concept of a claw, e.g. [2, 3], the other considering the eigenvalues of the corresponding strongly regular graph, e.g. [4]. In this note we suggest another method, namely, to study the structure of the set of treatments which are first associates of a given treatment, using all the information contained in the parameters n_1, p_{11}^1, p_{11}^2. It appears convenient to discuss the problems in terms of the strongly regular graph obtained from the association scheme by joining two treatments iff they are first associates. Henceforth we shall adopt the graphtheoretic language. To give a concrete example let us assume the graphs satisfy $p_{11}^2 \leqq 2$. This now implies the subgraph $A(x)$ generated by all points adjacent to a given point x does not contain a cycle of length four unless it is embedded in a complete graph on four points. This requirement considerably restricts the class of possible graphs $A(x)$, and in certain cases readily yields the solution of the characterization problem. To give an application we shall exhibit the characterization of the line graph of the complete bipartite graph [5], [6], [7]. The advantage of the present approach is that it lends itself to generalization to several associate classes (see [1]), admits variable p_{11}^1, p_{11}^2 and also produces all the exceptions.

2. A class of graphs. It is our goal to characterize the line graph of the complete bipartite graph on sets with m and n vertices, denoted by $L(B_{m,n})$, by the following properties:

For $m \geqq n \geqq 2$

(P1) $L(B_{m,n})$ has $m \cdot n$ vertices.

(P2) It is regular of degree $m+n-2$.

(P3) Exactly $n \cdot \binom{m}{2}$ pairs of adjacent points are mutually adjacent to $m-2$ points, the remaining $m \cdot \binom{n}{2}$ pairs of adjacent points are mutually adjacent to $n-2$ points.

(P4) Any two nonadjacent points are mutually adjacent to two points.

We now define for $m \geqq n \geqq 2$ the following class $\mathscr{G}(m, n)$ of graphs G:

(Q1) G contains $m+n-2$ points.[1]

(Q2) The degree of a point in G is either $m-2$ or $n-2$, but there are at least $m-1$ points of degree $m-2$.

(Q3) There is no cycle of length 4 in G unless it is embedded in a complete graph on 4 points.

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[1] It may be noted that (Q1)–(Q3) and thus the following theorem are also applicable to the line graph of a BIB design with m replications and block size n.

We shall denote the complete graph on i points by K_i; $G = A \cup B$ shall mean that the point set of G is the union of the two disjoint point sets of A and B. (There may be edges between A and B, however.) Using this terminology, it is clear that the graph $K_{m-1} \cup K_{n-1}$ belongs to $\mathscr{G}(m, n)$ for all $m \geqq n \geqq 2$. We shall refer to this graph as the normal graph $N(m, n)$ of the class $\mathscr{G}(m, n)$.

THEOREM 1. *$\mathscr{G}(m, n) = \{N(m, n)\}$, except for $m = 4$, $n = 3$; $m = 5$, $n = 4$; $m = n = 4$.*

$$\mathscr{G}(4, 3) = \{N(4, 3), G_1, \textit{simple } 5\textit{-cycle}\},$$
$$\mathscr{G}(5, 4) = \{N(5, 4), G_2, G_3\},$$
$$\mathscr{G}(4, 4) = \{N(4, 4), \textit{simple } 6\textit{-cycle}\}.$$

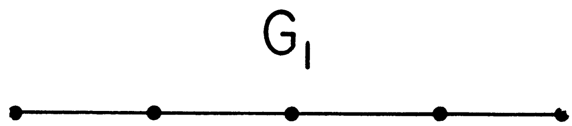

FIG. 1.

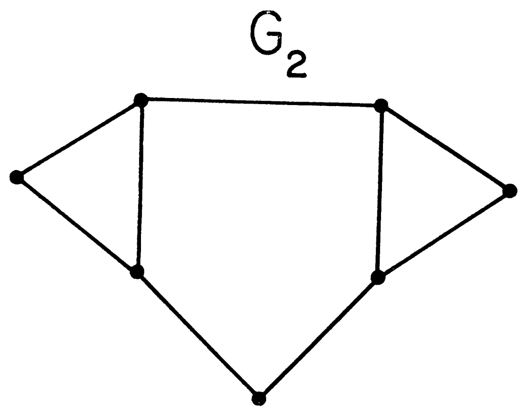

FIG. 2.

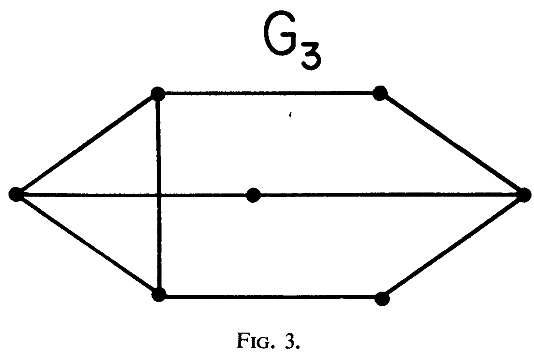

FIG. 3.

PROOF. First we note that the theorem is trivially true for $m = n = 2$, $m = 3$ and $n = 2$ or 3. Hence we assume $m \geqq 4$. Let G be an arbitrary member of $\mathscr{G}(m, n)$, we shall refer to points in G of degree $m-2$ as m-points, to the others as n-points. $A(x)$ shall denote the set of all points adjacent to x. Let $i \geqq 2$ be maximal such that there exists a K_i containing an m-point. The following three facts are all implied by (Q3).

(1) $A(x) = K_{i_1} \cup \cdots \cup K_{i_t}$, no lines between K_{i_j} and $K_{i_{j'}}$, $j \neq j'$,

(2) $y \notin A(x)$ implies $|A(x) \cap A(y)| \leqq 1$,

(3) $y, y' \notin K_i$ and y, y' adjacent to distinct points of K_i imply y is not adjacent to y'.

Let x be an arbitrary m-point and suppose x is contained in k K_i's, then

(4) $$k(i-1) \leqq m-2.$$

Suppose there are a m-points in the k K_{i-1}'s of $A(x)$, b m-points in the other K_j's ($j < i-1$), then counting the edges from $A(x)$ into the set of the remaining $n-1$ points other than x we obtain, using (2),

$$a(m-i-1)+(k(i-1)-a)\cdot(n-i-1)+b(m-i) +(m-2-k(i-1)-b)\cdot(n-i) \leqq n-1.$$

Making use of (4) this is easily rearranged to

(5) $$n-i \leqq \frac{n-1-(a+b)\cdot(m-n)}{m-2}+1.$$

As the right-hand side of (5) is less than 3 for $m > 3$, we are faced with three possibilities as to whether $n-i \leqq 0$, $= 1$, or $= 2$.

CASE A. $n-i \leqq 0$.
Here $m \geqq i+1$, and K_i entirely consists of m-points. Let $G = K_i \cup R$, $|R| = m+n-2-i$. Counting the number of edges from K_i into R, we get by (2)

(6) $$i(m-i-1) \leqq m+n-2-i, \quad \text{or}$$
$$(m-i)\cdot(i-1) \leqq i+n-2 \leqq 2(i-1),$$

and thus $1 \leqq m-i \leqq 2$, $|R| \leqq n$.

(a) $i = m-1$. Here $|R| = n-1$, and $N(m, n)$ results.

(b) $i = m-2$. In this case, (6) yields $i = n = |R|$, and every point of K_i is adjacent to exactly one point of R. Recalling (2) and (3), this means R is completely disconnected but since there must be at least one m-point in R (by (Q2)), we would obtain $m-2 = n = 1$, a contradiction.

CASE B. $n-i = 1$.
We have $G = K_{n-1} \cup R$, $|R| = m-1$, $n \geqq 3$. If all points of K_{n-1} are n-points (this includes the case $m = n$), we plainly arrive at $N(m, n)$. So assume there are $c \geqq 1$

m-points in K_i, hence $c \cdot (m-n)$ edges from K_i into R. Let $R = R' \cup R''$, where R' consists of the points in R adjacent to some point in K_{n-1}. Because of $m > n$, there must be m-points in R. If there were two or more m-points in R'', then applying (1) we note that R would have to be a K_{m-1}, contradicting the maximality of i. Hence the following two possibilities arise:

(a) There is exactly one m-point in R'', call it z.
(b) There is no m-point in R''.

(a) Since z is adjacent to every point in R, we invoke (2) to obtain $m = n+1$. Further R' is completely disconnected, and hence counting the number of edges from R' into $R'' - \{z\}$ we have

$$c(n-4) \leqq n-1-c \quad \text{or} \tag{7}$$

$$c \leqq (n-1)/(n-3) < 4 \quad \text{for} \quad n \geqq 4. \tag{8}$$

Since $c \leqq n-1$, (8) holds for $n = 3$ as well.

$c = 1$. (Q2) implies the number of m-points in R' must be at least $m - 3$, hence $m \leqq 4$, and the only possible case $m = 4$ readily yields the graph G_1.

$c = 2$. If both points in R' are m-points, then refining the count (7), we obtain $n = 3$, and the simple 5-cycle results. If one is an m-point, the other an n-point, then appealing once more to (7), we have $n = 4$ as the only case, and G_2 is easily seen to be only possible graph. Finally the case that neither point of R' is an m-point would imply $n \geqq 4$, but by (Q2) we also have $m \leqq 4$, a contradiction.

$c = 3$. (7) yields $n = 4$, all points of R' clearly must be n-points, thus G_3 results.

(b) By employing the counting argument (7), it is easily seen that this case does not produce any new graphs.

CASE C. $n-i = 2$.
Going back to (5), we infer that either $m = n$, or $m = n+1$ and $a = b = 0$. The second alternative is quickly disposed of by noting that the $m-2$ points not adjacent to x must all be m-points. By (2), they must form a K_{m-2}, which violates the maximality of i. Let us then examine the possibility $m = n$. Here $G = K_{n-2} \cup R$, $|R| = n$, $n \geqq 4$, and every point of K_{n-2} is adjacent to exactly one point of R. The set of these $n-2$ points in R is completely disconnected, hence any such point has degree at most 3, or $n \leqq 5$. The case $n = 5$ is readily shown to contradict (Q3), whereas for $n = 4$ the simple 6-cycle results.

3. Characterization of $L(B_{m,n})$. We may represent $L(B_{m,n})$ as the graph having as point set all ordered pairs (i, j), $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, with two pairs adjacent iff they have a coordinate in common.

LEMMA. *Given a graph G satisfying* (P1)–(P4),[2] *and assume there exists a point x with* $A(x) = N(m, n)$, *then* $G \cong L(B_{m,n})$.

[2] In fact, we could relax (P3) to (P3′): any pair of adjacent points are mutually adjacent to either $m-2$ or $n-2$ other points.

PROOF. Let us denote x by $(1, 1)$, the points contained in K_{m-1} by $(i, 1)$, $2 \leqq i \leqq m$, those in K_{n-1} by $(1, j)$, $2 \leqq j \leqq n$. Let

$$A((i, 1)) = \bigcup_{j \neq i} \{(j, 1)\} \cup B_i, \qquad 2 \leqq i \leqq m,$$

then by (P3) the sets B_i are mutually exclusive, and since

$$\sum_{i=2}^{m} |B_i| = (m-1)(n-1) = mn - m - n + 1,$$

they exhaust the set of all points not adjacent to x. Now $(i, 1)$ and an arbitrary point of B_i are mutually adjacent to at least $n-2$ points. As none of $(j, 1), j \neq i$, can be such a point, we conclude B_i generates a K_{n-1}. Every $(1, j), j \neq 1$, is adjacent to exactly one point of B_i (by (P4)). Let us call this point (i, j) with $C_j = \{(i, j), 2 \leqq i \leqq m\}$. The same argument as above then shows C_j generates a K_{m-1}. Finally $C_j \cap C_{j'} = \phi$ for $j \neq j'$, as the opposite would violate (P4) when applied to x and a point common to C_j and $C_{j'}$, and the proof is complete.

THEOREM 2. *Let G be a graph satisfying* (P1)–(P4), *then* $G \cong L(B_{m,n})$, *except for* $m = n = 4$, *in which case there is exactly one other graph.*

PROOF. From (P3), we infer that the average number of m-points in $A(x)$ equals $n \cdot \binom{m}{2} / m \cdot n = m - 1$. Hence we may apply Theorem 1 which, together wth the preceding lemma, establishes our assertion except possibly for $m = 4, n = 3$; $m = 5, n = 4$; $m = n = 4$. The five additional graphs of Theorem 1 have to be treated separately. A straightforward argument shows the impossibility of G_1, G_2, G_3 or the simple 5-cycle as candidates for $A(x)$. Let us just verify this claim for G_1. Here $m = 4$, $n = 3$, $|G| = 12$, and there are 6 points nonadjacent to x, call this set $B(x)$. Let u, v be the two endpoints of $A(x) = G_1$, w the center point. Now there are 2 points in $B(x)$ adjacent to w, and by (P4) neither one can be joined to u or v. Since of the remaining 4 points of $B(x)$, 3 are joined to u, 3 to v, at least 2 of them must be adjacent to both, in violation of (P4). Finally, in the case $m = n = 4$ and $A(x)$ being the simple 6-cycle (using the fact that in view of the lemma $A(y)$ must be a simple 6-cycle for every $y \in G$), it is easily shown that there is just one exception.

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