The Annals of Mathematical Statistics
1971, Vol. 42, No. 2, 802–808

INFINITE DIVISIBILITY AND VARIANCE MIXTURES OF THE NORMAL DISTRIBUTION

BY DOUGLAS KELKER
Washington State University

1. Introduction. In this paper when we say "mixture of normal distributions," we mean a variance mixture of $n(0, u)$ distributions with characteristic functions of the form $C(t) = \int \exp(-t^2u/2)\, dG(u)$, with G a distribution on $[0, \infty)$. The corresponding density is $f(x) = \int (2\pi u)^{-\frac{1}{2}} \exp(-x^2/(2u))\, dG(u)$. To investigate whether or not $C(t)$ is infinitely divisible (id), we use Laplace transform theory on $C(2^{\frac{1}{2}}t^{\frac{1}{2}})$, the Laplace transform of G. A function h is completely monotone (cm) on $(0, \infty)$ if and only if $(-1)^n h^{(n)}(x) \geqq 0$ for $x > 0$ and $n = 0, 1, 2, \cdots$. Bernstein's representation theorem for cm functions (see Feller [1] page 415) states that h is cm on $(0, \infty)$ if and only if $h(x) = \int \exp(-xu)\, dP(u)$ with P a non-decreasing function on $[0, \infty)$. From this we see that $C(u^{\frac{1}{2}})$ and $f(u^{\frac{1}{2}})$ are cm on $(0, \infty)$ as functions of u. Or we can say that a characteristic function is a mixture of normal characteristic functions (densities) if and only if $h(u)$ is an even function and $h(u^{\frac{1}{2}})$ is cm on $(0, \infty)$. The Cauchy density $(\pi(1+x^2))^{-1}$ is easily seen to be a mixture of normal densities, since $(1+x)^{-1}$ is obviously cm on $(0, \infty)$. For the same reason, the characteristic function of the Laplace distribution $(1+t^2)^{-1}$ is a mixture of normal characteristic functions. Student's t-distribution and the symmetric stable distributions are also mixtures of normal distributions.

Another type of characterization is contained in a result of I. J. Schoenberg [5]. Stated in probabilistic language, his theorem is as follows:

THEOREM 1. *A necessary and sufficient condition for a univariate characteristic function C to be a variance-mixture of normal characteristic functions is that there exists a function ψ such that $C(u) = \psi(u^2)$ and that, with $\mathbf{t}' = (t_1, t_2, \cdots, t_p)$, $C_p(\mathbf{t}) = \psi(|\mathbf{t}|^2)$ is a p-dimensional characteristic function for each p $(p = 1, 2, \cdots)$.*

2. Infinite divisibility and normal mixtures. An attempt was made to characterize all distributions G on $[0, \infty)$ such that $\int (2\pi u)^{-\frac{1}{2}} \exp(-x^2/(2u))\, dG(u)$ is the density of an id distribution. The attempt was only partially successful.

It is known (see Feller [1] page 427) that if G is id, then the mixture is id. However, not all mixtures are id, as the following theorem shows.

THEOREM 2. *If G is a non-degenerate distribution on* $[0, \infty)$ *and* $G(b) = 1$ *for some finite b, then the characteristic function* $C(t) = \int \exp(-t^2u/2)\, dG(u)$ *is not infinitely divisible.* (G is a "finite" distribution according to the terminology of Lukacs [4]).

PROOF. Let g be the characteristic function of G, $g(t) = \int e^{itu}\, dG(u)$. It is well known (see Lukacs [4] page 141) that g has infinitely many zeros in the complex

Received April 7, 1970.

plane. Noting that $g(it^2/2) = C(t)$, it follows easily that C also has zeros in the complex plane. Hence C cannot be id since C is an entire characteristic function (see Lukacs [4] page 187).

A. Tortrat [8] proves a particular case of Theorem 2.

But there are id normal mixtures in which the mixing distribution is not id. The following is an example of a mixture of normal characteristic functions of the form $C(t) = \int \exp(-t^2u/2)\,dG(u)$ which is id even though the distribution G is not id.

Define

$$H(x) = \begin{cases} 0 & x < 1 \\ .26 & 1 \leqq x < 2 \\ .52 & 2 \leqq x < 3 \\ .48 & 3 \leqq x < 4 \\ .74 & 4 \leqq x < 5 \\ 1.0 & x \geqq 5. \end{cases}$$

The function H was constructed to satisfy the conditions:

(i) H is not a distribution.

(ii) $H * H$ is a distribution.

(iii) $\int (2\pi u)^{-\frac{1}{2}} \exp(-x^2/(2u))\,dH(u)$ is a density.

Define the mixing distribution to be $G(x) = e^{-1}\sum_0^\infty (k!)^{-1}H^{*k}(x)$. Routine calculations show that G is a distribution function. We also have

$$\begin{aligned} &\int \exp(-t^2u/2)\,dG(u) \\ (1)\qquad &= e^{-1}\textstyle\sum_0^\infty (k!)^{-1}\left(\int \exp(-t^2u/2)\,dH(u)\right)^k \\ &= \exp\left[\int \exp(-t^2u/2)\,dH(u) - 1\right] \\ &= \exp\left[\int (\cos(tx) - 1)(2\pi)^{-\frac{1}{2}} \int u^{-\frac{1}{2}} \exp(-x^2/(2u))\,dH(u)\,dx\right]. \end{aligned}$$

The last expression in the string of equalities is a form of Kolmogorov's canonical representation (Gnedenko–Kolmogorov [2] page 85) for id characteristic functions, provided that $\int u^{-\frac{1}{2}} \exp(-x^2/(2u))\,dH(u) \geqq 0$ for all $x > 0$. But H was constructed to satisfy this condition. Thus $C(t) = \int \exp(-t^2u/2)\,dG(u)$ is an id characteristic function.

But G is not id, since H is the spectral measure in the Lévy representation of the characteristic function of G and H is not non-decreasing.

The function H was originally constructed to be used in the following example to show that the factors of a mixture of normal distributions do not have to be mixtures of normal distributions. Let $C_1(t) = \int \exp(-t^2u/2)\,d(H * H)(u)$, $C_2(t) = (C_1(t))^{\frac{1}{2}}$. The characteristic function C_1 is a mixture of normals, and C_2 is a characteristic function but is not a mixture of normal characteristic functions since $C_2(t) = \int \exp(-t^2u/2)\, dH(u)$, and H is not a distribution function.

The following lemma can be used as a quick proof in certain cases to show that a real-valued characteristic function is id. The procedure works this way.

LEMMA 1. *For a real-valued characteristic function C define $h_n(t) = (C(t^{\frac{1}{2}}))^{1/n}$ for $t > 0$ $(n = 1, 2, \cdots)$. If h_n is completely monotone on $(0, \infty)$ for each n, then C is infinitely divisible.*

PROOF. The fact that h_n is cm for each n implies that h_n is the Laplace transform of a probability distribution for each n. So according to the Laplace transform definition of infinite divisibility, we have $C(t^{\frac{1}{2}}) = \int \exp(-tu)\, dG(u)$ with G an id distribution on $[0, \infty)$. Then $C(t) = \int \exp(-t^2u/2)\, dG(u/2)$. Since G is id, so is the characteristic function C.

As an example, the density $\frac{1}{2}e^{-|x|}$ of the Laplace distribution has characteristic function $C(t) = (1+t^2)^{-1}$. Since $(1+t)^{-1/n}$ is cm for each n, we conclude that C is an id characteristic function. Similarly, all the Student-t densities considered as characteristic functions are id characteristic functions.

THEOREM 3. *The Student-t distribution is infinitely divisible for three, five and seven degrees of freedom (and, of course, also for one degree of freedom, since it is then a Cauchy distribution).*

PROOF. It is known that for odd degrees of freedom the characteristic function of the t-distribution can be expressed in closed form (D. Starkey [6]). If n is odd, the characteristic function is

$$C_n(t) = \exp(-n^{\frac{1}{2}}|t|)\, \frac{k!}{(2k)!} \sum_{r=0}^{k} \frac{(k+r)!\,|n^{\frac{1}{2}}t|^{k-r} 2^{k-r}}{(k-r)!\, r!}$$

with $k = (n-1)/2$.

We will need a theorem proved by Schoenberg [5] (or see Feller [1] page 425).

THEOREM 4. *The function ψ is the Laplace transform of an infinitely divisible distribution if and only if $\psi = e^{-h}$ with h a function with a completely monotone derivative and with $h(0) = 0$.*

Define $\psi_n(t) = C_n((|t|/n)^{\frac{1}{2}})$ for $n = 3$, 5, and 7. For $t \geqq 0$

$$\psi_3(t) = \exp(-t^{\frac{1}{2}})(1+t^{\frac{1}{2}})$$

$$\psi_5(t) = \exp(-t^{\frac{1}{2}})\left(1+t^{\frac{1}{2}}+\frac{t}{3}\right)$$

$$\psi_7(t) = \exp(-t^{\frac{1}{2}})\left(1+t^{\frac{1}{2}}+\tfrac{2}{5}t+\frac{t^{\frac{3}{2}}}{15}\right).$$

Since $\psi_n = \exp(\log \psi_n)$, ψ_n is the Laplace of an id distribution if and only if $-\log \psi_n$ has a cm derivative. A knowledge of two properties of cm functions will be helpful at this point. If f and g are cm, so is fg. If f is cm and g is nonnegative with a cm derivative, then $f(g)$ is cm.

Let $n = 3$. Then

$$-\frac{\psi_3'(t)}{\psi_3(t)} = \frac{\frac{1}{2}\exp(-t^{\frac{1}{2}})}{\exp(-t^{\frac{1}{2}})(1+t^{\frac{1}{2}})} = \tfrac{1}{2}\cdot\frac{1}{1+t^{\frac{1}{2}}}.$$

Since $1/(1+t^{\frac{1}{2}})$ is cm, by Theorem 4 we get that $\psi_3(t) = \int \exp(-tu)\, dG(u)$, with G id. Hence

$$C_3(t) = \psi_3(3t^2) = \int \exp(-t^2u/2)\, dG(u/6).$$

Since a normal mixture with an id mixing distribution is id, we conclude that $C_3(t)$ is an id characteristic function.

For $n = 5$ we have

$$-\frac{\psi_5'(t)}{\psi_5(t)} = \frac{\frac{1}{6}\exp(-t^{\frac{1}{2}})(1+t^{\frac{1}{2}})}{\exp(-t^{\frac{1}{2}})(1+t^{\frac{1}{2}}+t/3)} = \frac{1}{6}\cdot\frac{1}{1+t/(3(1+t^{\frac{1}{2}}))}.$$

The last expression is cm if $t/3(1+t^{\frac{1}{2}})$ has a cm derivative.

$$\frac{d}{dt}\frac{t}{3(1+t^{\frac{1}{2}})} = \frac{1}{6(1+t^{\frac{1}{2}})^2}+\frac{1}{6(1+t^{\frac{1}{2}})},$$

which is a cm expression. Accordingly,

$$\frac{1}{6}\cdot\frac{1}{1+t/3(1+t^{\frac{1}{2}})} \quad \text{is cm.}$$

Therefore, as with $n = 3$, we have that $C_5(t)$ is a mixture of normal characteristic functions with the mixing distribution id, so C_5 is id.

For $n = 7$,

$$\frac{\psi_7'(t)}{\psi_7(t)} = \frac{1}{10}\cdot\frac{1}{1+[t(1+t^{\frac{1}{2}})/15(1+t^{\frac{1}{2}}+t/3)]}.$$

The last expression is cm if $t(1+t^{\frac{1}{2}})/(1+t^{\frac{1}{2}}+t/3)$ has a cm derivative. The first derivative is positive and

$$\frac{d^2}{dt^2}\frac{t(1+t^{\frac{1}{2}})}{1+t^{\frac{1}{2}}+t/3} = -\frac{1}{12}\left[\frac{1}{1+[t/(3(1+t^{\frac{1}{2}}))]}\cdot\frac{1}{1+t^{\frac{1}{2}}+t/3}+\frac{5}{(1+t^{\frac{1}{2}}+t/3)^2}+\frac{2}{(1+t^{\frac{1}{2}}+t/3)^2}\cdot\frac{1}{1+[t/(3(1+t^{\frac{1}{2}}))]}\right].$$

We see that each of the summands is cm (1, $t^{\frac{1}{2}}$, t and $t/(1-t^{\frac{1}{2}})$ all have cm derivatives). Therefore $t(1+t^{\frac{1}{2}})/(1+t^{\frac{1}{2}}+t/3)$ has a cm derivative, and as with $n = 3, 5$, we conclude that C_7 is id.

Therefore the Student-t distribution is id for three, five, or seven degrees of freedom. Although the author suspects the result to be true for any odd n, the method used was too cumbersome for $n > 7$.

COROLLARY 1. *Let* $g_n(x) = \Gamma(n/2)^{-1}(n/2)^{(n/2)}x^{-(n+2/2)} \exp(-n/2x)$. *Then* g_n *is an infinitely divisible density for* $n = 3, 5,$ *or* 7.

PROOF.

$$(n\pi)^{-\frac{1}{2}}\Gamma(n/2)^{-1}\Gamma\left(\frac{n+1}{2}\right)\left(1+\frac{x^2}{n}\right)^{-(n+1/2)}$$
$$= (2\pi)^{-\frac{1}{2}} \int \exp(-x^2/(2u))(n/2)^{n/2}\Gamma(n/2)^{-1}u^{-(n+2/2)} \exp(-(n/2u))\, du.$$

The function g_n ($n = 3, 5, 7$) is the density of the id mixing distribution, the existence of which was shown in the proof of the theorem.

Goldie [3] has shown that any mixture of exponential densities of the form $\int u \exp(-xu)\, dG(u)$ is id; i.e., any cm density is id. Thus any mixture of normal distributions with the mixing distribution cm is id. Steutel [7] has shown that mixtures of Laplace distributions of the form $\int u/2 \exp(-|x|u)\, dG(u)$ are id. We can get this result by noting that

$$\int u/2 \exp(-|x|u)\, dG(u)$$
$$= \int (2\pi y)^{-\frac{1}{2}} \exp(-x^2/(2y)) \int \exp(-yu^2/2)(u^2/2)dG(u)\, dy.$$

This is a mixture of normal densities with the mixing distribution a mixture of exponential densities, and hence the original mixture is id.

The following theorem is equivalent to Theorem 1 of Tortrat [8].

THEOREM 5. *All scale parameter mixtures of Cauchy distributions are infinitely divisible.*

PROOF. We will show that the nth root of the characteristic function is convex on $(0, \infty)$ and hence is a characteristic function. $C(t) = \int \exp(-|t|u)\, dG(u)$. Clearly, $(d/dt)C(t)^{1/n} < 0$.

$$\frac{d^2}{dt^2} C(t)^{1/n} = n^{-1} \left(\int \exp(-tu)\, dG(u)\right)^{1/n-2} \Big[\int \exp(-tu)\, dG(u)$$
$$\cdot \int u^2 \exp(-tu)\, dG(u) - (1-1/n)\left(\int u \exp(-tu)\, dG(u)\right)^2\Big].$$

For t fixed consider the distribution $F(x) = \int_0^x \exp(-tu)\, dG(u)/\int_0^\infty \exp(-tu)\, dG(u)$. Since

$$E_F(X^2) \geqq (E_F(X))^2 > (1-1/n)(E_F(X))^2,$$

we conclude that

$$\frac{d^2}{dt^2}(C(t))^{1/n} > 0$$

and thus $(C(t))^{1/n}$ is convex on $(0, \infty)$ and therefore is a characteristic function.

Some of the mixtures of normal distributions present an interesting duality. Consider the functions $h(u) = k_0(1+u^2)^{-1}$, $g(u) = k_1 \exp(-|u|)$, $-\infty < u < \infty$. With $k_0 = \pi^{-1}$ and $k_1 = 1$, h is a density and g is the characteristic function of h. With $k_0 = 1$ and $k_1 = \frac{1}{2}$, h is the characteristic function of the density g. But the interesting fact is that both sets are id. Also, let $f_a(u) = k \exp(-|u|^a)$, $0 < a \leqq 1$. For $k = 1$, f_a is an id characteristic function. With k a normalizing constant, f_a is an id density, since it is cm. The next theorem gives a class of mixtures which have this property.

THEOREM 6. *Let* $h(t) = \int_0^\infty \exp(-t^2u)g(u)\,du$ *with* $h(0) = 1$ *and* $\int_{-\infty}^\infty h(t)\,dt = K < \infty$. *If* g *is completely monotone or if* $u^{-\frac{3}{2}}g(1/u)$ *is completely monotone, then* $h(t)$ *is an infinitely divisible characteristic function and* $K^{-1}h(x)$ *is an infinitely divisible density.*

PROOF. If g is cm, then $\int \exp(-t^2u)g(u)\,du$ is an id characteristic function, as was noted previously. As a density

$$K^{-1}\int \exp(-x^2u)g(u)\,du = K^{-1}\int \exp(-x^2u)\int \exp(-uy)\,dP(y)\,du$$
$$= K^{-1}\int (y+x^2)^{-1}\,dP(y),$$

which is a scale parameter mixture of Cauchy densities and therefore id.

If $u^{-\frac{3}{2}}g(1/u)$ is cm, then $K^{-1}\int \exp(-x^2u)g(u)\,du = K^{-1}\int y^{-\frac{1}{2}}\exp(-x^2/y)\cdot y^{-\frac{3}{2}}g(y)\,dy$, which is an id density. As a characteristic function

$$\int \exp(-t^2u)g(u)\,du = \int y^{-\frac{1}{2}}\exp(-t^2/y)\int \exp(-yu)\,dP(u)\,dy$$
$$= \int (\pi/u)^{\frac{1}{2}}\exp(-2|t|u^{\frac{1}{2}})\,dP(u)$$
$$= \int \pi^{\frac{1}{2}}\exp(-|t|v)2v^{-1}\,dP(v^2/4).$$

This is a mixture of Cauchy characteristic functions and hence is id.

But if the id density f is a mixture of normal distributions, it is not true in general that $f(t)/f(0)$ is an id characteristic function. To see this, let $g(x) = \exp(1-x)$ for $x \geqq 1$. The Laplace transform of g is $(1+t)^{-1}e^{-t}$, and we see that the nth root of this will be cm, so that g is an id density. Therefore $f(x) \equiv \int (2\pi u)^{-\frac{1}{2}}\exp(-x^2/(2u))g(u)\,du$ is an id density. As a characteristic function, $f(t)/f(0) = f(0)^{-1}\int (2\pi)^{-\frac{1}{2}}\exp(-t^2y/2)y^{-\frac{3}{2}}g(1/y)\,dy$. The support of $y^{-\frac{3}{2}}g(1/y)$ is (0, 1), and therefore from Theorem 2 we conclude that $f(t)/f(0)$ is not an id characteristic function.

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