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SEMI STABLE LAWS AS LIMIT DISTRIBUTIONS

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Introduction. Characteristic functions which satisfy the equation

$$f(t) = \{f(bt)\}^a \tag{1}$$

where $0 < |b| < a$ are studied in this paper. Here it can be assumed that $a > 1$, since if $a \leqq 1$, f belongs to a degenerate law. These are the so-called semi-stable laws of P. Lévy (1937). The semi-stable laws with finite expectation constitute the class of all solutions of the regression equation

$$E(X_1 - \alpha X_2 / X_1 + \beta X_2) = 0 \qquad \text{with } \alpha\beta > 0 \text{ and} \tag{2}$$

$|\beta| < 1$ and X_1 and X_2 two independent and identically distributed random variables with finite expectation.

THEOREM 1. *Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables with common distribution function F and common characteristic function f satisfying* (1).

Let
$$\xi_n = \frac{X_1 + X_2 + \cdots + X_n}{B_n}.$$

Then $F_{\xi_{k_n}} \to_W F$, where $k_n = [a^n]$ and $B_{k_n} = b^{-n}$ and $[x]$ = the largest integer $\leqq x$. Conversely, if for any subsequence $\{k_n\}$ $F_{\xi_{k_n}} \to_W F$, then the law F is semi-stable and its characteristic function satisfies (1).

REMARK. Every semi-stable law can appear as a limit distribution of the sums ξ_{k_n} and conversely, every such sub-sequence of sums of independent and identically distributed random variables whenever it converges weakly is a semi-stable law.

PROOF. By iteration it follows from (1) that

$$f(t) = \{f(b^n t)\}^{a^n}.$$

Let $f_n(t) = \{f(b^n t)\}^{[a^n]}$. Evidently $f(t) = f_n(t)\{f(b^n t)\}^{\theta_n}$ where $\theta_n = a^n - [a^n]$ and $0 \leqq \theta_n < 1$. Therefore,

$$|f(t) - f_n(t)| = |\{f(b^n t)\}^{[a^n]}| \cdot |1 - f^{\theta_n}(b^n t)| \leqq |1 - f^{\theta_n}(b^n t)| \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore,

$$\mathscr{L}(b^n(X_1 + X_2 + \cdots + X_{[a^n]})) \to_W \mathscr{L}(X).$$

Conversely, suppose that $F_{\xi_{k_n}} \to F$ where $k_n' = [a^n]$.

Then $\xi_{k_{n+1}} = b^{n+1}\{X_1 + \cdots + X_{[a^{n+1}]}\}$ and

$$\xi_{k_{n+1}} = U_n + V_n$$

where $U_n = b^{n+1}(X_1 + \cdots + X_{[a^n]})$ and $V_n = b^{n+1}(X_{[a^n]+1} + \cdots + X_{[a^{n+1}]})$.

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Let $f_n(t) = \{f(b^n y)\}^{[a^n]}$. Then we have $f_n(t) \to f(t)$; since $F_{\xi_{k_n}}$ converges weakly to F.

Write $\theta_n + [a^n] = a^n$. Then $[a^{n+1}] - [a^n] = a^n(a-1) + \theta_n - \theta_{n+1}$. The characteristic function of V_n is given by

$$\{f(b^{n+1}t)\}^{[a^{n+1}]-[a^n]} = \{f^{a^n(a-1)}(b^{n+1}t)\} \frac{f^{\theta_n}(b^{n+1}t)}{f^{\theta_{n+1}}(b^{n+1}t)}.$$

It is now clear that the characteristic function of V_n tends to $\{f(bt)\}^{a-1}$ as $n \to \infty$, since the characteristic function of U_n tends to $f(bt)$ as $n \to \infty$ and the characteristic function of V_n tends to $\{f(bt)\}^{a-1} = \{f(bt)\}^a$. Hence $f(t)$ is semi-stable.

NOTE. It is interesting to note that in the statement of the converse of Theorem 1, it is enough to have a sequence k_n such that

$$\frac{k_{n+1}}{k_n} \to a \quad \text{and} \quad \frac{B_{k_{n+1}}}{B_{k_n}} \to b^{-1}.$$

THEOREM 2. *The following assertion holds for a semi-stable law F represented by equation* (1). *Given $\varepsilon > 0$ there exists X_0 such that for all $x \geqq X_0$*

$$\frac{1}{ab^{-pr}}(1+\varepsilon)^{-1} \leqq \frac{1-F(x)+F(-x)}{1-F(b^r x)+F(-b^r x)} \leqq \frac{a}{b^{-rp}}(1+\varepsilon),$$

where r is a given but quite arbitrary positive integer and p is a unique solution of the equation $a|b|^p = 1$.

PROOF. According to Gnedenko and Kolmogoroff (1954, Theorem 4 of Section 25), if for suitable constants A_n and B_n, the distribution functions of the sums

$$\xi_n = \xi_{n_1} + \xi_{n_2} + \cdots + \xi_{nk_n} - A_n$$

of independent infinitesimal random variables converge to the distribution function F, then at continuity points of $M(u)$ and $N(u)$

$$\textstyle\sum_{k=1}^{k_n} F_{n_k}(x) \to M(x) \quad \text{for} \quad x < 0, \qquad \text{and}$$

$$\textstyle\sum_{k=1}^{k_n} (F_{n_k}(x) - 1) \to N(x) \quad \text{for} \quad x > 0$$

as $n \to \infty$ where $M(u)$ and $N(u)$ are determined by Lévy's formula for $F(x)$.

Accordingly,

$$a^n F(b^{-n}x) \to M(x) \quad \text{for} \quad x < 0, \tag{3}$$

$$a^n(1 - F(b^{-n})) \to -N(x) \quad \text{for} \quad x > 0 \tag{4}$$

at the continuity points of M and N respectively. For simplicity we assume that $b > 0$. The equation (4) in the characteristic function induces a similar equation in the functions $M(u)$ and $N(u)$ as follows:

$$M(u) = aM(u/b) \quad \text{and} \quad N(u) = aN(u/b). \tag{5}$$

On iterating (5) n times, we get

$$N(u) = N(b^n u)/a^n = N(b^n u)/b^{-np},$$

where p is the unique solution of $a = b^{-p}$. Let $k(u) = u^p|N(u)|$ and $h(u) = |u|^p M(u)$. Then we get $h(u) = h(b^n u)$ and $k(u) = k(b^n u)$ for all n. Therefore, in place of (3) and (4), we can write

$$a^n F(b^{-n}x) \to |x|^{-p} h(x) \quad \text{for} \quad x < 0, \tag{6}$$

$$a^n(1 - F(b^{-n}x)) \to x^{-p} k(x) \quad \text{for} \quad x > 0. \tag{7}$$

Let $y > 0$ be large. Choose n so large that for given $x > 0$, $b^{-n}x \leqq y \leqq b^{-(n+1)}x$. We now have

$$F(-b^{-(n+1)}x) \leqq F(-y) \leqq F(-b^{-n}x) \qquad \text{and}$$

$$1 - F(b^{-(n+1)}x) \leqq 1 - F(y) \leqq 1 - F(b^{-n}x).$$

For $s > 0$,

$$F(-sb^{-(n+1)}x) \leqq F(-sy) \leqq F(-b^{-n}sx),$$

$$1 - F(b^{-(n+1)}sx) \leqq 1 - F(sy) \leqq 1 - F(b^{-n}sx).$$

Therefore we have

$$\frac{1 - F(b^{-(n+1)}x) + F(-b^{-(n+1)}x)}{1 - F(b^{-n}sx) + F(-b^{-n}sx)}$$

$$\leqq \frac{1 - F(y) + F(-y)}{1 - F(sy) + F(-sy)}$$

$$\leqq \frac{1 - F(b^{-n}x) + F(-b^{-n}x)}{1 - F(b^{-(n+1)}sx)F + (-b^{-(n+1)}sx)}.$$

Let $s = b^r$, where r is a positive integer. Letting $n \to \infty$ and using (6) and (7), we have

$$\frac{1}{a} b^{rp}(1+\varepsilon)^{-1} \leqq \frac{1 - F(x) + F(-x)}{1 - F(b^{+r}x) + F(-b^{-r}x)} \leqq ab^{rp}(1+\varepsilon)$$

for all $x \geqq X_0(\varepsilon)$. This completes the proof.

THEOREM 3. *For a semi-stable law with exponent p, $E(|x|^\delta)$ exists for* $0 \leqq \delta < p$.

REMARK. There is a different proof of this theorem in Ramachandran and Rao (1968) with the stronger result that $E|X|^\delta$ does not exist for $\delta \geqq p$.

PROOF. By Theorem 2, for $\varepsilon > 0$ and $k = b^{-r}$, r a positive integer,

$$\frac{k^p}{a}(1+\varepsilon)^{-1} < \frac{P\{|\xi| > k^s X_0\}}{P\{|\xi| > k^{s+1} X_0\}} < (1+\varepsilon)ak^p \tag{8}$$

for $s = 1, 2, \cdots$. We suppose that ε is so small that $((1+\varepsilon)a/k^{p-\delta}) < 1$.

We have

$$\int |X|^{\delta}\, dF(x) \leqq \int_{-X_0}^{X_0} |X|^{\delta}\, dF(x) + \sum_{s=1}^{\infty} \int |X|^{\delta}\, dF(x) k^{s-1} X_0 \leqq |X| < k^s X_0$$

$$\leqq \int_{-X_0}^{X_0} |X|^{\delta}\, dF(x) + X_0{}^{\delta} \sum_{s=1}^{\infty} k^{s^{\delta}} P\{|\xi| > k^{s-1} S_0\}. \tag{9}$$

Let U_n be the nth term of the infinite series. Then

$$\frac{U_{n+1}}{U_n} = k^{\delta} \frac{P\{|\xi| > k^n X_0\}}{P\{|\xi| > k^{n-1} X_0\}} \quad \text{and}$$

$$\frac{U_{n+1}}{U_n} \leqq k^{\delta} a k^{-p}(1+\varepsilon) \quad \text{by} \quad (8)$$

$$= \frac{a(1+\varepsilon)}{k^{p-\delta}} < 1.$$

Therefore, the series on the right-hand side of (9) is convergent. Hence $E|X|^{\delta} < \infty$. This completes the proof.

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