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A NOTE ON CONVERGENCE OF MOMENTS

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In this note, some results in [4] on convergence of even-integer moments in a central limit situation are extended in Theorem B, Section 2 to cover the case of non-even-integer absolute moments. In Section 1 we note that related results of the author ([2], [3]) were first proved by S. N. Bernstein [1], a long time ago.

1. Results of Bernstein. Let $X_1, X_2, \cdots$ be independent random variables (rv's) with $EX_n = 0, EX_n^2 = \sigma_n^2 < \infty, S_n = X_1 + \cdots + X_n$, and $s_n^2 = ES_n^2$, for $n = 1, 2, \cdots$. Among the results in [2] and [3] is the following

THEOREM A. *For each $v > 2$, the condition*

$$\lim_{n\to\infty} s_n^{-v} \sum_{j=1}^n E|X_j|^v = 0 \tag{1}$$

is necessary and sufficient for

$$\mathscr{L}(S_n/s_n) \to N(0, 1) \quad \textit{as} \quad n \to \infty,$$

$$\lim_{n\to\infty} s_n^{-2} \max_{j\leq n} \sigma_j^2 = 0 \qquad \textit{and}$$

$$\lim_{n\to\infty} E|S_n/s_n|^v = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} |x|^v \exp(-\tfrac{1}{2}x^2)\, dx.$$

Dr. G. K. Eagleson has pointed out that this theorem is by no means new; in fact it was proved by S. N. Bernstein [1] fully thirty years ago, using symmetrization methods which do not rely on characteristic functions (ch.f's) as in [3]. The condition (1), called a *Lindeberg* condition of order v in [2] and [3], was given the possibly more apt name of "convergence to zero of the *Liapounov* parameter of order v," by Bernstein in [1]. The question of nomenclature is avoided in the present work by referring to such conditions simply as "L_v".

2. Convergence of moments. Following the notation of [4], let $X_{n1}, X_{n2}, \cdots, X_{nj_n}$, $n = 1, 2, \cdots$ be an elementary system of zero mean independent rv's, i.e., each X_{nj} has mean zero, distribution function $F_{nj}(\cdot)$, and variance DX_{nj}, with

$$\lim_{n\to\infty} \max_j DX_{nj} = 0, \qquad \text{and}$$

$$\sum_j DX_{nj} \leq \text{ some } C < \infty \qquad \text{for all } n = 1, 2, \cdots. \tag{2}$$

Let $S_n = \sum_j X_{nj}$. We assume throughout that S_n converges in law as $n \to \infty$ to an infinitely divisible rv. T_0 with ch.f. $\phi_0(\cdot)$ given by

$$\log \phi_0(t) = \int_{-\infty}^{\infty} (e^{itx} - 1 - itx) x^{-2}\, dG_0(x),$$

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where $G_0(\cdot)$ is a non-decreasing function of bounded variation. A necessary and sufficient condition for the convergence in law as $n \to \infty$ of S_n to T_0 is

(3) $G_n(\cdot)$ converges weakly as $n \to \infty$ to $G_0(\cdot)$ where $dG_n(x) = \sum_j x^2 dF_{nj}(x)$. We note that T_0 has rth cumulant $K_r(T_0)$, where $K_1(T_0) = 0$ and for $r = 2, 3, \cdots$

$$K_r(T_0) = \int_{-\infty}^{\infty} x^{r-2}\, dG_0(x).$$

LEMMA 1. (Theorem 3 of [4]). *Let* (3) *hold and let* $ET_0^{2k} < \infty$ *for some* $k = 1, 2, \cdots$. *Then the condition*

$$\lim_{n\to\infty} \sum_j EX_{nj}^{2k} = K_{2k}(T_0),$$

i.e.,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} x^{2k-2}\, dG_n(x) = \int_{-\infty}^{\infty} x^{2k-2}\, dG_0(x),$$

is necessary and sufficient for $\lim_{n\to\infty} ES_n^{2k} = ET_0^{2k}$.

Following [4] we can define the νth *absolute cumulant* of T_0, for $\nu \geqq 2$, as

$$B_\nu(T_0) = \int_{-\infty}^{\infty} |x|^{\nu-2}\, dG_0(x).$$

The condition L_ν is said to hold if in addition to (3),

$$\lim_{n\to\infty} \sum_j E\,|X_{nj}|^\nu = B_\nu(T_0) < \infty$$

or equivalently

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |x|^{\nu-2}\, dG_n(x) = \int_{-\infty}^{\infty} |x|^{\nu-2}\, dG_0(x). \tag{4}$$

THEOREM B. *Let* (3) *hold and let* $E|T_0|^\nu < \infty$ *for some* $\nu \geqq 2$. *Then the condition* L_ν *is necessary and sufficient for*

$$\lim_{n\to\infty} E|S_n|^\nu = E|T_0|^\nu.$$

The details of the proof will follow those of the proof of Theorem 5 of [3] provided the following modifications are made: replace Theorem 1.1 of [2] with our Lemma 1, replace the Lindeberg condition of order 2 (i.e. the CLT) by our equation (3), and replace the sequence $X_1 s_n^{-1}, \cdots, X_n s_n^{-1}$ with $X_{n1}, \cdots, X_{nj_n}$. To finally ensure that the proof can be followed exactly, it remains to check two items, namely

(i) $L_\nu \Rightarrow L_\alpha$ for $\nu \geqq \alpha \geqq 2$. This follows from (4), which holds together with (3).

(ii) As an analogue of Lemma 2 of [3], establish the following

LEMMA 2. *Let* $\nu > 2$ *be not an even integer, with*

$$V(n, \varepsilon) = \sum_j E|X_{nj}|^\nu A(\varepsilon|X_{nj}|),$$

where

$$A(x) = (-1)^\lambda \int_0^x \mathscr{R}l f_{m-2}(u) u^{-(\nu-1)}\, du,$$

$$f_n(u) = e^{iu} - \sum_{j=0}^{n} (iu)^j / j!,$$

and where m, λ are the greatest integers which are (strictly) less than v and $\frac{1}{2}v$, respectively. Let (3) *hold. Then L_v holds if and only if*

(5) $V(n, \varepsilon) \to 0$ *as* $\varepsilon \to 0$ *uniformly in* $n = 1, 2, \cdots$.

PROOF.

$$V(n, \varepsilon) = \sum_j \int_{-\infty}^{\infty} |x|^v A(\varepsilon|x|)\, dF_{nj}(x),$$

$$(6) \qquad = \int_{-\infty}^{\infty} |x|^{v-2} A(\varepsilon|x|)\, dG_n(x),$$

$$= (-1)^\lambda \int_0^\varepsilon t^{-(v-1)}\, dt \int_{-\infty}^{\infty} \mathscr{R}lf_{m-2}(tx)\, dG_n(x),$$

by interchanging the order of integration, a justifiable step since the integrand is of constant sign. Therefore the condition (5) is necessary and sufficient for the convergence to zero as $\varepsilon \to 0$ of the right-hand side of (6), uniformly in $n = 1, 2, \cdots$. But this latter condition is necessary and sufficient for

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |x|^{v-2}\, dG_n(x) = \int_{-\infty}^{\infty} |x|^{v-2}\, dG_0(x)$$

(which is L_v) by noting that (3) holds and applying Theorem 4 of [3], which can be used here because of (2), which implies that

$$\int_{-\infty}^{\infty} dG_n(x) \leqq C < \infty, \qquad \text{all} \quad n = 1, 2, \cdots.$$

The lemma, and hence the theorem, is proved.

REFERENCES

[1] BERNSTEIN, S. N. (1939). Several comments concerning the limit theorem of Liapounov. *Dokl. Akad. Nauk. SSSR.* **24** 3–7.

[2] BROWN, BRUCE M. (1969). Moments of a stopping rule related to the central limit theorem. *Ann. Math. Statist.* **40** 1236–1251.

[3] BROWN, B. M. (1970). Characteristic functions, moments and the central limit theorem. *Ann. Math. Statist.* **41** 658–664.

[4] BROWN, B. M. and EAGLESON, G. K. (1970). Behaviour of moments of row sums of elementary systems. *Ann. Math. Statist.* **41** 1853–1860.