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DISCRIMINATION OF POISSON PROCESSES[1]

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0. Introduction. In [3] Gikhman and Skorokhod obtained necessary and sufficient conditions for absolute continuity of multidimensional independent increment processes. By the Lévy–Itô decomposition, an independent increment process $\{X(t),\ 0 \leqq t \leqq T\}$ can be decomposed into two independent components, an independent increment Gaussian process, and a process determined by the jumps of $\{X(t),\ 0 \leqq t \leqq T\}$, the jumps forming a Poisson process on $[0, T] \times E^n$. Thus in solving their problem, the authors obtained necessary and sufficient conditions for absolute continuity of Poisson processes on $[0, T] \times E^n$, for which the expected number of jumps of norm $> \varepsilon$ is finite for all $\varepsilon > 0$.

In this paper we consider the problem of absolute continuity of Poisson processes with σ-finite mean measures over general measure spaces. Then Gikhman–Skorokhod conditions for absolute continuity generalize to our case, but the proof of sufficiency does not, and a different proof is presented. We also obtain conditions for singularity of Poisson processes and show that two Poisson processes with mutually absolutely continuous mean measures are either mutually absolutely continuous or singular.

1. Define a Poisson $(\mathscr{X}, \mathscr{C}, \mu)$ process where $(\mathscr{X}, \mathscr{C})$ is a measurable space and μ a measure over $(\mathscr{X}, \mathscr{C})$, to be a random nonnegative integer valued (including $+\infty$) set function N on $(\mathscr{X}, \mathscr{C})$ having the property that for any k and corresponding nonnegative integers $r_1, \cdots, r_k$ and nonoverlapping $\mathscr{C}$ sets $C_1, \cdots, C_k$:

$$\Pr(N(C_j) = r_j, \quad j = 1, \cdots, k) = \textstyle\prod_1^k p(\mu(C_j), r_j) \tag{1}$$

where

$$p(\lambda, \alpha) = \frac{\lambda^\alpha e^{-\lambda}}{\alpha!}, \qquad \lambda < \infty, \qquad \alpha < \infty$$
$$= 1 \qquad \lambda = \infty, \alpha = \infty \text{ or } \lambda = 0, \alpha = 0$$
$$= 0 \qquad \text{elsewhere.}$$

It is easy to prove the existence of a Poisson $(\mathscr{X}, \mathscr{C}, \mu)$ process for μ σ-finite (see [1] page 1939).

Each realization of a Poisson $(\mathscr{X}, \mathscr{C}, \mu)$ process with μ σ-finite is of the form $N(C, \omega) = \sum_{t_i' \subset C} N(t_i'(\omega), \omega)$ where $\{t_i', i = 1, 2, \cdots\}$ is a random countable, collection of chunks of $\mathscr{X}$ (a chunk of $\mathscr{X}$ is a set $C \in \mathscr{C}$ such that $C' \subset C$

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$C' \in \mathscr{C} \Rightarrow C' = C$ or $C' = \phi$), and $N(t_i'(\omega), \omega)$ is a positive finite integer. Form a random collection of chunks $\{t_i, i = 1, \cdots, N(\mathscr{X})\}$ consisting of the chunks t_i', each such chunk repeated $N(t_i')$ times.

We can regard a Poisson $(\mathscr{X}, \mathscr{C}, \mu)$ process as a probability measure over $(A, \mathscr{A})$ where A is the set of all countable subsets of chunks of $\mathscr{X}$ (multiple occurrences of chunks are permitted) and $\mathscr{A}$ is the minimal σ-algebra containing all sets of the form $\{a: N(C, a) = k\}$ $k = 0, 1, \cdots, \infty$, $C \in \mathscr{C}$ where $N(C, a)$ for $C \in \mathscr{C}$, $a \in A$, is defined as the number of chunks of $\mathscr{X}$ which belong to a (counting multiple occurrences of individual chunks) and which are also contained in C.

LEMMA 1. *If P_μ and P_ν are Poisson processes over $(\mathscr{X}, \mathscr{C})$, both $\mu(\mathscr{X})$ and $\nu(\mathscr{X})$ are finite, and $\mu \ll \nu$ with $f = d\mu/d\nu$, then $P_\mu \ll P\nu$ with*

$$\frac{dP_\mu}{dP_\nu} = \exp[-(\mu(\mathscr{X}) - \nu(\mathscr{X}))] \prod_{i=1}^{N(\mathscr{X})} f(t_i).$$

PROOF. $E_{P_\nu}(\pi f(t_i) \cdot I_{N(A)=k} \mid N(\mathscr{X}) = n)$

$$= \left[\frac{1}{\nu(A)} \int_A f(x)\nu(dx)\right]^k \left[\frac{1}{\nu(\tilde{A})} \int_{\tilde{A}} f(x)\nu(dx)\right]^{n-k} \left[\frac{\nu(A)}{\nu(\mathscr{X})}\right]^k \left[\frac{\nu(\tilde{A})}{\nu(\mathscr{X})}\right]^{n-k} \binom{n}{k}$$

$$= \binom{n}{k} \left[\frac{\mu(A)}{\nu(\mathscr{X})}\right]^k \left[\frac{\mu(\tilde{A})}{\nu(\mathscr{X})}\right]^{n-k} \qquad \text{for all} \quad n \geqq k,\ A \in \mathscr{C}.$$

Thus:

$$\int_{N(A)=k} \exp[-(\mu-\nu)\mathscr{X}] \pi f(t_i)\, dP_\nu = EP_\nu[I_{N(A)=k} \exp[-(\mu-\nu)\mathscr{X}] \pi f(t_i)]$$

$$= E_{P_\nu}[E_{P_\nu}(I_{N(A)=k} \exp[-(\mu-x)\mathscr{X}] \pi f(t_i) \mid N(\mathscr{X})]$$

$$= [\mu(A)]^k \frac{\exp[-\mu(\mathscr{X})]}{k!} \sum_{n=k}^{\infty} \frac{[\mu(\tilde{A})]^{n-k}}{(n-k)!} = \frac{[\mu(A)]^k \exp[-\mu(A)]}{k!}$$

$$= P_\mu(N(A) = k).$$

LEMMA 2. *Let P and Q be probability measures over $(A, \mathscr{A})$ and let $\mathscr{A}_n$ be an increasing sequence of σ-subalgebras of $\mathscr{A}$ with $\cup_n \mathscr{A}_n = \mathscr{A}$. Let P_n and Q_n be the restrictions of P and Q to $\mathscr{A}_n$ and suppose that $P_n \ll Q_n$ for all n. Let $g_n = dP_n/dQ_n$. Then under Q, g_n converges* a.s. *$[Q]$ to an $\mathscr{A}$ measurable function g. Moreover $P \ll Q$* iff *$E_Q g = 1$ in which case $g = dP/dQ$.*

PROOF. Neveu [4] page 144.

Define

$$A_c = \{x: 0 < |f(x) - 1| \leqq c\}$$

$$B_c = \{x: |f(x) - 1| > c\}.$$

THEOREM 1. *Let P_μ and P_ν be Poisson processes over $(\mathscr{X}, \mathscr{C})$ with σ-finite mean measures μ and ν. Then $P_\mu \ll P_\nu$* iff *the following conditions hold*:

(i) $\mu \ll \nu$.

(ii) $\mu(B_c) < \infty$, $\nu(B_c) < \infty$ *for all* $c > 0$,

or equivalently $\int_{B_c} |f-1|\, d\nu < \infty$ *for all* $c > 0$.

(iii) *For some* $c > 0$, $\int_{\tilde{B}_c} (f-1)^2\, d\nu < \infty$.

PROOF. The proof of necessity follows easily from the proof of Theorem 7.3 in [3].

Suppose the above conditions hold. Choose $c \in (0, 1)$ such that $\int_{\tilde{B}_c} (f-1)^2\, d\nu < \infty$. Define $C_m = B_{c/m+1} \cap \tilde{B}_{c/m}$, $m = 1, 2, \cdots$, and $D_n = \bigcup_1^n C_m$, $n = 1, 2, \cdots$. The sets $\{C_m\}$ are disjoint and $\bigcup_1^\infty C_m = A_c$. Define $\mathscr{A}_n = \mathscr{C} \cap D_n$, $n = 1, \cdots, \infty$, where $A_\infty = \mathscr{C} \cap \mathscr{A}_c$. Define $P_{\mu,n}(P_{v,n})$ to be the restriction of $P_\mu(P_v)$ to $\mathscr{A}_n$, $n = 1, \cdots, \infty$. It follows from conditions (i) and (ii) and Lemma 1, that $P_\mu \ll P_v$ iff $P_{\mu,\infty} \ll P_{v,\infty}$. By condition (i) and Lemma 1:

$$g_n = \frac{dP_{\mu,n}}{dP_{v,n}} = \prod_{m=1}^{n} Y_m, \qquad n = 1, 2, \cdots \tag{1}$$

where $Y_m = \exp[-(\mu(C_m) - v(C_m))] \prod_{t_i \in C_m} f(t_i)$. Now $\log g_n = \sum_{m=1}^n \log Y_m$ is a sum of propor ($f(x) > 0$ on D_n) independent random variables, and thus $\log g_n$ converges a.s. iff it converges in distribution, [5] page 251. Let $\beta_n(t) = \log E_{p_v}[\exp(it \log g_n)]$. Then:

$$\beta_n(t) = \int_{D_n} (\exp(it \log f(x)] - 1 - it(f(x)-1))v(dx). \tag{2}$$

Since $\exp[it \log f(x)] = 1 + it(f(x)-1) - \frac{1}{2}(t^2 + it)(f(x)-1)^2 + o(f(x)-1)^2$ as $f(x) \to 1$, it follows from condition (iii) that $\beta_n(t)$ converges pointwise. The dominated convergence theorem and condition (iii) show that the limit of β_n is continuous at 0. Thus $\log g_n$ converges in distribution and thus a.s. $[P_v]$ to an $\mathscr{A}_\infty$ measurable function which we label $\log g$. By the dominated convergence theorem and condition (iii), the moment generating function of $\log g$ exists and

$$\log E_{p_v}(\exp(t \log g)) = \int_{A_c} (f^t - 1 - t(f-1))\, dv,$$

so that $E_{p_v}(g) = E_{p_v}[\exp(t \log g)]\,|_{t=1} = 1$. By Lemma 2, $P_{\mu,\infty} \ll P_{v,\infty}$, and thus by our previous remark $P_\mu \ll P_v$.

THEOREM 2. *Let P_μ and P_v be Poisson processes over $(\mathscr{X}, \mathscr{C})$ with σ-finite mean measures μ and v. Let $\mu = \lambda + w$ be the Lebesgue decomposition of μ with respect to $v(\lambda \ll v,\ w \perp v)$. Then $P_\mu \perp P_v$* iff *one of the following three conditions hold*:

(i) $w(\mathscr{X}) = \infty$.

(ii) $\int_{B_c} |f-1|\, dv = \infty$ *for some* $c > 0$.

(iii) $\int_{\tilde{B}_c} (f-1)^2\, dv = \infty$ *for all* $c > 0$.

PROOF. The sufficiency of condition (i) is obvious. The sufficiency of (ii) and (iii) follow from the proof of necessity of Theorem 7.3 in [3].

Suppose that none of the above conditions hold. Note that λ and w are concentrated on disjoint subsets C_λ and C_w of $\mathscr{X}$ and $v(C_w) = 0$. Since $w(\mathscr{X}) < \infty$, $P_\mu(N(C_w) = 0) > 0$. Conditional on $N(C_w) = 0$, N has a Poisson $(\mathscr{X}, \mathscr{C}, v)$ distribution under v and a Poisson $(\mathscr{X}, \mathscr{C}, \lambda)$ distribution under μ. By Theorem 1, a Poisson $(\mathscr{X}, \mathscr{C}, v)$ process dominates a Poisson $(\mathscr{X}, \mathscr{C}, \lambda)$ process. Therefore P_μ and P_v are not singular.

Corollary 1 below is an immediate consequence of Theorem 1 and Theorem 2.

COROLLARY 1. *Let P_μ and P_v be Poisson processes over $(\mathscr{X}, \mathscr{C})$ with σ-finite mean measures μ and v, with $\mu \equiv v$. Then either $P_\mu \equiv P_v$ or $P_\mu \perp P_v \cdot P_\mu \equiv P_v$ iff there exists $c > 0 \ni \int_{B_c} |f-1| \, dv + \int_{\bar{B}_c} (f-1)^2 \, dv < \infty$. $P_\mu \perp P_v$ iff for all $c > 0$ $\int_{B_c} |f-1| \, dv + \int_{\bar{B}_c} (f-1)^2 \, dv = \infty$.*

2. Example. Let $\mathscr{X} = \{1, 2, 3, \cdots\}$ and $v(m) = m$, $\mu(m) = m+1$ for all m. Let P_μ and P_v be Poisson μ and v processes. Then $\mu \ll v$ with $(d\mu/dv)](m) = f(m) = 1+1/m$. Now $\mu(B_c) < \infty$, $v(B_c) < \infty$ for all $c > 0$ so that Condition (ii) of Theorem 1 holds; but:

$$\int_{\bar{B}_c} (f(x)-1)^2 \, v(dx) = \sum_{m > 1/c} 1/m = \infty$$

so that condition (iii) does not hold and thus $P_\mu \perp P_v$.

This example has been included because the result was surprising to the author. His intuition was that P_μ and P_v were equivalent.

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