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STRONG LAWS FOR RULED SUMS

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0. Introduction. Let $\{X_k : k \geqq 1\}$ be a sequence of independent, identically distributed random variables where $E(X_1) = 0$ and $E(X_1{}^2) = 1$ when these quantities exist. By a rule $(\ )$, we mean a function: $I^+ \to$ set of all subsets of I^+ where (n) is some collection of n distinct positive integers for each n, and its ruled sum is defined as $S_{(n)} \equiv \sum_{k\in(n)} X_k$. We let $\mathbb{R}$ denote the set of all rules. If

$(n) = \{1, 2, \cdots, n\}$ then we denote $S_{(n)}$ by S_n, and if

$(n) \cap (m) = \varnothing$ for $n \neq m$, then we denote $(\ )$ by $\langle\ \rangle$.

Clearly the weak laws for all ruled sums are the same as those for S_n; however the strong laws can be quite different. This note will attempt to study how some of the strong properties for $S_{(n)}$ compare to those for S_n. For instance, we will show that for symmetric X_k, and most well-behaved sequences $\{a_n\}$

$$\limsup (S_n/a_n) \leqq \limsup (S_{(n)}/a_n) \leqq \limsup (S_{\langle n\rangle}/a_n) \tag{0.1}$$

for all $(\ ) \in \mathbb{R}$. Laws of the iterated logarithm type, strong convergence laws and convergence rates will also be compared.

1. In all that follows we assume that $\liminf a_n = +\infty$. It then follows by standard arguments that $\limsup S_{(n)}/a_n$ is a.e. a constant. The right-hand inequality of (0.1) follows immediately from the Borel-Cantelli Lemma and so henceforth (0.1) will refer to the left-hand inequality.

The bulk of this section is devoted to showing that (0.1) is true for symmetric random variables. At present it is not clear to us how to drop the assumption of symmetry and prove (0.1) in general. However, in case $EX_1{}^2 < \infty$, we can drop the symmetry assumption and prove (0.1) for the interesting sequence $a_n = (2n \lg \lg n)^{\frac{1}{2}}$ corresponding to the law of the iterated logarithm. The method of proof in this first result will serve as a model for the method to be used in proving (0.1) for symmetric random variables.

THEOREM 1. *Let* $E(X_1{}^2) < \infty$ *and let* $\varepsilon > 0$, *then for all* $(\ ) \in \mathbb{R}$, $P[S_{(n)} > (1-\varepsilon)(2n \lg \lg n)^{\frac{1}{2}}$ i.o.$] = 1$.

PROOF. Choose N so large that $(1-\varepsilon/3)/(1-\alpha)^{\frac{1}{2}} < 1$ and $\varepsilon/(3(\alpha)^{\frac{1}{2}}) > 1+\varepsilon$ where

$$\lim_k \left(\textstyle\sum_{j=1}^k N^j/N^{k+1}\right) \nearrow \alpha.$$

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Let $(\;) \in \mathbb{R}$. By the definition of α we see that $S_{(N^k)}$ is the sum of N^k random variables, where at least $[(1-\alpha)N^k]$($[\;\;] \equiv$ largest integer function) of these variables did not appear in $S_{(N)}, S_{(N^2)}, \cdots, S_{(N^{k-1})}$. Denote the sum of $[(1-\alpha)N^k]$ of the variable that did not previously appear by $S_k^{(1)}$ and the sum of the rest of the variables $S_k^{(2)}$. Thus $S_{(N^k)} = S_k^{(1)} + S_k^{(2)}$ where $\{S_k^{(1)}\}$ is a sequence of independent random variables.

Now, letting σ_k^2 denote the variance of X_1 truncated at $N^{k/2}$, Theorem 4 of [3] implies that

$$\sum_{k=1}^{\infty} P[S_k^{(1)} > \sigma_k(1-\varepsilon/3)(2N^k \lg\lg N^k)^{\frac{1}{2}}] = \infty$$

and

$$\sum_{k=1}^{\infty} P[S_k^{(2)} < -\varepsilon/3\sigma_k(2N^k \lg\lg N^k)^{\frac{1}{2}}] < \infty.$$

Thus by the Borel–Cantelli lemma

$$P[S_k^{(1)} > \sigma_k(1-\varepsilon/3)(2N^k \lg\lg N^k)^{\frac{1}{2}} \text{ i.o.}] = 1$$

and

$$P[S_k^{(2)} < -\varepsilon/3\sigma_k(2N^k \lg\lg N^k)^{\frac{1}{2}} \text{ i.o.}] = 0.$$

Thus $P[S_{(N^k)} = S_k^{(1)} + S_k^{(2)} > (1-2\varepsilon/3)\sigma_k(2N^k \lg\lg N^k)^{\frac{1}{2}} \text{ i.o.}] = 1$. But $\sigma_k \nearrow 1$, and the theorem follows.

For the rest of this section the X_k's are assumed symmetric, and (0.1) is proved under this assumption and the assumption that the a_n are non-decreasing.

LEMMA 1. *Let $\{A_k\}$ and $\{B_k\}$ be two sequences of events, where $\sum P[A_k] = \infty$ and $P[B_k] \geqq \alpha$ for all k. Further assume A_n is independent of $\{A_i\}_{j=1}^{n-1}$ and $\{B_j\}_{j=1}^{n}$. Then $P[A_k B_k \text{ i.o.}] \geqq \alpha$.*

PROOF. If the conclusion is false, then $\exists\ \varepsilon > 0$, and m so that $k \geqq m+1$ implies $P[\bigcup_m^{k-1} A_j B_j] < \alpha - \delta$.

But

$$P[\bigcup_m^{\infty} A_k B_k] \geqq \sum_{k=m}^{\infty} P[A_k](\alpha - P[B_k \cap (\bigcup_m^{k-1} A_j B_j)])$$

by our assumption of independence and thus a contradiction results from the fact $\sum P[A_k] = \infty$.

LEMMA 2. *If for some $c > 1$, $\sum_{n=1}^{\infty} P[S_{c^n} > a_{c^{n+1}}] = \infty$, then*

$$P[\limsup (S_{(c^n)}/a_{c^n}) \geqq 1] = 1 \quad \textit{for all} \quad (\;) \in \mathbb{R}$$

(c^n as a subscript means $[c^n]$.)

PROOF. Since $c > 1$, there exists m so that $c-1 > c/(c^m-1)$. By hypothesis, there is a $j \in \{0, 1, 2, \cdots, m-1\}$ so that $\sum_{n=1}^{\infty} P[S_{c^{nm+j}} > a_{c^{nm+j+1}}] = \infty$, and for convenience we assume this $j = 1$.

By the definition of m, $\sum_{l=1}^{k-1} c^{ml+1} \leqq (c-1)c^{mk}$, so we see $S_{(c^{km+1})}$ contains at least $[c^{km}]$ random variables that are not in $S_{(c^{lm+1})}$, $l = 1, 2, \cdots, k-1$. Let the sum of exactly $[c^{km}]$ of these random variables be denoted by $S_{\langle c^{km}\rangle}$. Let $A_k \equiv [S_{\langle c^{km}\rangle} > a_{c^{km+1}}]$ and $B_k \equiv [S_{(c^{km+1})} - S_{\langle c^{km}\rangle} \geqq 0]$. Then since the X_k's are symmetric, Lemma 1 shows $P[\limsup (S_{(c^n)}/a_{c^n} \geqq 1] \geqq \frac{1}{2}$. But $\limsup S_{(n)}/a_n$ is an a.e. constant and so the lemma follows.

LEMMA 3. *Let* $P[S_n \geqq a_n \text{ i.o.}] = 1$, *let* $2^{\frac{1}{3}} > c > 1$ *and let* $\liminf a_{c^{n+1}}/a_{c^n} = L$, *and* $\limsup a_{c^{n+1}}/a_{c^n} = M$. Then $\limsup (S_{(c^n)}/a_{c^n}) \geqq \max(L/2, 1/M^2) \geqq \frac{1}{2}$ *for all* $(\ ) \in \mathbb{R}$.

PROOF.

Claim (i) $\sum_{n=1}^{\infty} P[S_{c^n} > \frac{1}{2}a_{c^{n+2}}] = \infty$ and

Claim (ii) $\sum_{k=1}^{\infty} P[S_{c^n} > a_{c^{n-1}}] = \infty$.

If (i) were false, then the Lévy inequalities would imply

$$\sum_{k=1}^{\infty} P[\max_{k \leqq c^n}(S_{c^n+k} - S_{c^n}) > \tfrac{1}{2}a_{c^{n+2}}] < \infty \quad \text{and}$$

$$\sum_{k=1}^{\infty} P[\max_{k \leqq c^n} S_k > \tfrac{1}{2}a_{c^{n+2}}] < \infty,$$

which combined with $2^{\frac{1}{3}} > c$ implies

$$P[\max_{c^{n+2} \leqq j \leqq c^{n+3}} S_j > a_{c^{n+2}} \text{ i.o.}] = 0.$$

But the a_n are non-decreasing and a contradiction results. (ii) is just a similar application of the Lévy inequalities. But (i), (ii) and Lemma 2 verify this lemma. So from Lemma 3, we see that $\limsup (S_{(n)}/a_n) \geqq \frac{1}{2} \limsup (S_n/a_n)$ and thus

COROLLARY 1. *If* $\limsup (S_n/a_n) = \{{}^{+\infty}_{0}$ a.e., *then* $\limsup S_{(n)}/a_n \geqq \limsup S_n/a_n$ a.e. *for all* $(\ ) \in \mathbb{R}$.

(We note that the condition that $\limsup (S_n/a_n)$ is $+\infty$, or 0 a.e. can hold for all $a_n \to \infty$, for instance, if X_i does not belong to the domain of partial attraction of the normal distribution (see Heyde [4]).)

Let $A = [\{a_n\} : a_n$ is non-decreasing and $\alpha(c) \equiv \lim (a_{c^{n+1}}/a_{c^n})$ exists, finite or infinite; for all $c > 1]$.

Standard arguments show that $\alpha(c) \nearrow$ and either for all $c > 1$, $\alpha(c) = \infty$, or for all $c > 1$, $\alpha(c) < \infty$. The latter happens if and only if $\lim_m \alpha(c^{1/m}) = 1$. Note that most of the interesting sequences are in A—e.g., $a_n = n$, $\lg n$, n^α, $\lg \lg n$, etc.

COROLLARY 2. *Let* $\{a_n\} \in A$ *with* $\alpha(c) < \infty$. $\sum_{n=1}^{\infty} P[S_{c^n} > (1-\varepsilon)a_{c^n}] = \infty$ *for* $2^{\frac{1}{3}} > c > 1$ *and* $\varepsilon > 0$ *if and only if* $\limsup \acute{S}_n/a_n \geqq 1$ a.e.

PROOF. First note that the above series being infinite implies

$$\sum_{n=1}^{\infty} P\left[S_{c^n} > \frac{(1-\varepsilon)}{\alpha(c)} a_{c^{n+1}}\right] = \infty$$

and so Lemma 2 and $\lim_m \alpha(c^{1/m}) = 1$ give the desired conclusion. Conversely, Lemma 3 implies $\limsup S_{c^n}/a_{c^n} \geqq (1-\varepsilon)/\alpha^2(c)$ for all $\varepsilon > 0$ and $2^{\frac{1}{2}} > c > 1$. Thus

$$\sum_{n=1}^{\infty} P\left[S_{c^n} \geqq \frac{(1-\varepsilon)}{\alpha^2(c)} a_{c^n}\right] = \infty$$

by the Borel–Cantelli Lemma and since $\lim_m \alpha(c^{1/m}) = 1$ the proof is complete.

THEOREM 2. *Let* $\{a_n\} \in A$. *Then* $\limsup S_{(n)}/a_n \geqq \limsup S_n/a_n$.

PROOF. Straightforward application of Lemma 3.

2. Stronglaws. In this section strong laws for $S_{(n)}$, $(\,) \in \mathbb{R}$, are considered.

PROPOSITION 1.

(a) $S_{(n)}/n \to 0$ a.e. *for all* $(\,) \in \mathbb{R}$ iff $EX_1^{\,2} < \infty$.

(b) *Let* $1 < r \leqq 2$. *There exists* $(\,) \in \mathbb{R}$ *so that* $S_{(n)}/n \to 0$ a.e. iff $E|X_1|^r < \infty$.

(c) *Let* $1 < r \leqq 2$ *and* Y_1 *be such that* $E\,|Y_1|^r = \infty$, $E|Y_1|^q < \infty$, $q < r$. *Then there exists* $(\,) \in \mathbb{R}$ *so that* $\lim S_{(n)}/n \to 0$ a.e. *implies* $E|X_1|^q < \infty$ *for all* $q < r$ *and* $\sum_{i\in(n)} Y_i/n \to 0$ a.e.

PROOF. (a) $S_{\langle n\rangle}/n \to 0$ a.e. iff $\sum_{n=1}^{\infty} P[|S_n| > n\varepsilon] < \infty$ all $\varepsilon > 0$, but this is equivalent to $EX_1^{\,2} < \infty$. Clearly, $S_{\langle n\rangle}/n \to 0$ a.e. implies $S_{(n)}/n \to 0$ a.e. all $(\,) \in \mathbb{R}$. (b) and (c) involve constructions. That the constructions work is an immediate consequence of $S_{\langle k^\alpha\rangle}/[k^\alpha] \to 0$ a.e. iff $E|X_1|^{\alpha+1/\alpha}$. This last equivalence follows by the methods of [1].

To see (b) let $\alpha = 1/(r-1)$ and define $(\,) \in \mathbb{R}$ as follows:

$$(n) = \{\text{1st } n \text{ even integers}\} \qquad n \neq [k^\alpha] \quad \text{some} \quad k.$$

$$= \left\{\text{1st}\frac{[k^\alpha]([k^\alpha]+1)}{2}\text{odd integers}\right\} - \left\{\text{1st}\frac{[k^\alpha]([k^\alpha]-1)}{2}\text{ odd integers}\right\} \qquad n = [k^\alpha].$$

Clearly $S_{(n)}/n \to 0$ a.e. iff $E|X_1|^r < \infty$.

The following exhibits a rule for (c). Let $r_j \nearrow r$, $k(j) \equiv [k^{1/(r_j-1)}]$ and define $n(j)$ so that

$$\textstyle\sum_{k(j)\geqq n(j)} P[|\sum_{i=1}^{k(j)} Y_i| > k(j)\, 3^{-j}] > 2^{-j}.$$

Let

$$(n) = \left\{\text{1st } \frac{k(j)(k(j)+1)}{2} \text{ odd integers}\right\} - \left\{\text{1st } \frac{k(j)(k(j)-1)}{2} \text{ odd integers}\right\}$$

if $n = k(j)$ and $n \geqq n(j)$ where if there is more than one choice for k and j choose the representation with the smallest k.

$= \{\text{1st } n \text{ even integers}\}$ otherwise.

Analogs of the convergence rate theorems of [1] for S_n are easily exhibited for $S_{\langle n\rangle}$. This follows since

$$P\left\{\sup_{k\geqq n}\frac{|S_{\langle k\rangle}|}{k}>\varepsilon\right\}\leqq\sum_{k=n}^{\infty}P\left[\frac{|S_k|}{k}>\varepsilon\right].$$

If $EX_1^{\ 2}<\infty$ then lim sup $S_n/(2n\lg\lg n)^{\frac{1}{2}}=1$ and if $EX_1^{\ 4}<\infty$ then lim sup $S_{\langle n\rangle}/(2n\lg n)^{\frac{1}{2}}=1$ (see [2]).

Let $2<q<4$ and define

$A_q=\{\{a_n\}:a_n-(q-2)\lg n\to-\infty$ and for each $n\ \exists\ N_n$ so that $2\lg n\leqq a_{N_n}\leqq 2\lg(n+1)$. N_n is smallest such integer$\}$.

PROPOSITION 2. *If* $E|X_1|^q<\infty$, $2\lg\lg n\leqq a_n\leqq 2\lg n$ *and* $\{a_n\}\in A_q$ *there exists a rule in* $\mathbb{R}$ *so that* lim sup $S_{(n)}/(na_n)^{\frac{1}{2}}=1$.

The rules are obtained by a construction that depends on a result of Pinsky [5], which is slightly in error as stated. In [5] it is actually shown that if $E|X|^q<\infty$ for $2<q<4$, then for $\varepsilon>0$ and a sequence $\{a_n\}$ such that $a_n-(q-2)\lg n\to-\infty$

$$\exp\{-(1+\varepsilon)a_n^{\ 2}/2\}\leqq P[S_n/n^{\frac{1}{2}}\geqq a_n]\leqq\exp\{-(1-\varepsilon)a_n^{\ 2}/2\},\qquad n\geqq n_0.$$

PROOF OF PROPOSITION 2. Define ( ) as follows:

$$(n)=\left\{1\text{st }\frac{m(m+1)}{2}\text{ odd integers}\right\}-\left\{1\text{st }\frac{m(m-1)}{2}\text{ odd integers}\right\}m\in\{N_n\}$$
$$=\{1\text{st } m \text{ even integers}\}\quad\text{if}\quad m\notin\{N_n\}.$$

Note $\lim_{m\notin\{N_n\}}\sup S_{(m)}/(ma_m)^{\frac{1}{2}}\leqq 1$ by law of iterated logarithm for S_n.
By the above result of Pinsky, for sufficiently small $\varepsilon>0$

$$\sum_{n=n_0}^{\infty}P\left[S_{N_n}/N_n^{\frac{1}{2}}\geqq\left(\frac{(1+\varepsilon)a_{N_n}}{(1-\varepsilon)}\right)^{\frac{1}{2}}\right]\leqq\sum_{n=n_0}^{\infty}\exp\{-(1+\varepsilon)a_{N_n}/2\}$$

and

$$\sum_{n=n_0}^{\infty}P\left[S_{N_n}/N_n^{\frac{1}{2}}\geqq\left(\frac{(1-\varepsilon)}{1+\varepsilon}a_{N_n}\right)^{\frac{1}{2}}\right]\geqq\sum_{n=n_0}^{\infty}\exp\{-(1-\varepsilon)a_{N_n}/2\}.$$

However, note that by the definition of ( ), $\{S_{(N_n)}\}$ is a collection of independent random variables and the result follows by the Borel–Cantelli Lemma.

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