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ASYMPTOTIC BEHAVIOR OF A CLASS OF CONFIDENCE REGIONS BASED ON RANKS IN REGRESSION

BY HIRA LAL KOUL

Michigan State University

0. Summary. Asymptotic behavior of a class of confidence regions, based on rank statistics, for the regression parameter vector is considered. These regions are shown to be asymptotically bounded and ellipsoidic in probability. Asymptotic normality of their center of gravities is also proved. It is noted that the asymptotic efficiencies of these regions when defined in terms of ratio of Lebesgue measures corresponds to that of corresponding test statistics that are used to define these regions. Similar conclusion holds for their center of gravities, where now asymptotic efficiency is defined as inverse ratio of their generalized limiting variances. Also a class of consistent estimators is given for some functionals of the underlying distributions. Finally simultaneous confidence intervals, based on the above center of gravity, for linear parametric functions are shown to have asymptotic coverage probability $1-\alpha$. Basic to this work are two papers, one by the author [4] and one by Jurečková [3].

1. Introduction, notation and assumptions. Theory of rank statistics has made several advances in fields such as testing of hypotheses and point estimation. Comparatively very little work using rank statistics has been done in confidence regions. Perhaps the first work in this direction is due to Lehmann [5] in 1963. Here he constructs confidence intervals using Wilcoxon rank statistic for shift parameter in one- and two-sample problems. He proved that the length of interval tends to a finite and positive limit. His work was generalized by Sen [7] (1966) to a class of rank statistics in two-sample problems involving shift parameter. In [4] this author gave similar results using Wilcoxon type signed rank statistic for regression parameter vector. The current work might be considered the generalization of the above works. Theorem 2.3 here is precisely the generalization of Theorem 1 of [5] and Theorem 2 of [7] to multiple regression model and to a wider class of rank statistics. In addition to this we prove that the corresponding center of gravities have limiting normal distribution. This work also generalizes the results of [4] to a much wider class of rank statistics but on the other hand puts somewhat restrictive condition on regression scores (see 1.4b below). However the assumption on the underlying distribution are not as restrictive as in [4].

Let $\{Y_{in} 1 \leqq i \leqq n\} n \geqq 1$ be independent rv's such that

$$\Pr[Y_{in} \leqq y] = F(y-\boldsymbol{\theta}'\mathbf{X}_{in}) \qquad 1 \leqq i \leqq n \tag{1.1}$$

where $\boldsymbol{\theta}' = (\theta_1, \cdots, \theta_p)$ is the parameter of interest and

$$\mathbf{X}_{in} = (x_{in}(1), \cdots, x_{in}(p))$$

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are some known regression constants. We shall usually write $x_{i\nu}$ for $x_{in}(\nu)$, $1 \leqq \nu \leqq p$. F is a cdf. Let, for each $\boldsymbol{\theta}$, R_{in}' be the rank of $Y_{in}-\boldsymbol{\theta}'\mathbf{X}_{in}$ among $\{Y_{in}-\boldsymbol{\theta}'\mathbf{X}_{in}, 1 \leqq i \leqq n\}$. Let φ be a given score function. We make the following assumptions on the above set of given things.

F is absolutely continuous cdf with $F' = f, f'$ existing almost everywhere such that

(1.2) $$I(f) = \int_{-\infty}^{\infty} \left(\frac{f'}{f}\right)^2 dF < \infty.$$

Let $\mathbf{X}_n = ((x_{i\nu}-\bar{x}_\nu))$, $i = 1, \cdots, n$, $\nu = 1, \cdots, p$, with $\bar{x}_\nu = n^{-1}\sum_{i=1}^n x_{i\nu}$.

Assume

(1.3) $$\lim n^{-1}\mathbf{X}_n'\mathbf{X}_n = \lim \textstyle\sum_n^* = \sum^*$$

exists and is a positive definite matrix, where the limit of a matrix is the matrix formed by the limit of each entry in the matrix. The limit will be always taken as $n \to \infty$.

Notice that (1.3) implies that

$$\frac{1}{n}\sum_{i=1}^{n}(x_{i\nu}-\bar{x}_\nu)^2 > 0, \qquad 1 \leqq \nu \leqq p$$

for all but finitely many n's. We also assume that

(1.4a) $$\lim \max_{1 \leqq i \leqq n}(x_{i\nu}-\bar{x}_\nu)^2\left[\textstyle\sum_{i=1}^n (x_{i\nu}-\bar{x}_\nu)^2\right]^{-1} = 0 \qquad 1 \leqq \nu \leqq p, \text{ and}$$

(1.4b) $$(x_{i\nu}-x_{j\nu})(x_{i\gamma}-x_{j\gamma}) \geqq 0 \quad \text{or} \quad (x_{i\nu}-x_{j\nu})(x_{i\gamma}-x_{j\gamma}) \leqq 0$$
$$\text{for all} \quad 1 \leqq i, j \leqq n; 1 \leqq \nu \neq \gamma \leqq p.$$

So far φ function is concerned we assume

(1.5) $$\varphi \in L^2[0, 1],$$

(1.6) $$\sigma_\varphi^{\,2} = \textstyle\int_0^1[\varphi(u)-\bar{\varphi}]^2 > 0 \qquad \text{where } \bar{\varphi} = \int_0^1 \varphi(u)\,du,$$

(1.7) $$\varphi \quad \text{is nondecreasing on} \quad [0, 1].$$

Also define

(1.8) $$\varphi(u,f) = -f'(F^{-1}(u))/f(F^{-1}(u)) \qquad 0 \leqq u \leqq 1.$$

Note that (1.2) is equivalent to

(1.9) $$\textstyle\int_0^1 \varphi^2(u,f)\,du < \infty.$$

Observe that (1.5), (1.9) and usual Schwartz inequality implies that

(1.10) $$b(\varphi,f) = \textstyle\int_0^1 \varphi(u)\varphi(u,f)\,du < \infty.$$

We also note that (1.3) and (1.4b) yields

(1.11) $$\max_{1 \leqq i \leqq n} n^{-1}(x_{iv}-\bar{x}_v)^2 \to 0, \qquad 1 \leqq v \leqq p.$$

We further define $\mathscr{L}_\theta(X)$ as law of rv X when $\boldsymbol{\theta}$ is the true parameter. For any two sets A, B; $A \triangle B$ will denote symmetric difference of the two sets. For event E depending on $\{Y_{in}\}$, $P_n[E]$ will mean probability of E when $\mathbf{0}$ is the true parameter. λ will denote the p dimensional Lebesgue measure. For a vector

$$\boldsymbol{\theta}, \; \|\boldsymbol{\theta}\| = \textstyle\sum_{i=1}^p |\theta_i|.$$

Next define, for $Y = (Y_{in}, 1 \leqq i \leqq n)$,

(1.12) $$S_{nv}(Y, \boldsymbol{\theta}) = n^{-1}\textstyle\sum_{i=1}^n x_{iv}\varphi(R'_{in}/(n+1)) \qquad 1 \leqq v \leqq p,$$

(1.13) $$\mathbf{S}_n'(Y, \boldsymbol{\theta}) = (S_{n1}(Y, \boldsymbol{\theta}), \cdots, S_{np}(Y, \boldsymbol{\theta})),$$

(1.14) $$\textstyle\sum_n = \sum_n^* \sigma_\varphi^2 ,$$

and finally

(1.15) $$M_n(Y, \boldsymbol{\theta}) = n\mathbf{S}_n'(Y, \boldsymbol{\theta})\textstyle\sum_n^{-1}\mathbf{S}_n(Y, \boldsymbol{\theta}).$$

Many times we shall write $M_n(\boldsymbol{\theta})$ for $M_n(Y, \boldsymbol{\theta})$. The same applies to $S_{nv}(\boldsymbol{\theta})$.

Suppose one accepts the hypothesis $H(\boldsymbol{\theta}_0)$: $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ at level α when $M_n(Y, \boldsymbol{\theta}_0) \leqq k_{\alpha,n}$. This test is distribution free and $k_{\alpha,n}$ can be determined from $\chi_p{}^2$ tables for large n. Consequently $k_{\alpha,n}$ is determined for all n. The confidence region one gets from the above test is

(1.16) $$R_n(Y) = \{\boldsymbol{\theta};\, M_n(Y, \boldsymbol{\theta}) \leqq k_{\alpha,n}\}$$

Also define

(1.17) $$n^{\frac{1}{2}}\hat{\boldsymbol{\theta}}_n = \{\lambda[n^{\frac{1}{2}}R_n(Y)]\}^{-1}\int \mathbf{t}I[n^{\frac{1}{2}}R_n(Y)]\,d\lambda(\mathbf{t})$$

where the integral is to be interpreted as a vector valued integral. I is the indicator function.

Throughout this paper we shall assume that the true parameter point is $\mathbf{0}$. This will not change our conclusion, because $R_n(Y)$ is such that

$$\boldsymbol{\theta} \in R_n(Y+\mathbf{b}'\mathbf{X}) \Leftrightarrow \boldsymbol{\theta}+\mathbf{b} \in R_n(Y)$$

for any vector b and all n. We now define

(1.18) $$V_{nv}(Y, \boldsymbol{\theta}) = S_{nv}(\mathbf{0}) - \theta' b_{nv}, \qquad 1 \leqq v \leqq p$$

where

(1.19) $$\mathbf{b}_{nv} = n^{-1}\textstyle\sum_{i=1}^n (x_{iv}-\bar{x}_v)\mathbf{X}_{in}\cdot b(\varphi, f).$$

Note that if $\boldsymbol{\theta}_0$ were true parameter, one would have $(\boldsymbol{\theta}-\boldsymbol{\theta}_0)$ in place of $\boldsymbol{\theta}$ and $\mathbf{0}$ would be replaced by $\boldsymbol{\theta}_0$ in (1.18). Furthermore define

$$B_n = \sigma_\varphi^{-2} \textstyle\sum_n b(\varphi, f), \tag{1.20}$$

$$\mathbf{V}_n'(Y, \boldsymbol{\theta}) = (V_{n1}(Y, \boldsymbol{\theta}), \cdots, V_{np}(Y, \boldsymbol{\theta}))$$

$$\begin{aligned} T_n(Y, \boldsymbol{\theta}) &= n\mathbf{V}_n'(Y, \boldsymbol{\theta}) \textstyle\sum_n^{-1} \mathbf{V}_n(Y, \boldsymbol{\theta}) \\ &= n(\mathbf{S}_n(\mathbf{0}) - \boldsymbol{\theta}' B_n)' \textstyle\sum_n^{-1} (\mathbf{S}_n(\mathbf{0}) - \boldsymbol{\theta}' B_n), \end{aligned} \tag{1.21}$$

$$D_n(Y) = \{\boldsymbol{\theta};\ T_n(Y, \boldsymbol{\theta}) \leqq k_{\alpha,n}\}, \tag{1.22}$$

and finally.

$$n^{\frac{1}{2}}\hat{\hat{\boldsymbol{\theta}}}_n = \{\lambda[n^{\frac{1}{2}} D_n(Y)]\}^{-1} \textstyle\int \mathbf{t} I[n^{\frac{1}{2}} D_n(Y)] \lambda(d\mathbf{t}) \tag{1.23}$$

where again $\int$ is p-vector valued integral and I is indicator.

$\hat{\boldsymbol{\theta}}_n$ as defined by (1.17) and (1.16) may be considered as an estimator of the parameter $\boldsymbol{\theta}$. Adichie in [1] considered a class of estimators for the case $p = 1$ using the statistics S_{n1} defined by (1.12) above. If in (1.16) and (1.17) above one takes $\alpha = 1$, $p = 1$ and use the appropriate M_n for this case, one can see that $\hat{\boldsymbol{\theta}}_n$ reduces to Adichie type point estimate of θ_1. By reducing regression problem to two-sample problem in usual fashion similar relation holds between $\hat{\boldsymbol{\theta}}_n$ and Sen's [7] estimators.

2. Boundedness of $n^{\frac{1}{2}}R_n(Y)$. Our main problem is to see how the regions $R_n(Y)$ behave asymptotically. It turns out, as will be seen in this section, the regions $n^{\frac{1}{2}}R_n(Y)$ behave like $n^{\frac{1}{2}}D_n(Y)$ in probabilistic sense and consequently $n^{\frac{1}{2}}\hat{\boldsymbol{\theta}}_n$ behaves like $n^{\frac{1}{2}}\hat{\hat{\boldsymbol{\theta}}}_n$. In order to make these statements and their proofs precise we need the following

THEOREM 2.1. (Jurečková, J. (1969)).

Under the conditions (1.2) *through* (1.7)

$$P_n[\max_{||\boldsymbol{\theta}|| \leqq a} n^{\frac{1}{2}} |S_{n\nu}(\boldsymbol{\theta} n^{-\frac{1}{2}}) - V_{n\nu}(\boldsymbol{\theta} n^{-\frac{1}{2}})| \geqq \varepsilon] \to 0, \qquad 1 \leqq \nu \leqq p \tag{2.1}$$

for every $\varepsilon > 0$ *and any* $0 < a < \infty$, *fixed.*

Now one of the immediate consequences of the above theorem is the following

LEMMA 2.1. *Under the conditions* (1.2) *through* (1.7), *for every* $\varepsilon > 0$

$$\lim P_n[\max_{||\boldsymbol{\theta}|| \leqq a} |M_n(Y, \boldsymbol{\theta} n^{-\frac{1}{2}}) - T_n(Y, \boldsymbol{\theta} n^{-\frac{1}{2}})| \geqq \varepsilon] = 0 \tag{2.2}$$

for any $0 < a < \infty$, *fixed.*

PROOF. The proof follows by noting that

$$\begin{aligned} |M_n(\boldsymbol{\theta} n^{-\frac{1}{2}}) - T_n(\boldsymbol{\theta} n^{-\frac{1}{2}})| &\leqq n^{\frac{1}{2}} ||\mathbf{S}_n(\boldsymbol{\theta} n^{-\frac{1}{2}}) - \mathbf{V}_n(\boldsymbol{\theta} n^{-\frac{1}{2}})|| \\ &\cdot \{||\textstyle\sum_n^{-1} n^{\frac{1}{2}} \mathbf{S}_n(Y, \boldsymbol{\theta} n^{-\frac{1}{2}})|| + ||n^{\frac{1}{2}} \textstyle\sum_n^{-1} \mathbf{V}_n(Y, \boldsymbol{\theta} n^{-\frac{1}{2}})||\} \end{aligned} \tag{2.3}$$

and applying Theorem 2.1 to the first term of the right-hand side of (2.3) and using the fact that $\max_{\|\theta\| \leq a} \|n^{\frac{1}{2}}\sum_n^{-1}\mathbf{S}_n(Y, \theta n^{-\frac{1}{2}})\|$ has a limiting distribution, for $\max_{\|\theta\| \leq a}\|n^{\frac{1}{2}}\sum_n^{-1}\mathbf{V}_n(Y, \theta n^{-\frac{1}{2}})\|$ has limiting distribution. The latter fact follows from the definition of $\mathbf{V}_n$ and the fact that $\|n^{\frac{1}{2}}\sum_n^{-1}\mathbf{V}_n(Y, \theta n^{-\frac{1}{2}})\|$ has a limiting normal distribution (see [2]), and therefore the former statement follows in view of (2.1.).

Our next result shows that the regions $n^{\frac{1}{2}}R_n(Y)$ are bounded for large n. Define

$$V(a) = \{\theta; \|\theta\| < a\}. \tag{2.4}$$

THEOREM 2.2. *Under the conditions* (1.2) *to* (1.7), *for every* $\varepsilon > 0$ *there is an* n_ε *and* $0 < a < \infty$ *large such that*

$$P_n[y; n^{\frac{1}{2}}R_n(y) \subset V(a)] > 1-\varepsilon \tag{2.5}$$

for $n \geqq n_\varepsilon$

PROOF. The proof of this theorem will be split among several lemmas.

We introduce two functions g_n and h_n as follows.

For any real number $-\infty < r < +\infty$ and vector θ such that $\|\theta\| = 1$, define

$$\begin{aligned} g_n(r, \theta) &= n^{\frac{1}{2}}[\theta'\mathbf{S}_n(\mathbf{0}) - r\theta' B_n\theta]/(\theta'\textstyle\sum_n\theta)^{\frac{1}{2}} \\ &= n^{\frac{1}{2}}[\theta'\mathbf{S}_n(\mathbf{0}) - rb\theta'\textstyle\sum_n\theta]/(\theta'\textstyle\sum_n\theta)^{\frac{1}{2}} \end{aligned} \tag{2.6}$$

$$h_n(r, \theta) = n^{\frac{1}{2}}[\theta'\mathbf{S}_n(r\theta)]/(\theta'\textstyle\sum_n\theta)^{\frac{1}{2}}, \tag{2.7}$$

where $b = \sigma_\varphi^{-2}b(\varphi, f)$.

Assume the conditions (1.2)–(1.7) hold.

LEMMA 2.2. *For every* $\varepsilon > 0$ *and a given* $d > k_{\alpha,n}$ *there exists an* $0 < a < \infty$ *large enough and* n_ε *such that* $n \geqq n_\varepsilon$ *yields*

$$P_n[\inf_{\|\theta\|=1} \inf_{|r|=an^{-1/2}} |h_n(r, \theta)| \geqq d] \geqq 1-\varepsilon. \tag{2.8}$$

PROOF. It will be enough, in view of Theorem 2.1, to prove the statement like (2.8) for g_n function. For, implications of Theorem 2.1 are that for every $\varepsilon > 0$ there is n_ε such that

$$P_n[|\inf_{\substack{|r|=an^{-1/2}\\ \|\theta\|=1}} |g_n(r, \theta)| - \inf_{\substack{|r|=an^{-1/2}\\ \|\theta\|=1}} |h_n(r, \theta)|| \leqq \varepsilon] \geqq 1-\varepsilon \tag{2.9}$$

for any $0 < a < \infty$, $n \geqq n_\varepsilon$.

Thus we set to prove the following.

For every $\varepsilon > 0$ and a given $d > k_{\alpha,n}$ there exist n_ε and $0 < a < \infty$ large such that for $n \geqq n_\varepsilon$

$$P_n[\inf_{\|\theta\|=1} \inf_{|r|=an^{-1/2}} |g_n(r, \theta)| \geqq d] \geqq 1-\varepsilon \tag{2.10}$$

Now it is well known that the lim $\mathscr{L}(n^{\frac{1}{2}}\mathbf{S}_n'(\mathbf{0})\sum_n^{-1})$ is a normal law (e.g. see Hájek and Šidák [2]). Consequently since $\sum_n$ has limit, it follows $n^{\frac{1}{2}}\|S_n(0)\|$ has a limiting distribution also. Also note that since $\lim \sum_n$ is positive definite, there are two constants $\eta > 0$ and $k < \infty$ such that

$$0 < \eta \leqq (\theta' \textstyle\sum_n \theta)^{\frac{1}{2}} \leqq k < \infty \tag{2.11}$$

uniformly in all $\|\theta\| = 1$ and for n large. Therefore for any $\|\theta\| = 1$ we have

$$b^{-1}n^{\frac{1}{2}}[\theta' S_n(0)][\theta' \textstyle\sum_n^{-1}\theta]^{-\frac{1}{2}} \leqq b^{-1}\eta^{-1}n^{\frac{1}{2}}\|S_n(0)\|. \tag{2.12}$$

Combining (2.12) with the above remarks, for every $\varepsilon > 0$ there is a "c" and n_ε such that $n \geqq n_\varepsilon$ yields

$$P_n[|\inf_{\|\theta\|=1} |b^{-1}[\theta' \textstyle\sum_n \theta]^{-\frac{1}{2}} n^{\frac{1}{2}}|\theta' S_n(0)|| \leqq c] \geqq 1-\varepsilon.$$

Choose

$$a \geqq (d+|b|c)(\eta|b|)^{-1}. \tag{2.13}$$

Then for this "a" using (2.11), (2.12) and (2.6), it is easy to see that

$$\begin{aligned}
P_n[\inf_{\|\theta\|=1} \inf_{|r|=an^{-1/2}} |g_n(r, \theta)| \geqq d]
&= P_n[|b| \inf_{\|\theta\|=1} \inf_{|r|=a} |b^{-1}(\theta' \textstyle\sum_n \theta)^{-\frac{1}{2}} n^{\frac{1}{2}}\theta' S_n(0) - r(\theta' \textstyle\sum_n \theta)^{\frac{1}{2}}| \geqq d] \\
&\geqq P_n[|b|\eta \inf_{\|\theta\|=1} \inf_{|r|=a} |b^{-1}(\theta' \textstyle\sum_n \theta)^{-1} n^{\frac{1}{2}}\theta' S_n(0) - r| \geqq d] \\
&\geqq P_n[|b|\eta \inf_{\|\theta\|=1} |b^{-1}(\theta' \textstyle\sum_n \theta)^{-1} n^{\frac{1}{2}}\theta' S_n(0)| \leqq -d + a|b|\eta] \\
&\geqq P_n[|b|\eta(\eta^{-1}) \inf_{\|\theta\|=1} |b^{-1}(\theta' \textstyle\sum_n \theta)^{-\frac{1}{2}} n^{\frac{1}{2}}|\theta' S_n(0)|| \leqq -d + a|b|\eta] \\
&\geqq P_n[\inf_{\|\theta\|=1} |b(\theta' \textstyle\sum_n \theta)^{-\frac{1}{2}} n^{\frac{1}{2}}|\theta' S_n(0)|| \leqq c] \\
&\geqq 1-\varepsilon
\end{aligned}$$

for $n \geqq n_\varepsilon$. This proves (2.10) and hence the lemma.

LEMMA 2.3. *For every $\varepsilon > 0$ and a given $d > k_{\alpha,n}$ there exists an "a" and n_ε, $0 < a < \infty$, such that*

$$P_n[\inf_{\|\theta\|=1} \inf_{|r|\geqq an^{-1/2}} |h_n(r, \theta)| \geqq d] \geqq 1-\varepsilon \tag{2.14}$$

for $n \geqq n_\varepsilon$.

PROOF. Since φ is non-decreasing, one can see, after rewriting h_n as

$$(\theta' \textstyle\sum_n \theta)h_n(r, \theta) = n^{-1}\sum_{i=1}^n d_{in}\varphi(R_{in}/n+1)$$

where $d_{in} = \theta' X_{in}$ and R_{in} is the rank of $Y_{in} - r\, d_{in}$, that $h_n(r, \theta)$ is a non-increasing function of r for every θ. Hence the lemma follows from (2.8).

PROOF OF THE THEOREM. Note that (2.5) will follow if we show that for every $\varepsilon > 0$ and a given $d > k_{\alpha,n}$, there is an "a" $< \infty$ large and n_ε such that $n \geqq n_\varepsilon$ implies

$$P_n[\inf_{\|\theta\| \geqq an^{-\frac{1}{2}}} |M_n(Y, \boldsymbol{\theta})| \geqq d] \geqq 1-\varepsilon. \tag{2.15}$$

However, since $\sum_n$ is positive definite for all but finitely many n, one uses inequality

$$M_n(Y, \boldsymbol{\theta}) \geqq \frac{|n^{\frac{1}{2}}\boldsymbol{\theta}'\mathbf{S}_n(\boldsymbol{\theta})|^2}{(\boldsymbol{\theta}'\sum_n \boldsymbol{\theta})} \tag{2.16}$$

and the fact that for any vector $\boldsymbol{\theta}$, there is a vector $\boldsymbol{\theta}^*$ and a real number r such that $\|\boldsymbol{\theta}^*\| = 1$ and

$$\boldsymbol{\theta} = r\boldsymbol{\theta}^* \tag{2.17}$$

Hence using (2.17), (2.16) one concludes the claimed theorem using Lemma 2.3 and definition (2.7). The proof is terminated.

REMARK. As may be seen from the above proof, one can also say that for every $\varepsilon > 0$ there is an n_ε and $0 < a < \infty$ such that $n \geqq n_\varepsilon$ implies

$$P_n[y;\, n^{\frac{1}{2}}D_n(Y) \subset V(a)] \geqq 1-\varepsilon. \tag{2.18}$$

This "a" is the same as given by (2.13).

THEOREM 2.3. *$\lambda[n^{\frac{1}{2}}R_n(Y)\Delta n^{\frac{1}{2}}D_n(Y)] \to 0$ in P_n-probability and consequently*

$$\lambda[n^{\frac{1}{2}}R_n(Y)] \to C_{p,\alpha}/\{\sigma_\varphi^{-1}b(\varphi, f)\}^p|\textstyle\sum^*|^{\frac{1}{2}} \tag{2.19}$$

in probability, where $C_{p,\alpha} = (\pi k_\alpha)^{p/2}/\Gamma((p/2)+1))$, $\sum^*$ is defined by (1.3) *and k_α is limit of $k_{\alpha,n}$ determined by $\chi_p{}^2$.*

PROOF. Last statement of the theorem follows, because the right-hand side is precisely the Lebesgue measure of the ellipsoid $n^{\frac{1}{2}}D_n(Y)$.

In order to prove the first statement we introduce

$$W_n(y) = n^{\frac{1}{2}}R_n(y) \triangle n^{\frac{1}{2}}D_n(y) \tag{2.20}$$

$$K_n(a, \varepsilon) = \{y;\, \sup_{\theta \in V(a)} |M_n(y, \boldsymbol{\theta}n^{-\frac{1}{2}}) - T_n(y, \boldsymbol{\theta}n^{-\frac{1}{2}})| \geqq \varepsilon\} \tag{2.21}$$

$$U_n(y, \varepsilon) = \{\boldsymbol{\theta};\, k_{\alpha,n} - \varepsilon < T_n(y, \boldsymbol{\theta}n^{-\frac{1}{2}}) \leqq k_\alpha + 2\varepsilon\} \tag{2.22}$$

$$Q_n(a) = \{y;\, W_n(y) \subset V(a)\}. \tag{2.23}$$

Observe that from (2.5), (2.18) and Lemma 2.1 it follows that for every $\varepsilon > 0$ there is an $0 < a < \infty$ and n_ε such that

$$P_n[K_n{}^c(a, \varepsilon) \cap Q_n(a)] \geqq 1-\varepsilon \tag{2.24}$$

for $n \geqq n_\varepsilon$.

But for a $y \in K_n^c(a, \varepsilon) \cap Q_n(a)$, a $\boldsymbol{\theta}$ in $W_n(y)$ is such that for every $\varepsilon > 0$ it also belongs to $U_n(y, \varepsilon)$. Therefore for every $\varepsilon > 0$ there is n_ε such that

$$P_n[y;\ W_n(y) \subset U_n(y, \varepsilon)] \geqq 1-\varepsilon \tag{2.25}$$

for $n \geqq n_\varepsilon$, which implies

$$P_n[y;\ \lambda[W_n(y)] \leqq \lambda[U_n(y, \varepsilon)]] \geqq 1-\varepsilon. \tag{2.26}$$

But it is easy to see that

$$\lambda[U_n(y, \varepsilon)] = \delta_n\{(k_{\alpha,n}+2\varepsilon)^{p/2}-(k_{\alpha,n}-\varepsilon)^{p/2}\} \tag{2.27}$$

where δ_n is a constant depending on n only through the determinant $|\sum_n|$ and hence $\lim \delta_n < \infty$. (2.27) can be thus made arbitrarily small for arbitrarily small ε. This concludes the proof.

Asymptotic efficiency of the regions. Suppose ψ is any other score function satisying the conditions of Section 1. Let $R_n(Y, \psi)$ be the corresponding region.

DEFINITION.

$$\begin{aligned} E_{\psi,\varphi} &= \lim e(R_n(Y, \psi), R_n(Y, \varphi)) \\ &= \lim [\lambda[n^{\frac{1}{2}}R_n(Y, \psi)]/\lambda[n^{\frac{1}{2}}R_n(Y, \varphi)]]^2 \end{aligned}$$

will be called the relative asymptotic efficiency of the region corresponding to ψ function relative to the one corresponding to φ function. From Theorem 2.3 it follows that

$$\begin{aligned} E_{\psi,\varphi} &= \left[\frac{\sigma_\varphi^{-1}b(\varphi,f)}{\sigma_\psi^{-1}b(\psi,f)}\right]^{2p} \\ &= \left\{\frac{\int \varphi(u)\varphi(u,f)(\int \psi(u)-\bar{\psi})^2)^{\frac{1}{2}}}{(\int(\varphi(u)-\bar{\varphi})^2)^{\frac{1}{2}}\int \psi(u)\varphi(u,f)}\right\}^{2p} \end{aligned} \tag{2.28}$$

Now suppose $p = 1$, $\psi(u) = \varphi(u,f) = \varphi_0(u)$ then from (2.28) one observes that

$$E_{\varphi_0,\varphi} = \left\{\frac{\int \varphi(u)\varphi_0(u)}{[\int(\varphi(u)-\bar{\varphi})^2\int \varphi_0{}^2(u)]^{\frac{1}{2}}}\right\}^2$$

which is precisely the asymptotic efficiency of the test based on S_1 statistic, generated by φ function, to test $\theta_1 = 0$ against $\theta_1 > 0$. This expression appears e.g. in Hájek and Šidák [2] page 268.

One can, in general, write

$$\begin{aligned} E_{\psi,\varphi} &= \left[\frac{\sigma_\varphi^{-1}I^{-\frac{1}{2}}(f)b(\varphi,f)}{\sigma_\psi^{-1}I^{-\frac{1}{2}}(f)b(\psi,f)}\right]^{2p} \\ &= \left[\frac{\rho(\varphi, \varphi_0)}{\rho(\psi, \varphi_0)}\right]^{2p} \end{aligned}$$

where $\rho(\varphi, \varphi_0) = \int \varphi\varphi_0[\int(\varphi(u)-\bar{\varphi})^2.\ (\int \varphi_0{}^2)]^{-\frac{1}{2}}$.

3 Asymptotic normality of $n^{\frac{1}{2}}\hat{\boldsymbol{\theta}}_n$. The following result gives asymptotic normality of $n^{\frac{1}{2}}\hat{\boldsymbol{\theta}}_n$; the center of gravity of the confidence region $n^{\frac{1}{2}}R_n(Y)$.

THEOREM 3.1. *Under the conditions* (1.2)–(1.7) *on the underlying quantities*

$$\lim \mathscr{L}_{\theta_0}(n^{\frac{1}{2}}(\hat{\boldsymbol{\theta}}_n-\boldsymbol{\theta}_0)) = N(\mathbf{0}, \textstyle\sum^{-1} b^{-2}) \tag{3.1}$$

where $b = \sigma_\varphi{}^{-2}b(\varphi,f)$ *and* $\sum^{-1} = \lim \sum_n^{-1}$, *and N stands for the multivariate normal distribution.*

PROOF. Since regions $n^{\frac{1}{2}}R_n(Y)$ are invariant under translation in the sense mentioned at the end of Section 1, entailing invariance of $n^{\frac{1}{2}}\hat{\boldsymbol{\theta}}_n$ it will be enough to prove (4.1) with $\boldsymbol{\theta}_0$ replaced by $\mathbf{0}$.

We first prove that

$$n^{\frac{1}{2}}||\hat{\boldsymbol{\theta}}_n-\hat{\hat{\boldsymbol{\theta}}}_n|| \to 0 \tag{3.2}$$

in P_n-probability.

For, by definition (1.17) and (1.23) and the usual properties of the norm $\|\cdot\|$, we have

$$n^{\frac{1}{2}}||\hat{\boldsymbol{\theta}}_n-\hat{\hat{\boldsymbol{\theta}}}_n|| \leqq \frac{|\lambda(n^{\frac{1}{2}}R_n(y))-\lambda(n^{\frac{1}{2}}D_n(y))|}{\lambda[n^{\frac{1}{2}}R_n(y)]\lambda[n^{\frac{1}{2}}D_n(y)]}\int_{n^{1/2}R_n(y)}||\mathbf{t}||\,d\lambda(\mathbf{t}) + \lambda[n^{\frac{1}{2}}D_n(y)]^{-1}\int||\mathbf{t}||I[W_n(y)]\,d\lambda(\mathbf{t}). \tag{3.3}$$

Now note that by Theorem 2.3 the first factor in the first term of the right-hand side tends to zero in P_n probability and by Theorem 2.1 the second factor is at most $a\cdot\lambda[V(a)]$ for an $a < \infty$. Hence the first term on the right-hand side of (3.2) tends to zero in P_n probability.

Again by Theorem 2.3, $\lambda[n^{\frac{1}{2}}D_n(Y)]$ having finite limit, the second term tends to zero for $\lambda[W_n(Y)] \to 0$, in P_n probability.

Now next observe that by definition (1.24)

$$n^{\frac{1}{2}}\hat{\hat{\boldsymbol{\theta}}}_n = n^{\frac{1}{2}}b^{-1}\textstyle\sum_n^{-1}\mathbf{S}_n(\mathbf{0}) \tag{3.4}$$

The claimed asymptotic normality now follows by using the well-known fact that $\mathscr{L}_0(n^{\frac{1}{2}}\mathbf{S}_n(\mathbf{0})) \to N(\mathbf{0}, \sum)$ and (3.2). The proof is terminated.

Asymptotic efficiency of the center of gravities. If we define asymptotic efficiency of the estimator $n^{\frac{1}{2}}\hat{\boldsymbol{\theta}}_n(\varphi)$, generated by φ function, relative to another estimator $n^{\frac{1}{2}}\hat{\boldsymbol{\theta}}_n(\psi)$ the one generated by ψ function, as an inverse ratio of their generalized asymptotic variances and denote it by $C(F, \psi, \varphi)$, we have

$$C(F, \psi, \varphi) = \left[\frac{b(\varphi,f)\sigma_\varphi{}^{-1}}{b(\psi,f)\sigma_\psi{}^{-1}}\right]^{2p}.$$

Consistent estimators of $b(\varphi,f)$. Theorem 2.3 enables one to define a class of consistent estimators of the functional $b(\varphi,f)$ as follows.

Define

$$\hat{b}_n(\varphi) = [C_{p,\alpha} \cdot \sigma_\varphi{}^p\{\lambda[n^{\frac{1}{2}}R_n(Y)]\}^{-1}|\textstyle\sum_n|^{-\frac{1}{2}}]^{1/p} \tag{3.5}$$

Then $\hat{b}_n(\varphi)$ is consistent for $b(\varphi, f)$ follows from (2.26).

4. Rank competitors of Scheffé's S-method. Let $C = ((c_{ij}))$ be a $q \times p$ matrix of known real numbers whose rows are linearly independent so that rank of C is q. Define

$$C_i = \textstyle\sum_{j=1}^{p} c_{ij}\theta_j \qquad i = 1, \cdots, q \tag{4.1}$$

$$\psi' = (\psi_1, \cdots, \psi_q).$$

Let L be a q dimensional linear space generated by ψ. Let $\hat{\psi}_n = C\hat{\theta}_n$. Clearly, in view of Theorem 3.1 $\mathscr{L}_\Psi(n^{\frac{1}{2}}(\hat{\psi}_n - \psi)) \to N(\mathbf{0}, B)$ where

$$B = \lim C \textstyle\sum_n^{-1} C'b^{-1} = \lim B_n. \tag{4.2}$$

Suppose C is such that $C \sum_n^{-1} C'$ has inverse, then

$$\mathscr{L}_\psi(n(\hat{\psi}_n - \psi)' B_n{}^{-1}(\hat{\psi}_n - \psi)) \to \chi_q{}^2 \tag{4.3}$$

and the following inequality $n(\hat{\psi}_n - \psi)' B_n{}^{-1}(\hat{\psi}_n - \psi) \leqq K_{\alpha,{}^nq}$ defines an ellipsoid in q dimensional space.

For any $\psi \in L$, there is a vector $\mathbf{h}_{q \times 1}$ such that $\psi = \mathbf{h}'\psi$ and we define $\hat{\psi}_n = \mathbf{h}'\hat{\psi}_n$ as its rank estimator. Let $\sigma^2_{\hat{\psi}_n} = \text{Var}(\hat{\psi}_n)$ and $K = (K_{\alpha,q})^{\frac{1}{2}}$, where $K_{\alpha,q}$ is such that $\text{Prob}\,[\chi_q{}^2 \geqq K_{\alpha,q}] = \alpha$. Then, following the proof of Theorem 3.5.1 of Scheffé [6], one can conclude

THEOREM 4.1. *Under the assumptions* (1.2)–(1.7), *probability that simultaneously for all* $\psi \in L$,

$$\hat{\psi}_n - K\sigma_{\hat{\psi}_n} \leqq \psi \leqq \hat{\psi}_n + K\sigma_{\hat{\psi}_n} \tag{4.4}$$

tends to $1 - \alpha$ *as* $n \to \infty$.

Consequently one can construct simultaneous confidence intervals for linear contrasts using rank statistics such that for large n one has coverage probability $1 - \alpha$.

REMARK. It must be noted that all the above results are also valid for tests based on the scores of the type $E\varphi(U^{(i)})$, where $U^{(i)}$ is ith ordered statistic of a sample from uniform distribution.

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