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A LIMIT THEOREM FOR A BRANCHING PROCESS WITH STATE-DEPENDENT IMMIGRATION

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Let $\{p_j\}$ and $\{a_j\}$ be probabilities defined on the non-negative integers. Consider a Markov chain $\{Z_n; n = 0, 1, \cdots\}$ defined on the non-negative integers, with stationary transition probabilities given by

$$(1)\qquad p_{ij} = P(Z_{n+1} = j \mid Z_n = i) = p_j^{(i*)}, \qquad i \geqq 1$$
$$= a_j, \qquad i = 0$$

where $p_j^{(i*)}$ is the jth term in the ith fold convolution of the sequence $\{p_j\}$. $\{Z_n\}$ represents the sizes of successive generations in a Galton–Watson process, modified to allow an immigration of particles whenever the zero state is reached. After entering the system in accordance with the probabilities $\{a_j\}$, immigrating particles reproduce with offspring law $\{p_j\}$, independently of each other and of particles already present. Z_0 is considered to be positive and non-random.

The case in which the offspring distribution has mean one and finite variance is of particular interest because the limit behavior of $\{Z_n\}$ is unlike that usually observed in branching processes (with or without immigration), in that a non-linear normalization is required in order to get a proper non-zero limit distribution. If the offspring mean is greater than or less than one, the appropriate normalization for Z_n is the same as for the process in which the immigration is not state dependent (Pakes, Theorems 3 and 10).

From now on it will be assumed that:

1. $\sum jp_j = 1$;
2. $\sum j^2 p_j < \infty$;
3. $\sum ja_j \equiv a < \infty$;
4. $p_j < 1$ for all j;
5. $a_0 < 1$.

The last two assumptions are made to avoid trivial cases. An example will be given to show that Z_n can have an entirely different kind of limit behavior if a is allowed to be infinite.

Set $f(s) = \Sigma p_j s^j$, $h(s) = \Sigma a_j s^j$, $|s| \leqq 1$, and let $f_n(s)$ denote the nth functional iterate of $f(s)$ (i.e., $f_0(s) = s$ and $f_{n+1}(s) = f(f_n(s))$ for $n \geqq 0$). A simple computation

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using (1) shows that the generating functions $g_n(s) = \Sigma P(Z_n = j)s^j$, $|s| \leqq 1$, are given recursively by

$$g_{n+1}(s) = g_n(f(s)) - (1-h(s))g_n(0),$$
$$g_0(s) = s^{Z_0}.$$

Repeated application of the first relation gives

$$g_{n+1}(s) = f_{n+1}^{Z_0}(s) - \sum_{k=0}^{n} [1-h(f_k(s))]g_{n-k}(0). \tag{2}$$

The moments of Z_n can be obtained by differentiating (2):

$$EZ_{n+1} = g'_{n+1}(1-) = Z_0 + a\sum_{k=0}^{n} g_k(0)$$
$$EZ_{n+1}^2 = g'_{n+1}(1-) + g''_{n+1}(1-)$$
$$= Z_0^{\,2} + Z_0(n+1)\sigma^2 + \sum_{k=0}^{n} (ak\sigma^2 + a + b)g_{n-k}(0)$$

where $b = h''(1-)$, $\sigma^2 = f''(1-)$. The higher moments of Z_n can be computed in a similar fashion.

To determine the asymptotic behavior of the last two quantities, set $F(t) = \Sigma[1-f_n^{\,Z_0}(0)]t^n$, $H(t) = \Sigma[1-h(f_n(0))]t^n$, $G(t) = \Sigma g_n(0)t^n$, $|t| \leqq 1$. Noting that $G(0) = g_0(0) = 0$ and $F(0) = 1-f_0^{\,Z_0}(0) = 1$, relation (2) with $s = 0$ becomes $G(t) = (1-t)^{-1} - F(t) - tH(t)G(t)$, which can be solved for $G(t)$ to give

$$G(t) = [(1-t)^{-1} - F(t)]/[1+tH(t)].$$

Assumptions 1, 2 and 4 imply (see, for example, Theorem 1 of Kesten, Ney and Spitzer (1966), that $\lim_{n\to\infty} (n\sigma^2/2)[1-f_n(0)] = 1$, so $F(t) \sim Z_0\Sigma[1-f_n(0)]t^n \sim (2Z_0/\sigma^2) \log(1-t)^{-1}$ as $t \to 1-$. Similarly, $H(t) \sim (2a/\sigma^2)\log(1-t)^{-1}$. Thus $G(t) \sim (\sigma^2/2a)[(1-t)\log(1-t)^{-1}]^{-1}$. A standard Tauberian theorem (Feller (1966) page 423) now gives the result

$$\sum_{k=0}^{n} g_k(0) \sim (\sigma^2/2a)n/\log n \qquad \text{as } n \to \infty. \tag{3}$$

The formulas for the first and second moments of Z_n now yield

$$EZ_{n+1} \sim n\sigma^2/2\log n$$
$$EZ_{n+1}^2 \sim n^2\sigma^4/4\log n.$$

In order to investigate the limit behavior of Z_n, the asymptotic behavior of the individual terms $g_n(0)$ is needed. The following result will be required: *The Markov chain* $\{Z_n; n = 0, 1, \cdots\}$ *is aperiodic.* The proof is as follows: Assumptions 1 and 4 imply that $p_0 > 0$ and that there exists $j \geqq 1$ such that $p_j > 0$. Assumption 5 implies that there exists $i \geqq 1$ such that $a_i > 0$. From the definition of the transition probabilities of the process, $P(Z_{n+2} = 0 \mid Z_n = 0) \geqq a_i p_0^{\,i} > 0$ and $P(Z_{n+3} = 0 \mid Z_n = 0) \geqq a_i p_j^{\,i} p_0^{\,ij} > 0$.

The elapsed time T until the first arrival of $\{Z_n\}$ at the zero state has distribution

$$P(T = n) = f_n^{\,Z_0}(0) - f_{n-1}^{Z_0}(0), \qquad n = 1, 2, \cdots.$$

The distribution of the time between successive returns to the zero state is given by

$$b_n = \sum_{i=1}^{\infty} a_i[f_{n-1}^i(0) - f_{n-2}^i(0)], \qquad n = 2, 3, \cdots$$
$$= a_0, \qquad n = 1.$$

This formula is obtained by conditioning on the first state to be visited after leaving zero. Note that $\sum_{n=1}^{\infty} b_n = a_0 + \sum_{i=1}^{\infty} a_i = 1$, so that $\{Z_n\}$ retuns to zero infinitely often with probability one. Successive returns to zero thus constitute a discrete renewal process with the tail distribution of the first return times satisfying

$$\sum_{k=n+1}^{\infty} b_k = \sum_{i=0}^{\infty} [1 - f_{n-1}^i(0)]a_i = 1 - h(f_{n-1}(0))$$
$$\sim a[1 - f_{n-1}(0)] \sim 2a/n\sigma^2.$$

A result of Erickson (1970) concerning the behavior of the renewal measure $u_n = \sum_{k=1}^{\infty} b_n^{(k*)}$ associated with an aperiodic distribution $\{b_n\}$ on the non-negative integers satisfying $\sum_{k=n+1}^{\infty} b_k \sim n^{-1}L(n)$, L slowly varying at infinity, states that $\lim_{n\to\infty} m(n)u_n = 1$, where $m(n) = \sum_{k=1}^{n} kb_k$. Thus

$$u_n \sim [\sum_{j=1}^{n} jb_j]^{-1} = [a_0 + \sum_{j=1}^{n} j[h(f_{j-1}(0)) - h(f_{j-2}(0))]]^{-1}$$
$$\sim [a \sum_{j=2}^{n} j(f_{j-1}(0) - f_{j-2}(0))]^{-1}$$
$$\sim \sigma^2/2a \log n.$$

The result $\lim_{j\to\infty} (2j^2/\sigma^2)(f_{j-1}(0) - f_{j-2}(0)) = 1$ (Kesten, Ney and Spitzer (1966) Corollary 1 to Theorem 1) has been used. Conditioning on T gives

$$g_n(0) = \sum_{k=1}^{n} u_{n-k}[f_k^{Z_0}(0) - f_{k-1}^{Z_0}(0)]$$
$$\sim (\sigma^2/2a \log n) \sum_{k=1}^{n} [f_k^{Z_0}(0) - f_{k-1}^{Z_0}(0)].$$

Since $\sum_{k=1}^{n} [f_k^{Z_0}(0) - f_{k-1}^{Z_0}(0)] = f_n^{Z_0}(0) \to 1$ as $n \to \infty$, we have

$$g_n(0) \sim \sigma^2/2a \log n. \tag{4}$$

THEOREM. *Under Assumptions* 1–5, $\lim_{n\to\infty} P((\log Z_n)/\log n \leqq \beta) = \beta$ *for* $0 < \beta < 1$.

PROOF. The following version of Spitzer's Comparison Lemma (Athreya and Ney (1972) Chapter 1, Section 9) for the iterates of a generating function $f(s) = \Sigma p_j s^j$ whose probabilities satisfy Assumptions 1, 2 and 4 is used:

Given $\varepsilon > 0$, there is a number $s_0 \in (0, 1)$ such that for any $k \geqq 0$ and for all $s \in [s_0, 1)$,

$$\left[\frac{1}{1-s} + k(\sigma^2 + \varepsilon)/2\right]^{-1} \leqq 1 - f_k(s) \leqq \left[\frac{1}{1-s} + k(\sigma^2 - \varepsilon)/2\right]^{-1}.$$

Also, given $\varepsilon \in (0, a)$ there exists $t_0 \in (0, 1)$ such that for all $s \in [t_0, 1)$, $(a - \varepsilon) \times (1 - s) \leqq 1 - h(s) \leqq a(1 - s)$. By (2) and the Comparison Lemma,

$$g_{n+1}(\exp(-s/n^\beta)) \leqq f_{n+1}^{Z_0}(\exp(-s/n^\beta)) - \sum_{k=0}^{n} (a - \varepsilon)[2n^\beta/s + k(\sigma^2 + \varepsilon)/2]^{-1} g_{n-k}(0),$$
$$g_{n+1}(\exp(-s/n^\beta)) \geqq f_{n+1}^{Z_0}(\exp(-s/n^\beta)) - \sum_{k=0}^{n} a[n^\beta/s + k(\sigma^2 - \varepsilon)/2]^{-1} g_{n-k}(0)$$

for fixed $s \in [t_0, 1)$ and n sufficiently large. By (4) we have

$$\limsup_{n\to\infty} g_{n+1}(\exp(-s/n^{\beta})) \leqq 1-(\sigma^2(1-\varepsilon/a)/2)\lim_{n\to\infty}\sum_{k=0}^{n-2} \cdot[2n^{\beta}/s+k(\sigma^2+\varepsilon)/2]^{-1}/\log(n-k),$$

$$\liminf_{n\to\infty} g_{n+1}(\exp(-s/n^{\beta})) \geqq 1-(\sigma^2/2)\lim_{n\to\infty}\sum_{k=0}^{n-2} \cdot[n^{\beta}/s+k(\sigma^2-\varepsilon)/2]^{-1}/\log(n-k).$$

An elementary computation, the details of which are omitted, shows that

$$\lim_{n\to\infty}\sum_{k=0}^{n-2}[n^{\beta}/s+kc]^{-1}/\log(n-k) = c^{-1}(1-\beta)$$

for any $s > 0$ and $c > 0$. Since ε was arbitrary, we have $\lim_{n\to\infty} g_{n+1}$ $(\exp\times(-s/n^{\beta})) = \beta$. It follows that $\lim_{n\to\infty} E(\exp(-sZ_n/n^{\beta})) = \beta$. This limit is the Laplace transform of a degenerate probability distribution with mass β at the origin and mass $1-\beta$ at infinity. Applying the continuity theorem for Laplace transforms (Feller (1966) page 408), we have $\lim_{n\to\infty} P(Z_n/n^{\beta} \leqq x) = \beta$ for each $x > 0$. Setting $x = 1$ and taking logarithms of both sides of the inequality in the last relation completes the proof of the theorem.

The following example shows that a different type of limit behavior is possible if a is allowed to be infinite. For fixed $\alpha \in (0, 1)$, let $a_0 = 0$ and $a_n = \binom{\alpha}{n}(-1)^{n+1}$ for $n \geqq 1$, so that $h(s) = 1-(1-s)^{\alpha}$ and $h'(1-) = \infty$. The first return times then satisfy $\sum_{k=n+1}^{\infty} b_k = (1-f_{n-1}(0))^{\alpha} \sim (2/n\sigma^2)^{\alpha}$. Note that $b_n = (1-f_{n-2}(0))^{\alpha} - (1-f_{n-1}(0))^{\alpha} = (1-f_{n-2}(0))^{\alpha}[1-(1-f(f_{n-2}(0)))^{\alpha}/(1-f_{n-2}(0))^{\alpha}]$ is decreasing with n. In this case it is known (Williamson (1968) Corollary 3-A) that the associated renewal measure u_n satisfies $u_n \sim (\sigma^2/2)^{\alpha}\pi^{-1}\sin\pi\alpha\, n^{\alpha-1}$. Conditioning on T then gives $g_n(0) \sim u_n$. It can then be shown, using (2), that

$$\lim_{n\to\infty} E(\exp(-sZ_n/n) = 1-\pi^{-1}\sin\pi\alpha\int_0^1(2/s\sigma^2+x)^{-\alpha}(1-x)^{\alpha-1}\,dx.$$

The details are omitted.

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