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ON THE ADMISSIBLE ESTIMATORS FOR CERTAIN FIXED SAMPLE BINOMIAL PROBLEMS

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0. Introduction. Let X be a binomial random variable, $b(n, p)$, and f a continuous real valued function on $[0, 1]$. We consider the problem of estimating $f(p)$ with, for example, squared error loss. It is well known (Wald (1950), Le Cam (1955)) that the class of Bayes procedures is a complete class. Further, since any unique Bayes estimator is admissible, our concern with admissibility centers on the non-unique Bayes estimators. In Section 1, this single parameter problem is considered. In Section 2, $X_1, \cdots, X_q$ are assumed independent binomials, $b(n_i, p_i)$, $i = 1, \cdots, q$, and each of $f_i(p_i)$, $i = 1, \cdots, q$, or $f(p_1, \cdots, p_q)$ is to be estimated with summed squared error loss for the first problem and squared error loss for the second, and in each case with a continuity assumption on the f. In Section 3, the $X_1, \cdots, X_q$ are assumed to have a multinomial distribution and the analogues of the problems of Section 2 are considered. The main result of this note is that for these problems the classes of admissible estimators are closed in the topology of pointwise convergence of the estimators. Also, a procedure is given for constructing the non-unique Bayes estimators. A few examples, for which the admissibility is generally known, are included to illustrate the construction. For the problem of estimating each $f_i(p_i)$ when the X_i are independent, it is shown that there is no Stein effect (Stein (1956)). That is, $\delta = (\delta_1, \cdots, \delta_q)$ is admissible if δ_i is admissible for the problem of estimating $f_i(p_i)$ based only on X_i. Section 4 contains some more or less obvious extensions to related problems.

The method employed is contained in the following simple observation. Let P_α, $\alpha \in A$, be a family of discrete probability densities for X. Suppose our interest is in estimators, $\delta(X)$, of $f(\alpha)$ when the loss is, say, squared error. Let A_0 be a closed subset of A, $D^c = (x : P_\alpha(X = x) > 0 \text{ for some } \alpha \in A_0)$, and D the remainder of the sample space, which we suppose is not empty. The risk of δ is

$$\rho(\alpha, \delta) = \sum_{x \in D^c} (\delta(x) - f(\alpha))^2 P_\alpha(X = x) + \sum_{x \in D} (\delta(x) - f(\alpha))^2 P_\alpha(X = x \mid D) P_\alpha(D).$$

Suppose the restriction of δ to D^c is admissible for the problem of estimating $f(\alpha)$ if α is restricted to A_0 and further that no other estimator for the restricted problem has the same risk. Then plainly the risk of δ can only be minorized by another estimator with the same determination on D^c. Also, δ is admissible if and only if its restriction to D is admissible for the problem of estimating $f(\alpha)$ if α is restricted to $A \sim A_0$ and the distributions of X are $P_\alpha(\ \mid D)$. Finally, if the class of distributions $P_\alpha(\ \mid D)$ is completed in a manner that leaves f well defined, the above may be iterated to construct admissible estimators. For the problems we consider, non-unique Bayes procedures occur exactly when the support of the a priori

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measure is a subset of the boundary of the parameter space. In this case D^c is exactly the subset of the sample space on which the procedure is uniquely determined and, hence, the above decomposition may be employed to construct admissible non-unique Bayes procedures. Although admissible, these estimators may reflect a tyranny of the boundary of the parameter space. Example 3 is of this type.

1. A single binomial parameter.

Let X have a binomial distribution $b(n, p)$, n known and $p \in [0, 1]$.

THEOREM 1. *Let f be a continuous real valued function on* [0, 1]. *Let $\mathscr{D}$ be the class of estimators of $f(p)$ with representation*

$$\begin{aligned}\delta(x) &= f(0), && \text{for } x \leqq r\\ &= \frac{\int_0^1 f(p)p^{x-r-1}(1-p)^{s-x-1}\pi(dp)}{\int_0^1 p^{x-r-1}(1-p)^{s-x-1}\pi(dp)}, && \text{for } r+1 \leqq x \leqq s-1,\\ &= f(1), && \text{for } x \geqq s\end{aligned}$$

where r and s are integers, $-1 \leqq r < s \leqq n+1$, and π is a probability measure with $\pi(\{0\} \cup \{1\}) < 1$.

(a) *$\mathscr{D}$ is precisely the class of admissible estimators of $f(p)$ relative to squared error loss.*

(b) *$\mathscr{D}$ is closed in the topology of pointwise convergence.*

PROOF. We first show that any $\delta_0 \in \mathscr{D}$ is admissible. The cases of $r = -1$, $s = n+1$, and $r+1 = s$ are trivial. Suppose $s \geqq 0$ or $r \leqq n$ and $r+1 < s$. Suppose δ' is another estimator with the risks of δ' and δ_0 satisfying $\rho(p, \delta') \leqq \rho(p, \delta_0)$ for all $p \in [0, 1]$. Let $r' = \max(m : \delta'(x) = f(0), x \leqq m)$ and $s' = \min(m : \delta(x) = f(1), x \geqq m)$. Considering the risks of δ' and δ_0 near zero and one, it is clear that $r' \geqq r$ and $s' \leqq s$. Let

$$\psi_{r,s}(p, \delta) = \textstyle\sum_{x=r+1}^{s-1} (f(p) - \delta(x))^2 \binom{n}{x} p^{x-r-1}(1-p)^{s-x-1}k(p)^{-1},$$

where $k(p) = P(r+1 \leqq X \leqq s-1 \mid p)/p^{r+1}(1-p)^{n-s+1}$. Clearly, $\rho(p, \delta') \leqq \rho(p, \delta_0)$ for all $p \in [0, 1]$ iff $\psi_{r,s}(p, \delta') \leqq \psi_{r,s}(p, \delta_0)$ for all $p \in [0, 1]$. Let π_0 be the measure in the representation and $\pi^*(dp) = k(p)\pi_0(dp)$. The problem of minimizing $\int_0^1 \psi(p, \delta)\pi^*(dp)$, in $(\delta(r+1), \cdots, \delta(s-1))$, has the unique solution $(\delta_0(r+1), \cdots, \delta_0(s-1))$, establishing the assertation.

Let δ' be any admissible estimator. Define r' and s' as above. Then the restriction of δ' is a Bayes procedure for the problem $\psi_{r',s'}$; that is, $(\delta'(r'+1), \cdots, \delta'(s'-1))$ minimizes $\int_0^1 \psi_{r',s'}(p, \delta)\pi'(dp)$, with π' some probability measure. This is so because the restriction of δ' must be admissible for the $\psi_{r',s'}$ problem and upon adjoining the limits of the conditional distribution of X as $p \to 0$ or 1 the parameter space for the problem enjoys the compactness required for the Bayes procedures to be complete. As $\delta'(r'+1) \neq f(0)$, $\delta'(s'-1) \neq f(1)$, $\pi'(\{0\} \cup \{1\}) < 1$ and δ' has the asserted representation.

To demonstrate $\mathscr{D}$ is closed, let $\delta_m \in \mathscr{D}$ and $\delta_m(x) \to \delta(x)$, $x = 0, 1, \cdots, n$. Let r_m, s_m and r, s be defined as above for δ_m and δ. For m large, we may suppose $r_m \leqq r$ and $s_m \geqq s$. Let

$$\begin{aligned} \delta_m'(x) &= f(0), && x \leqq r \\ &= \delta_m(x), && \textit{for } r < x < s \\ &= f(1), && x \geqq s. \end{aligned}$$

Clearly, $\delta_m' \in \mathscr{D}$ and $\delta_m' \to \delta$. Let π_m' be the measure in the representation of δ_m' and suppose, without loss of generality, that π_m' converges weakly to π. As $\delta(r+1) \neq \delta(s-1) \neq f(1)$, it follows easily that $\pi(\{0\} \cup \{1\}) < 1$ and that δ has the representation with π. □

The following well-known examples illustrate the representation:

EXAMPLE 1. For $f(p) = p$, taking $r = 0$ and $s = n$ and π the uniform measure, the estimator X/n results.

EXAMPLE 2. For $f(p) = p(1-p)$, taking $r = 0$ and $s = n$ and π the uniform measure, the estimator $X(n-X)/n(n+1)$ results.

It should be remarked that in the proof of Theorem 1 we have, in the cases $r \geqq s$ and $s \leqq n-1$, deviated slightly from the method enunciated above. For example, if both inequalities hold, the first stage of the construction would be to assign $\delta(0) = f(0)$ and $\delta(1) = f(1)$ which is unique Bayes for the restricted problem with parameter space $\{0\} \cup \{1\}$. Iteration of this type of decomposition reduces the sample space to $(r+1, \cdots, s-1)$ and the construction is completed by the iteration made in the proof.

2. Several binomial parameters. Let X_i, $i = 1, \cdots, q$, be independent with binomial distribution $b(n_i, p_i)$, n_i known and $p_i \in [0, 1]$. We consider the following estimation problems:

Problem I. Estimating $f_i(p_i)$, $i = 1, \cdots, q$, when the loss is the sum of squared errors and the f_i are continuous real valued functions on $[0, 1]$.

Problem II. Estimating $f(p_1, p_2, \cdots, p_q)$ when the loss is squared error and f is a continuous real valued function on $[0, 1]^q$.

From the point of view of proving the analogue of Theorem 1, it will become quite evident that the case $q = 2$ is completely representative of the general case. The details which follow are so restricted.

We begin by constructing admissible estimators for Problems I and II which are not unique Bayes. To fix notations, the development is in terms of Problem I.

In what follows, D, subscripted or not, will denote the subset of the lattice points D_0 of $[0, n_1] \times [0, n_2]$ which lie in or on the boundary of a certain convex solyhedron in $[0, n_1] \times [0, n_2]$. D^c will denote the complement of D in D_0. For any subset of G of D_0, $(\delta_1, \delta_2)_G$ will denote a function on G with values in R^2, real two-space. (δ_1, δ_2) will be said to be a completion of $(\delta_1, \delta_2)_G$ if it is an extension of $(\delta_1, \delta_2)_G$ to all of D_0.

At the kth stage of the construction we assume that $(\delta_1, \delta_2)_{D_k{}^c}$ has the property,

P: the risk function of any completion of $(\delta_1, \delta_2)_{D_k{}^c}$ can only be minorized by the risk function of another completion of $(\delta_1, \delta_2)_{D_k{}^c}$.

The $(k+1)$st stage of the construction entails finding a nonempty subset, D_{k+1}, of D_k and a function $(\delta_1, \delta_2)_{D_{k+1}^c}$, an extension of $(\delta_1, \delta_2)_{D_k{}^c}$, with P.

The densities for (X_1, X_2) at the kth stage are the conditional densities given D_k and their limits as p_1 and/or p_2 approach 0 or 1 appropriately. The densities obtained as limits concentrate their mass on lines of support of D_k. If $p_1 \to 0$ and $p_2 \to p \in (0, 1)$, then the mass is confined to the line $x_1 = \min(x_1 : (x_1, x_2) \in D_k)$. The density is of the form $H(p_2)h(x_2)p^{x_2 - x_2'}(1-p_2)^{x_2'' - x_2}$, where x_2' and x_2'' are, respectively, the smallest and largest values of x_2 in the intersection of this line and D_k. If $x_1 + ax_2 = b$, a and $b > 0$, is a line of support and p_1 and p_2 go to zero so that $p_2{}^a/p_1 \to \theta$, then the mass is confined to this line. The density is of the form $H(p)h(x_2)p^{x_2 - x_2'}(1-p)^{x_2'' - x_2}$, where x_2' and x_2'' are as above and $\theta = p/(1-p)$. For the purpose of defining a topology on the densities, we identify generic density by P_{p_1, p_2}, to be interrupted as follows. If $(p_1, p_2) \in (0, 1)^2$, the density is the conditional density determined by (p_1, p_2). If $p_1 = 0$ and $p_2 \in (0, 1)$, the density is of the first type considered above. If $p_1 = p_2 = 0$, the density is of the second type considered above. The interpretation of the other extreme values of p_1 and p_2 is similar. If P_{p_1, p_2} and $P'_{p_1', p_2'}$ are two densities, define

$$d(P_{p_1, p_2}, P'_{p_1', p_2'}) = \max_{x \in D_k} |P_{p_1, p_2}(x) - P'_{p_1', p_2'}(x)| + |p_1 - p_1'| + |p_2 - p_2'|.$$

The class of densities at the kth stage is clearly compact in the topology induced by d.

To complete the definition of the ancillary problem at the kth stage, f_1 and f_2 must be defined at each density. At P_{p_1, p_2} we take the determination $f_i(p_i)$, $i = 1, 2$. Clearly, the f_i are continuous.

A consequence of the above remarks is that the class of Bayes procedures is complete for the ancillary problem at the kth stage. If a unique Bayes procedure is chosen for this problem, the construction is terminated and the resulting estimator is admissible. Otherwise there is a closed subset of the boundary of the parameter or density space, A_k, which is the support of the a priori measure corresponding to the procedure chosen. Let Δ_k be the subset of D_k on which the procedure is determined by its Bayes character. Then $\Delta_k = (x : P(x) > 0$, for some $P \in A_k)$. In fact, the inequality must hold on a set of positive a priori measure. In both Problems I and II, it is easily verified that the restriction to Δ_k of the procedure chosen is a unique Bayes procedure for a correspondingly restricted version of the ancillary problem; that is, the problem with parameter space A_k. Thus, $D_{k+1} = D_k \sim \Delta_k$ and the extension provided has property P. Otherwise, the restriction to Δ_k of some completion of $(\delta_1, \delta_2)_{D_k{}^c}$ would have risk minorizing that of the unique Bayes procedure for the problem with parameter space A_k. Iteration of the above results in the construction of an admissible estimator for the original problem.

Let $\mathscr{D}_1$ be the class of estimators which can be constructed in this manner. That any admissible estimator, (δ_1, δ_2), belongs to $\mathscr{D}_1$ is implicit in the preceding discussion. Suppose that the restriction of (δ_1, δ_2) to D_k^c can be constructed. Then, in view of the hypothesized admissibility, the restriction of (δ_1, δ_2) to D_k^c must be admissible for the ancillary problem at the kth stage defined above. Thus, it is a Bayes procedure and unique or not the construction of (δ_1, δ_2) may be continued.

In the case of $q = 2$, three types of Δ_k arise in the construction of an estimator. They are identical for both Problems I and II.

(A) Suppose that $x_1 + ax_2 = b$, a and $b > 0$, is a line of support to D_k with no point of D_k below it. Take Δ_k to be the points of D_k on the line. The corresponding A_k is all densities denoted by $P_{0,0}$ and the determination of the estimator on Δ_k is $(f_1(0), f_2(0))$ (or $f(0, 0)$ for Problem II). From the point of view of construction it is evident that we may as well take $\Delta_k = (x : x_1 + ax_2 \leqq b) \cap D_k$, where a and b are positive and the line is not necessarily one of support, with the same determination of the estimator. The other corners of the parameter space are similarly handled.

(B) Consider the rectangle containing D_k with sides parallel to the x_1, x_2 axes and supports to D_k. The Δ_k consist of points in D_k and on the sides of the rectangle and are specified as follows. If $x \in \Delta_k$ and is on exactly one side of the rectangle, then all points in D_k on this side are in Δ_k. If x and x' are distinct points of Δ_k and belong to different sides, than all "intervening" lattice points on the sides of the rectangle in either the clockwise or counterclockwise direction belong to Δ_k. Four species of Δ_k arise depending on the number of sides involved. In all cases the corresponding A_k may be taken to be the densities for which Δ_k has mass one. Further, the determination of the estimator on Δ_k may be assumed to be that of a unique Bayes procedure for the ancillary problem with parameter space A_k.

(C) $\Delta_k = D_k$ and we may suppose that the determination of the estimator on Δ_k is that of a unique Bayes procedure for the ancillary problem.

For $q > 2$, the possibilities for the Δ_k are more numerous but are obvious generalizations of the above.

It is easily shown that $\mathscr{D}_I$ is closed. Let $(\delta_{1,m}, \delta_{2,m})$, $m = 1, \cdots$, be in D_I and converge to (δ_1, δ_2). By choosing an appropriate subsequence we may suppose that each can be constructed by applications of (A), (B), and (C) in the same order and with identical Δ_k and A_k. Thus we need establish the same result for the ancillary problems with parameter space A_k. For Δ_k of type (A), this is trivial. The proofs for Δ_k of types (B) and (C) are essentially the same. We examine the latter. Consider the ancillary problem with parameter space A_k and sample space D_k. We assume that the construction of each member of the subsequence is completed by an application of (C) to this problem. Suppose that the restriction of (δ_1, δ_2) to D_k may be constructed on $D_k \sim D^*$ by applications of (A) and (B) but that no further application of (A) or (B) is possible. Define

$$(\delta^*_{1,m'}, \delta^*_{2,m'})_{D_k} = (\delta_{1,m'}, \delta_{2,m'}) \qquad \text{on } D^*$$
$$= (\delta_1, \delta_2) \qquad \text{on } D_k \sim D^*.$$

It is easily seen that these estimators are admissible for the ancillary problem. In fact, $(\delta_{1,m'}, \delta_{2,m'})_{D^*}$, the restriction of $(\delta_{1,m'}, \delta_{2km'})$ to D^*, is a unique Bayes procedure for the ancillary problem determined by D^*. Let A^* be the parameter space for this problem and $\pi_{m'}$ the corresponding a priori measures. The class of probability measures on A^* is tight so we may suppose $\pi_{m'}$ converge weakly to π. $(\delta_1, \delta_2)_{D^*}$ is clearly Bayes relative to π. If unique Bayes, the assertion follows. Otherwise (δ_1, δ_2) can be constructed past $D_k \sim D^*$ by applications of (A) and (B).

We summarize the preceding for both problems in Theorem 2.

THEOREM 2. *The classes of admissible estimators for problems* I *and* II, $\mathscr{D}_I$ *and* $\mathscr{D}_{II}$, *are precisely those estimators which can be constructed.* $\mathscr{D}_1$ *and* $\mathscr{D}_{II}$ *are closed.*

Several examples of Problem I of interest are contained in the following theorem.

THEOREM 3. *Let the* f_i, $i = 1, \cdots, q$, *be as in Problem* I. *Suppose that* $\delta_i(X_i)$ *is admissible for* $f_i(p_i)$ *against squared error loss. Then* $(\delta_1, \cdots, \delta_q)$ *is admissible for Problem I.*

PROOF. Suppose the assertion established for 1 through $q-1$ parameters. In view of Theorem 1, there are integers r_i and s_i and a measure π_i such that δ_i has the representation therein. Suppose $r_1 \geqq 0$ and set $\Delta_0 = \{(x_1, \cdots, x_q): x_1 = 0\}$. Then by the induction hypothesis $(f_1(0), \delta_2, \cdots, \delta_q)$ is admissible for the ancillary problem with $A_0 = \{(p_1, \cdots, p_q): p_1 = 0\}$. Although it will not in general be a unique Bayes procedure for this problem, it is evident that the contribution to the risk of the original problem from this determination on Δ_0 cannot be minorized by any other determination. Thus property P prevails. Iteration of this decomposition continues the construction to the point that $D_k = \{(x_1, \cdots, x_q): r_1 < x_i < s_i$, for all $i\}$. An application of (C) with a priori measure $\pi = \prod_{i=1}^{q} \pi_i$ completes the construction. □

The following examples are illustrative of applications to Problem II:

EXAMPLE 3. Let $f(p_1, p_2) = \max(p_1, p_2)$. Then $\delta(X_1, X_2) = \max(X_1/n_1, X_2/n_2)$ is admissible. The estimator may be constructed on $(0, 0)$ and $\{(x_1, x_2): x_1 = n_1$ or $x_2 = n_2\}$ by applications of (A). Let $D_1{}^c$ consist of these points. Take $\Delta_1 = \{(x_1, 0): x_1 = 1, \cdots, n_1 - 1\}$ and $A_1 = \{(p_1, p_2): p_2 = 0\}$. An application of (B) with uniform a priori measure yields the estimator on Δ_1. The same is true for $\Delta_2 = \{(0, x_2): x_2 = 1, \cdots, n_2 - 1\}$. Suppose that $n_1 \geqq n_2$ and let a be the integer part of n_1/n_2. Then $\Delta_k = \{(k-2, x_2): x_2 = 1, \cdots, n_2 - 1\}$ for $k = 3, \cdots, a+2$, $\Delta_{a+3} = \{(x_1, 1): x_1 = a+1, \cdots, n_1 - 1\}$, and so on. In all cases an application of (B) suffices. The a priori measure for the kth stage is a Beta with parameters b and 1, b being the minimum value of the coordinate which is not fixed in Δ_k.

EXAMPLE 4. Let $f(p_1, p_2) = p_1 p_2$. Then $\delta(X_1, X_2) = (X_1/n_1)(X_2/n_2)$ is admissible. The estimator may be constructed on (n_1, n_2) and $\{(x_1, x_2): x_1 x_2 = 0\}$ by applications of (A). Let $D_1{}^c$ consist of these points. Take $\Delta_1 = \{(x_1, n_2) : x_1 = 1, \cdots, n_1 - 1\}$ and $\Delta_2 = \{(n_1, x_2) : x_2 = 1, \cdots, n_2 - 1\}$. In both cases an application of (B) with uniform a prior measure suffices. The construction is completed by an application of (C) with uniform a priori measure.

3. The multinomial case. Let $X = (X_1, \cdots, X_q)$ have a multinomial distribution with parameters n and $p_1, \cdots, p_q$, n known. Let I′ and II′ denote the analogues of Problems I and II. II′ with $q = 2$ and I with $q = 1$ are identical problems. The results are thus contained in Theorem 1. For Problem I′ with $q = 2$, the assertions of Theorem 1 hold with the representation

$$\delta_1(x_1, x_2) = f_1(0), \delta_2(x_1, x_2) = f_2(1), \qquad \text{for } x_1 \leqq r$$

$$\delta_1(x_1, x_2) = f_1(1), \delta_2(x_1, x_2) = f_2(0), \qquad \text{for } x_1 \geqq s$$

$$\delta_i(x_1, x_2) = \frac{\int_0^1 f_i(\hat{p})\hat{p}^{x_1 - r - 1}(1-\hat{p})^{s - x_1 - 1}\pi(dp)}{\int_0^1 \hat{p}^{x_1 - r - 1}(1-\hat{p})^{s - x_1 - 1}\pi(dp)}, \qquad \text{for } r < x_1 < s$$

where $\hat{p} = p$ if $i = 1$ and $1-p$ if $i = 2$ and π is a probability measure with $\pi(\{0\} \cup \{1\}) < 1$. Generalization to cases $q > 2$ follows much the lines of the preceding section. We consider briefly the case $q = 3$.

At the kth stage of construction we suppose that the determination on $D_k{}^c$ has the property P. The boundary densities for the ancillary problem concentrate their mass on lines of support to D_k, in the plain $x_1 + x_2 + x_3 = n$. The passage to stage $k+1$ is achieved by an application of one of the following.

(A′) For a and $b > 0$ take $\Delta_k = \{x : x_1 + a x_2 \leqq b\} \cap D_k$. The determination of the estimator on Δ_k is $\delta_i = f_i(0)$, $i = 1, 2$, and $\delta_3 = f_3(1)$ (or $f(0, 0, 1)$).

(B′) Consider the triangle in the plain $x_1 + x_2 + x_3 = n$, containing D_k, and with sides parallel to the intersection of $x_i + x_j = n$, $i \neq j$, and the plain and supports to D_k. The Δ_k are points on the boundary of the triangle subject to the same constraints applied in (B). The A_k may be taken to be all densities for which Δ_k has mass one. Further, the determination of the estimator on Δ_k may be assumed to be that of a unique Bayes procedure for the ancillary problem with parameter space A_k.

(C′) $\Delta_k = D_k$ and we may suppose that the determination of the estimator on Δ_k is that of a unique Bayes procedure for the ancillary problem.

The proof of the following is identical to that of Theorem 2.

THEOREM 4. *The classes of admissible estimators for Problems* I′ *and* II′, $D_{I'}$ *and* $\mathscr{D}_{II'}$, *are precisely those estimators which can be constructed.* $\mathscr{D}_{I'}$ *and* $\mathscr{D}_{II'}$ *are closed.*

We consider two examples:

EXAMPLE 5. Suppose $f_i(p_i) = p_i$. Then $(\delta_1, \cdots, \delta_2)$, where $\delta_i(X) = X_i/n$ is admissible. For $q = 3$, the vertices of D_0 are handled by applications of (A′), the three sides of D_0 by applications of (B′), and the construction is completed by an application of (C′). In the latter cases, the a priori measure is uniform on the appropriate densities. For $q > 3$ the construction is identical, proceeding from lowest to highest dimensional boundary sets of D_0 and concluding with an application of (C′).

EXAMPLE 6. Let $f(p_1, \cdots, p_q) = \max_i p_i$. We investigate the admissibility of $\delta(X) = \max_i X_i/n$. For $q = 2$ the admissible estimators are those with the representation of Theorem 1. For values of n of 1 through 5 and 7, such a representation is possible. For n even and larger than 4, such a representation is not possible. The value $\frac{1}{2}$ can only be realized when the a priori measure exhausts its mass on the points 0, $\frac{1}{2}$, and 1. For $q > 2$ a similar behavior may be observed. For example, if $q = 3$ and $n = 3j, j \geqq 4$, the estimator is not admissible. Consider its construction. The vertices of D_0 may be handled by applications of (A′). The construction cannot be completed by an application of (C′) because the estimator takes the value $\frac{1}{3}$. If n is even, no application of (B′) is possible because of the value $\frac{1}{2}$. Suppose that n is odd and that the sides of D_0 can be handled by applications of (B'). Again, the construction cannot be completed by an application of (C′). Further application of (B′) is not possible because the value $(n-1)/2n$ cannot be realized.

4. Concluding remarks. (a) The characterizations of the admissible estimators of the preceding sections for Problems II and II′ remain essentially unchanged if the loss is $K(p)W(\delta - f(p))$, where $K(p)$ is positive and finite and W is nonnegative and strictly convex. The same is true for Problems I and I′.

(b) The continuity assumed of f may be somewhat relaxed without the loss of the essentially unique Bayes nature of the admissible estimators or their closure property. For example, if f is continuous on (0, 1) and on [0, 1] with the compactification of $(-\infty, \infty)$, then the bounded admissible estimators are precisely those given and the class is boundedly closed. Also, if f has, for example, a finite number of discontinuities of the first type and is otherwise continuous on [0, 1], the parameter space may be compactified in the intrinsic sense by disconnecting it at the points of discontinuity and then completing it. The admissible estimators for the new problem are exactly the admissible estimators for the original problem. The former class is closed and admits of representations essentially identical to those obtained. In the absence of any continuity assumption, the method of decomposition may be used to construct admissible estimators.

(c) The constructive procedure is easily modified to problems involving truncated parameter spaces. For example, if the p_i in either the binomial or multinomial problems satisfy a known ordering, the construction of all admissible estimators differs only in that certain Δ_k's are disallowed and the a priori measures are appropriately restricted.

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