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ON THE FIRST PASSAGE OF THE INTEGRATED WIENER PROCESS

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The rate of first passage of the integrated Wiener process to $x>0$ is determined in terms of the "$\frac{1}{2}$-winding time" distribution of H. P. McKean, Jr. The probability that the integrated Wiener process is currently at its maximum is approximated.

1. Introduction. Let $e(t)$ be the standard representation of the 1-dimensional Wiener process ($e(0) = 0$). In [2] McKean studied the winding around the origin of the two-dimensional Markov process (u, v) where $v(t) = b+e(t)$ and $u(t) = a+\int_0^t v(s)\,ds$. In addition to many interesting strong limit theorems, he obtained the joint distribution of the "$\frac{1}{2}$ winding time" $\mathfrak{t}_1$ and the hitting place $\mathfrak{h}_1$

$$(1.1)\qquad P_{0,1}(\mathfrak{t}_1 \in dt, \mathfrak{h}_1 \in dh) = \frac{3h \exp[-2(1-h+h^2)/t]}{2^{\frac{1}{2}}\pi t^2} \times \left(\int_0^{4h/t} \frac{e^{-3\theta/2}}{(\pi\theta)^{\frac{1}{2}}}\,d\theta\right) dt\,dh$$

where $\mathfrak{t}_1 = \min(t; t > 0, u(t) = 0)$, $\mathfrak{h}_1 = |v(\mathfrak{t}_1)|$ and $a = 0$, $b = 1$. By means of the scaling $e(t) \to ce(t/c^2)$ for $c > 0$, the requirement $b = 1$ can be converted to $b \neq 0$. In [3] the integral in (1.1) is shown to be expressible in terms of theta-functions.

Whereas [2] deals with the first return of $u(t)$ to its initial value, this note takes up the first passage of $u(t)$ to $x > 0$ when $u(0) = a = 0$ and $v(0) = b \leqq 0$. The assumption $a = 0$ is not restrictive. Let $p(t, \xi, \eta, x, y)$ be the transition density for $(u(t), v(t))$, $\tau_x = \min(t; t > 0, u(t) = x)$ and $\phi_b(x, t) = (d/dt)P(\tau_x \leqq t)$. We show that

(a (Proposition 1)) $\phi_b(x, t) = \frac{1}{2}[3/(2\pi t^3)]^{\frac{1}{2}}(3xt^{-1}-b)\exp[-3(x-bt)^2/2t^3]+ \int_0^\infty d\xi \int_0^t \int_0^\infty \xi P_{0,\xi}(\mathfrak{t}_1 \in ds, \mathfrak{h}_1 \in dh)[p(t-s, 0, b, x, \xi)-p(t-s, 0, b, x, -\xi)]$.

(b (Proposition 2)) As $t \to \infty$, $\phi_0(x, t) \sim \text{const.} \times x^{1/6}t^{-(5/4)}$.

(c) $P(u(t) = \max_{0\leqq s\leqq t} u(s))$ is approximately 0.372 when $a = b = 0$.

2. Basics. We let $P_{a,b}$ be the probability measure of the process (u, v) where $(u(0), v(0)) = (a, b)$ and $g_{0,b}(t, x, y)\,dx\,dy = P_{a,b}(\max_{0\leqq s\leqq t} u(s) = u(t) \in dx, v(t) \in dy)$. As in [2] the transition density for $P_{a,b}(u(t) \in dx, v(t) \in dy)$,

$$p(t, a, b, x, y) = 3^{\frac{1}{2}}/\pi t^2 \exp\left[-\frac{2}{t}(y-b)^2+\frac{6}{t^2}(y-b)(x-a-bt) - \frac{6}{t^3}(x-a-bt)^2\right].$$

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It is well known that for $z \geqq 0$ and $t > 0$

$$(2.1)\quad P_{0,b}(u(t) \in dx,\, v(t) \in dy,\, \max_{0 \leqq s \leqq t} u(s) \leqq z)$$
$$= p(t, 0, b, x, y)dxdy - \int_0^t ds \int_0^\infty \xi g_{0,b}(t-s, z, \xi)p(s, z, \xi, x, y)d\xi\, dxdy.$$

If $x = z = 0$, $b < 0$, $y < 0$, (2.1) is equivalent to (3.1) of [2]. If $y > 0$, we may take the limit $z \to x^+$ and obtain

$$(2.2)\quad g_{0,b}(t, x, y) = p(t, 0, b, x, y) - \int_0^t ds \int_0^\infty \xi g_{0,b}(t-s, x, \xi)p(s, x, \xi, x, y)d\xi.$$

If (2.1) is integrated on x and y we obtain

$$(2.3)\quad P_{0,b}(\max_{0 \leqq s \leqq t} u(s) \leqq z) = 1 - \int_0^t ds \int_0^\infty \xi g_{0,b}(t-s, z, \xi)d\xi.$$

If $z > 0$, $t > 0$, p is a genuine density so $P_{0,b}(u(t) = z) = 0$ and

$$(2.4)\quad P(\max_{0 \leqq s \leqq t} u(s) \leqq z) = P(\tau_z > t) = 1 - P(\tau_z \leqq t).$$

When $-y < 0$, (2.2) can be interpreted as

$$(2.5)\quad 0 = -p(t, 0, b, x, -y) + \int_0^t ds \int_0^\infty \xi g_{0,b}(t-s, x, \xi)p(s, x, \xi, x, -y)d\xi.$$

Following [1] we let $p^*(t, a, b, x, y) = p(t, a, b, x, y) - p(t, a, b, x, -y)$ and $k^*(t, y, \xi) = p(t, x, \xi, x, -y) - p(t, x, \xi, x, y) = (3^{\frac{1}{2}}/\pi t^2)[\exp(-2(y^2 - y\xi + \xi^2)/t) - \exp(-2(y^2 + y\xi + \xi^2)/t)] = p^*(t, 0, -\xi, 0, y)$.
Upon adding (2.2) and (2.5) for $-y < 0$ we obtain

$$(2.6)\quad g_{0,b}(t, x, y) = p^*(t, 0, b, x, y) + \int_0^t ds \int_0^\infty \xi g_{0,b}(t-s, x, \xi)k^*(s, y, \xi)d\xi.$$

Since $k^*(t, y, \xi) \geqq 0$ for positive t, y, and ξ, we have by iteration

$$(2.7)\quad g_{0,b}(t, x, y) \geqq p^*(t, 0, b, x, y) + \int_0^t dr \int_0^\infty \xi p^*(t-r, 0, b, x, \xi)k^*(r, y, \xi)d\xi$$
$$+ \int_0^t ds \int_0^{t-s} dr \int_0^\infty d\xi \int_0^\infty \xi\eta p^*(t-s-r, 0, b, x, \xi)k^*(r, \eta, \xi)$$
$$\cdot k^*(s, \eta, y)d\eta + \cdots.$$

For $b < 0$ and each $x \geqq 0$ the expression on the right-hand side of (2.7), when muliplied by y and integrated on y and t from 0 to ∞, can be seen to be unity. From (2.3) $\int_0^\infty ds \int_0^\infty y g_{0,b}(s, x, y)dy$ cannot exceed unity. Therefore equality must hold in (2.7) almost everywhere (y, t). We do not carry out the above calculation.

Let $\psi(t, x, \xi) = \int_0^\infty y p^*(t, 0, -\xi, x, y)dy$. Since $k^*(t, y, \xi) = p^*(t, 0, -\xi, 0, y)$ so $\psi(t, 0, \xi) = \int_0^\infty y k^*(t, y, \xi)dy$. Therefore, (2.7) with the aforementioned equality a.e. implies

$$(2.8)\quad \int_0^\infty y g_{0,b}(t, x, y)dy = \psi(t, x, -b) + \int_0^t dr \int_0^\infty \xi p^*(t-r, 0, b, x, \xi)\psi(r, 0, \xi)d\xi$$
$$+ \int_0^t ds \int_0^{t-s} dr \int_0^\infty d\xi \int_0^\infty \xi\eta p^*(t-s-r, 0, b, x, \xi)k^*(r, \eta, \xi)$$
$$\cdot \psi(s, 0, \eta)d\eta + \cdots$$

and

$$\text{(2.9)}\quad \int_0^\infty y g_{0,b}(t, 0, y)dy = \psi(t, 0, -b)+\int_0^t dr \int_0^\infty \xi p^*(t-r, 0, b, 0, \xi) \cdot \psi(r, 0, \xi)d\xi+\int_0^t ds \int_0^{t-s} dr \int_0^\infty d\xi \int_0^\infty \xi\eta p^*(t-s-r, 0, b, 0, \xi) \cdot k^*(r, \eta, \xi)\psi(s, 0, \eta)d\eta+\cdots$$

In view of (2.9), (2.8) can be identified as

$$\text{(2.10)}\quad \int_0^\infty y g_{0,b}(t, x, y)dy = \psi(t, x, -b)+\int_0^t ds \int_0^\infty \left(\int_0^\infty y g_{0,-\xi}(s, 0, y)dy\right)p^*(t-s, 0, b, x, \xi)d\xi$$

in view of the associativity of convolution on the time-like variables. If $b < 0$, it is clear that equations (2.3) and (2.4) apply to $z = 0$ providing τ_z is interpreted as t_1 of Section 1 and [2]. Therefore $\int_0^\infty y g_{0,b}(t, 0, y)dy dt = P_{0,b}(\mathrm{t}_1 \in dt)$. By means of the scaling $e \to ce(t/c^2)$

$$P_{0,\xi}(\mathrm{t}_1 \in dt, \mathrm{h}_1 \in dh) = \frac{3h}{\pi 2^{\frac{1}{2}} t^2} \exp\left(-2(\xi^2-\xi h+h^2)/t\right) \int_0^{4\xi h/t} \frac{e^{-3\theta/2}}{(\pi\theta)^{\frac{1}{2}}}\, d\theta\, dt dh.$$

Furthermore, evaluation of $\psi(t, x, -b)$ is routine. Therefore, we have

PROPOSITION 1. *If $b \leqq 0$ and $x > 0$*

$$\text{(2.11)}\quad \phi_b(x, t) = \frac{d}{dt} P(\tau_x \leqq t) = \int_0^\infty y g_{0,b}(t, x, y)dy = \psi(t, x, -b)+\int_0^\infty d\xi \int_0^t \int_0^\infty \xi P_{0,\xi}(\mathrm{t}_1 \in ds, \mathrm{h}_1 \in dh) \cdot [p(t-s, 0, b, x, \xi)-p(t-s, 0, b, x, -\xi)]$$

where $\psi(t, x, -b) = (3/8\pi t^3)^{\frac{1}{2}}(3xt^{-1}-b) \exp(-3(x-bt)^2/2t^3)$,

$$P_{0,\xi}(\mathrm{t}_1 \in ds, \mathrm{h}_1 \in dh) = \int_0^{4\xi h/s} \frac{3h}{\pi 2^{\frac{1}{2}} s^2 (\pi\theta)^{\frac{1}{2}}} \exp\left[-\frac{2}{s}(\xi^2-\xi h+h^2)-\frac{3\theta}{2}\right] d\theta\, ds dh,$$

and

$$p(r, 0, b, x, \xi) = \frac{3^{\frac{1}{2}}}{(\pi r^2)} \exp\left[-\frac{2}{r}(\xi-b)^2+\frac{6}{r^2}(\xi-b)(x-bt)-\frac{6}{r^3}(x-bt)^2\right].$$

3. Asymptotic behavior of the first passage density. Since the integral in (2.11) is quite complicated we describe its asymptotic behavior as $t \to \infty$ in the case $x > 0$, $b = 0$. We have

$$\text{(3.1)}\quad \int_0^t ds \int_0^\infty d\xi \int_0^\infty dh \int_0^{4h\xi/s} \frac{3h\xi}{\pi 2^{\frac{1}{2}} s^2 (\pi\theta)^{\frac{1}{2}}} \exp\left(-\frac{2}{s}(\xi^2-\xi h+h^2)-\frac{3\theta}{2}\right) \cdot p^*(t-s, 0, 0, x, \xi)d\theta$$

where $p^*(r, 0, 0, x, \xi) = 2(3)^{\frac{1}{2}}/(\pi r^2) \exp(-2\xi^2/r - 6x^2/r^3) \sinh(6\xi x/r^2)$ by the previous definitions. By means of the successive substitutions $h \to s^{\frac{1}{2}}h$, $\xi \to s^{\frac{1}{2}}\xi$, $t-s = w^{-\frac{1}{3}}$, $h \to h\xi$, $\xi \to (tw^{\frac{1}{3}} - h + h^2)^{-\frac{1}{2}}\xi$, $h \to t^{\frac{1}{2}}h$, $\theta \to t^{-\frac{1}{2}}\theta$, and $h \to w^{\frac{1}{6}}h$ expression (3.1) can be seen to be

$$(3.2)\qquad \frac{3(6)^{\frac{1}{2}}}{\pi^2 t^{5/4}} \int_{t^{-3}}^{\infty} \frac{dw}{w} \int_0^{\infty} d\xi \int_0^{\infty} dh \int_0^{\alpha(w,\xi,h,t)} \frac{h\xi^3}{(1 - w^{-\frac{1}{6}}t^{-\frac{1}{2}}h + h^2)^2(\pi\theta)^{\frac{1}{2}}}$$

$$\cdot \exp\left(-2\xi^2 - 6wx^2 - \frac{3\theta}{2t^{\frac{1}{2}}}\right) \sinh\left(6x\xi w^{\frac{1}{2}}\left[\frac{1 - t^{-1}w^{-\frac{1}{3}}}{1 - w^{-\frac{1}{6}}t^{-\frac{1}{2}}h + h^2}\right]^{\frac{1}{2}}\right) d\theta$$

where $\alpha(w, \xi, h, t) = 4h\xi^2 w^{-\frac{1}{6}}(1 - w^{-\frac{1}{6}}t^{-\frac{1}{2}}h + h^2)^{-1}$. If $w \geqq t^{-3}$, $w^{-\frac{1}{6}}t^{-\frac{1}{2}} \leqq 1$ so $1 - w^{-\frac{1}{6}}t^{-\frac{1}{2}}h + h^2 \geqq (1+h^2)/2$. Also, $1 - t^{-1}w^{-\frac{1}{3}} \leqq 1 - w^{-\frac{1}{6}}t^{-\frac{1}{2}}h + h^2$. If I_t is the indicator function of the region where $w > t^{-3}$, the integrand in (3.2) is bounded by

$$(3.3)\qquad w^{-1}h^3(1+h^2)^{-2}\theta^{-\frac{1}{2}} \exp(-2\xi^2 - 6wx^2) \sinh(6x\xi w^{\frac{1}{2}})4I_t(w, \xi, h, \theta).$$

The limit as $t \to \infty$ of the integrand in (3.2) is a constant multiple of (3.3) so an integration of that limit shows the integrability of (3.3) and shows that limit as $t \to \infty$ and integration operations may be interchanged. The result of this is

$$\int_0^{\infty} dw \int_0^{\infty} d\xi \int_0^{\infty} dh \int_0^{4h\xi^2 w^{-\frac{1}{6}}(1+h^2)^{-1}} \frac{h\xi^3}{(1+h^2)^2(\pi\theta)^{\frac{1}{2}}} \exp(-2\xi^2 - 6wx^2)$$

$$\cdot \sinh\left(\frac{6x\xi w^{\frac{1}{2}}}{(1+h^2)^{\frac{1}{2}}}\right) d\theta.$$

The series for sinh and integration on θ give

$$\int_0^{\infty} dw \int_0^{\infty} d\xi \int_0^{\infty} \sum_{n=0}^{\infty} \frac{4h^{\frac{3}{2}}\xi^{2n+5}w^{n+5/12-1}(6x)^{2n+1}}{\pi^{\frac{1}{2}}(1+h^2)^{n+3}(2n+1)!} \exp(-2\xi^2 - 6wx^2)dh.$$

The ξ and w integrations result in $\Gamma(k)$ for various k; $\int_0^{\infty} 2h^{\frac{3}{2}}/(1+h^2)^{n+3}dh = B(n+\frac{7}{4}, \frac{5}{4})$; the first term in (2.11) tends to zero more rapidly than $t^{-\frac{5}{4}}$. We therefore have

Proposition 2. *If $x > 0$, as $t \to \infty$*

$$\phi_0(x, t) \sim t^{-1}\left(\frac{x^2}{t^3}\right)^{1/12} \frac{9(6)^{\frac{1}{12}}}{4\pi^2} \frac{\Gamma(5/4)\Gamma(7/4)\Gamma(5/12)}{\Gamma(3/2)} {}_2F_1(\tfrac{5}{12}, \tfrac{7}{4}; \tfrac{3}{2}; \tfrac{3}{4}).$$

4. The probability that u is at its maximum. From the definition of $g_{a,b}$

$$(4.1)\qquad P_{a,b}(u(t) = \max\nolimits_{0 \leqq s \leqq t} u(s)) = \textstyle\int_0^{\infty}\int_0^{\infty} g_{a,b}(x, y, t)\, dx dy.$$

If $a = b = 0$ this probability is independent of t and is the limit as $t \to \infty$ of (4.1) n other cases.

We integrate the first two terms on the right-hand side of (2.7)

$$\int_0^\infty \int_0^\infty p^*(t, 0, 0, x, y)\, dxdy =$$

$$\frac{3^{\frac{1}{2}}}{\pi t^2} \int_0^\infty \int_0^\infty \left[\exp\left(-\frac{6x^2}{t^3} + \frac{6xy}{t^2} - \frac{2y^2}{t}\right) - \exp\left(-\frac{6x^2}{t^3} - \frac{6xy}{t^2} - \frac{2y^2}{t}\right)\right] dxdy.$$

The substitutions $x \to 3^{-\frac{1}{2}} t^{\frac{3}{2}} x$, $y \to \frac{1}{2} t^{\frac{1}{2}} y$ yield

$$1/2\pi \int_0^\infty \int_0^\infty \left[\exp\left(-\tfrac{1}{2}(x^2 + (y - 3^{\frac{1}{2}}x)^2)\right) - \exp\left(-\tfrac{1}{2}(x^2 + (y + 3^{\frac{1}{2}}x)^2)\right)\right] dxdy.$$

A difference of two integrals may be formed, changed to polar coordinates and evaluated to give

$$\int_0^\infty \int_0^\infty p^*(t, 0, 0, x, y)\, dxdy = \tfrac{5}{12} - \tfrac{1}{12} = \tfrac{1}{3}.$$

The integral of the second term on the right-hand side of (2.7) is

$$\left(\frac{3}{\pi}\right)^2 \int_0^\infty dx \int_0^\infty dy \int_0^t ds \int_0^\infty \xi s^{-2}(t-s)^{-2} \exp\left[-\frac{2}{s}y^2 - \frac{2}{s}\xi^2 - \frac{6x^2}{(t-s)^3} - \frac{2\xi^2}{t-s}\right]$$

$$\cdot \left[2 \sinh\left(\frac{2y\xi}{s}\right)\right]\left[2 \sinh\left(\frac{6x\xi}{(t-s)^2}\right)\right] d\xi.$$

With the substitutions $y \to \frac{1}{2}s^{\frac{1}{2}}y$, $\xi \to (t-s)^{\frac{1}{2}}\xi$, $x \to ((t-s)/12)^{\frac{1}{2}}x$, $s \to ts$, and $(1-s)/s = w$, this is just

$$(4.2) \quad \frac{3^{\frac{1}{2}}}{4\pi^2} \int_0^\infty dx \int_0^\infty dy \int_0^\infty dw \int_0^\infty \frac{\xi w^{\frac{1}{2}}}{w+1} \exp\left(-\frac{y^2}{2} - \frac{x^2}{2} - 2(w+1)\xi^2\right)$$

$$\cdot [2 \sinh (y\xi w^{\frac{1}{2}})][2 \sinh (3^{\frac{1}{2}} x\xi)]\, d\xi.$$

If $c > 0$, $\int_0^\infty e^{-z^2/2} \sinh cz\, dz = e^{c^2/2} \int_0^c e^{-z^2/2} dz$ so (4.2) is

$$\frac{3^{\frac{1}{2}}}{\pi^2} \int_0^\infty \frac{w^{\frac{1}{2}} dw}{w+1} \int_0^\infty \xi \exp[-(3w+1)\xi^2/2] \left[\int_0^{\xi w^{\frac{1}{2}}} e^{-y^2/2} dy \int_0^{3^{\frac{1}{2}}\xi} e^{-x^2/2} dx\right] d\xi.$$

We may integrate by parts so that $\xi \exp[-(3w+1)\xi^2/2]$ is integrated and the product of integrals is differentiated. This gives

$$\frac{3^{\frac{1}{2}}}{\pi^2} \int_0^\infty \frac{w^{\frac{1}{2}} w^{\frac{1}{2}} dw}{(w+1)(3w+1)} \int_0^\infty d\xi \int_0^{3^{\frac{1}{2}}\xi} \exp[-\tfrac{1}{2}(4w+1)\xi^2 - \tfrac{1}{2}x^2]\, dx$$

$$+ \frac{3^{\frac{1}{2}}}{\pi^2} \int_0^\infty \frac{w^{\frac{1}{2}} 3^{\frac{1}{2}} dw}{(w+1)(3w+1)} \int_0^\infty d\xi \int_0^{\xi w^{\frac{1}{2}}} \exp[-\tfrac{1}{2}(3w+4)\xi^2 - \tfrac{1}{2}y^2]\, dy.$$

By means of the usual change to polar coordinates this is

$$\frac{3^{\frac{1}{2}}}{\pi^2} \int_0^\infty \frac{w \tan^{-1}[3/(4w+1)]^{\frac{1}{2}}}{(w+1)(3w+1)(4w+1)^{\frac{1}{2}}} dw + \frac{3}{\pi^2} \int_0^\infty \frac{w^{\frac{1}{2}} \tan^{-1}[w/(3w+4)]^{\frac{1}{2}}}{(w+1)(3w+1)(3w+4)^{\frac{1}{2}}} dw.$$

The rationalizing substitutions $\theta^2 = 3/(4w+1)$ and $\theta^2 = w/(3w+4)$ give

$$\frac{2}{\pi^2}\int_0^{3^{\frac{1}{2}}} \frac{(3-\theta^2)\tan^{-1}\theta}{(1+\theta^2)(9+\theta^2)}\,d\theta + \frac{24}{\pi^2}\int_0^{1/3^{\frac{1}{2}}} \frac{\theta^2\tan^{-1}\theta}{(1+\theta^2)(1+9\theta^2)}\,d\theta.$$

After integration by parts we have, finally,

$$\text{(4.3)}\qquad \frac{24}{\pi^2}\int_0^{3^{\frac{1}{2}}} \frac{\theta(\tan^{-1}\theta)^2}{(9+\theta^2)^2}\,d\theta + \frac{12}{\pi^2}\left[\frac{1}{12}\left(\frac{\pi}{6}\right)^2 - \int_0^{1/3^{\frac{1}{2}}} \frac{2\theta(\tan^{-1}\theta)^2}{(1+9\theta^2)^2}\,d\theta\right].$$

The summands in (4.3) are approximately 0.021 and 0.018, respectively. Since (4.1) is less than

$$P(u(t) > 0,\, v(t) > 0) = \textstyle\int_0^\infty \int_0^\infty p(t, 0, 0, x, y)\,dxdy = \frac{5}{12}.$$

and $0.333+0.021+0.018 = 0.372$, we see that

$$P(u(t) = \max_{0 \leq s \leq t} u(s) \mid u(t) > 0,\, v(t) > 0)$$

is quite close to unity.

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