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# BRANCHING PROCESSES WITH RANDOM ENVIRONMENTS, II: LIMIT THEOREMS[1]

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**0. Introduction.** We refer to [1] for the basic set up, notation and terminology. In this sequel to [1] we elaborate several analogs of the known limit theorems for simple Galton-Watson processes.

The B.P.R.E. will be labeled supercritical, critical, or subcritical according as $E \log \varphi'_{\zeta_0}(1) > 0, = 0$ or $< 0$ respectively. The supercritical case is distinguished in that extinction of the population is not a certain event for almost every realization of the environmental process. Here the classical Martingale theorem has a natural extension (see Theorem 1 of Section 1). In order to justify the rest of the terminology separating the critical from the subcritical case, we recall the following facts concerning one type Galton-Watson processes.

A subcritical simple branching process has the property that

$$\lim_{n\to\infty} P\{Z_n = k \mid Z_n \neq 0\} = a_k, \qquad k = 1, 2, \cdots \tag{1}$$

exists and $\{a_k\}$ determines a genuine discrete probability density while in the critical case the limit in (1) identically vanishes. In fact, in the latter case $E(Z_n \mid Z_n \neq 0) \sim cn$ provided $\varphi''(1) < \infty$ where $\varphi(s)$ is the progeny p.g.f. of the process. (Here $c$ is an appropriate positive constant.) More specifically, we have the limit law

$$E\left(\exp\left[-\lambda\frac{Z_n}{n}\right] \middle| Z_n \neq 0\right) \to \frac{1}{1+\lambda a} \tag{2}$$

for suitable $a > 0$. This is commonly known as Kolmogorov's limit law while priority for (1) is generally attributed to Yaglom.

In order to develop a version of (1) in the context of B.P.R.E. we impose additional conditions on the environmental process $\{\zeta_t, t \geqq 0\}$.

DEFINITION 1. The stationary ergodic process $\zeta_t$ is said to be *exchangeable* if the vector random variables $(\zeta_i, \zeta_{i+1}, \cdots, \zeta_{i+n})$ and $(\zeta_{n+i}, \zeta_{n+i-1}, \cdots, \zeta_i)$ are identically distributed for each $i \geqq 0$ and $n \geqq 0$.

When $\zeta_t$, $t \geqq 0$ consists of i.i.d. random variables then $\zeta_t$ is manifestly an exchangeable process. Another example of an exchangeable process arises when $(\zeta_t, t \geqq 0)$ is a stationary reversible ergodic Markov chain. Our result here is that the sequence of random probability distributions viz. $\{Z_n \mid Z_n \neq 0, \bar{\zeta}\}$ converges in law to a random probability distribution. More precisely,

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THEOREM 2. *Let $(\zeta_t, t \geqq 0)$ be an exchangeable process in accordance with Definition* 1. *Suppose $E|\log \varphi'_{\zeta_0}(1)| < \infty$. In the subcritical* B.P.R.E., *i.e., where $E(\log \varphi'_{\zeta_0}(1)) < 0$ holds, there exists a* p.g.f. $\tilde{Y}(s, \bar{\zeta})$ *for* a.e. $\bar{\zeta}$ *such that*

$$E(s^{Z_n} \mid Z_n \neq 0, \mathbb{F}(\bar{\zeta})) \to \tilde{Y}(s, \bar{\zeta}) \quad \text{in law} \tag{3}$$

*as $n \to \infty$. In the critical case, that is where $E(\log \varphi'_{\zeta_0}(1)) = 0$ then* (3) *persists but* $\tilde{Y}(s, \bar{\zeta}) \equiv 0$ a.s.

N. Kaplan [3] has provided examples to show that the limit law (3) cannot be strengthened to a probability one statement.

The subcritical Galton-Watson process is characterized by the property that the mean number of progeny per parent is less than 1. The analog of this property in the B.P.R.E. case is the content of the following theorem.

THEOREM 3. *Let $\zeta_t, t \geqq 0$ be an exchangeable process. Suppose $E|-\log(1-\varphi_{\zeta_0}(0))|$ and $E|\log \varphi'_{\zeta_0}(1)|$ are finite. Then*

$$m(\bar{\zeta}) = \lim_{n\to\infty} \frac{[1-\varphi_{\zeta_n}(\varphi_{\zeta_{n-1}}(\cdots \varphi_{\zeta_0}(0))\cdots)]}{[1-\varphi_{\zeta_n}(\varphi_{\zeta_{n-1}}(\cdots \varphi_{\zeta_1}(0))\cdots)]}$$

*exists and is $\leqq 1$ a.s. The* B.P.R.E. *is subcritical* iff $P(\bar{\zeta}, m(\bar{\zeta}) < 1) > 0$.

REMARK. Notice that for the special case $\varphi_{\zeta_i}(s) = \varphi(s)$, independent of $\zeta_i$, we have $m(\bar{\zeta}) = \varphi'(1)$.

We have also determined (subject to a mild moment condition) the exact rate of approach to zero of $1-\varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots(\varphi_{\zeta_n}(s)\cdots)))$ (see Theorem 4 of Section 2).

The anticipated generalization of (2) to the critical case B.P.R.E. probably reads

$$E\left(\exp\left[-\frac{\lambda Z_n}{a_n(\bar{\zeta})}\right] \middle| Z_n \neq 0, \mathbb{F}(\bar{\zeta})\right) \to \frac{1}{1+\lambda} \quad \text{in law} \tag{4}$$

as $n \to \infty$ where $a_n(\bar{\zeta})$ is an appropriate sequence of rv's which increases to $\infty$ w.p. 1. Subject to mild conditions on $\varphi'_{\zeta_0}(1)$ and $\varphi''_{\zeta_0}(1)$, the validity of (4) reduces equivalently to establishing that

$$\frac{1}{\sum_{i=1}^{n} P_i} \sum_{j=1}^{n} \frac{P_j{}^2}{\sum_{k=1}^{j} P_k} \to 0 \tag{5}$$

in probability as $n \to \infty$ where $\mathrm{P}_m = \prod_{i=0}^{m-1} \varphi'_{\zeta_i}(1)$. The details are set forth in Section 3. Again Kaplan [3] has shown that (5) in general cannot occur with probability 1. It follows, for the critical case, that $1-\varphi_{\zeta_n}(\varphi_{\zeta_{n-1}}(\cdots(\varphi_{\zeta_0}(0))$ may not tend to 0 w.p. 1 although convergence in probability is assured since the reversed sequence

$$1-\varphi_{\zeta_0}(\cdots(\varphi_{\zeta_{n-1}}(\varphi_{\zeta_n}(0))\cdots) = \text{probability of no extinction by generation } n$$

certainly converges to zero w.p. 1.

On the other hand, the quantity $1-\varphi_{\zeta_n}(\varphi_{\zeta_{n-1}}(\cdots(\varphi_{\zeta_0}(0))$ does converge to zero w.p. 1 in the subcritical case as is seen on the basis of Theorem 5.

**1. Supercritical case.** This section extends the basic limit theorem of Kesten and Stigum [2] on supercritical Galton-Watson process to a B.P.R.E. Recall that when a B.P.R.E. is supercritical $P\{\bar{\zeta}:q(\bar{\zeta}) < 1\} = 1$. (Smith and Wilkinson [4] refer to this as the immortal case.) We assume throughout this section that the hypothesis of Theorem 3 of [1] holds, namely, $E|\log(1-\varphi_{\zeta_0}(0))| < \infty$ and $E(\log \varphi'_{\zeta_0}(1))^- < E(\log \varphi'_{\zeta_0}(1))^+ \leqq \infty$. These stipulations, according to that Theorem 3, make the process supercritical. We know from the results of Section 3 in [1] that

$$P\{\omega: Z_n \to \infty \mid \bar{\zeta}\} = 1-q(\bar{\zeta}) > 0 \qquad \text{a.s.}$$

We shall now evaluate the rate of growth of $Z_n$ on the set of nonextinction.

In complete analogy with the theorem of Kesten and Stigum [3] on supercritical Galton-Watson process we obtain

THEOREM 1. *Let $W_n = Z_n P_n^{-1}$ where for $n \geqq 1$, $P_n = \prod_{j=0}^{n-1} \varphi'_{\zeta_j}(1)$ and $P_0 = Z_0 = 1$. Then, the family $\{W_n; (\mathbb{F}_n(\bar{\zeta}); n = 0, 1, 2, \cdots\}$ constitutes a nonnegative martingale and hence* $\lim_{n\to\infty} W_n = W$ *exists* a.s. *Suppose, in addition, that*

$$E\{(\varphi'_{\zeta_0}(1))^{-1} \textstyle\sum_{j=2}^{\infty} p_{\zeta_0}(j) j \log j\} < \infty.$$

*Then,*

(i) $\lim_{n\to\infty} E(e^{-uW_n} \mid \mathbb{F}(\bar{\zeta})) = \psi(u,\bar{\zeta})$ *where $\psi(u,\bar{\zeta})$ is the unique solution of the functional equation ($T$ is the shift operator on the environmental process)*

$$(6) \qquad \psi(u,\bar{\zeta}) = \varphi_{\zeta_0}\left(\psi\left(\frac{u}{\varphi'_{\zeta_0}(1)}, T\bar{\zeta}\right)\right) \qquad \text{a.s.}$$

*among those satisfying* $\lim_{u\downarrow 0} u^{-1}[1-\psi(u,\bar{\zeta})] = 1$,

(ii) $E(W \mid \mathbb{F}(\bar{\zeta})) = 1$

(iii) $P(W = 0 \mid \mathbb{F}(\bar{\zeta})) = q(\bar{\zeta})$ a.s.

REMARK. The proof of (i) and (ii) given below is not the shortest we know but the method is used again later in the analysis of the subcritical case (see Theorem 5).

PROOF. The martingale property is readily verified from the definition of the process. Also (iii) follows easily from (i) and (ii) by letting $u \to \infty$ in (6) and using Theorem 6 of [1]. We now turn to (i) and (ii). Define

$$g_n(u,\bar{\zeta}) = E(e^{-uW_n} \mid \bar{\zeta}) \equiv \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots(\varphi_{\zeta_{n-1}}(e^{-u/P_n})), \qquad \text{for } n \geqq 1,$$

$$g_0(u,\bar{\zeta}) = e^{-u},$$

$$\psi_n(u,\bar{\zeta}) = u^{-1}|g_{n+1}(u,\bar{\zeta})-g_n(u,\bar{\zeta})| \qquad \text{for } n \geqq 0$$

and

$$\textstyle\sum_{n=0}^{\infty} \psi_n(u,\bar{\zeta}) = K(u,\bar{\zeta}).$$

We shall show that $K(u,\bar{\zeta}) < \infty$ for $u > 0$ a.s. $\bar{\zeta}$ and

$$(7) \qquad \lim_{u\downarrow 0} K(u,\bar{\zeta}) = 0.$$

Assuming (7) proved for the moment, it follows that

(a) for almost all $\bar{\zeta}$, $\lim_{n\to\infty} g_n(u, \bar{\zeta}) = \psi(u,\bar{\zeta})$ exists. (This is, of course, manifest from the fact $W_n \to W$ a.s.)

(b) $\lim_{u\downarrow 0} |u^{-1}(1-\psi(u,\bar{\zeta}))-1| \leqq \limsup_{u\downarrow 0} |u^{-1}(1-e^{-u})-1|$

$$+\limsup_{u\downarrow 0} u^{-1}|g_0(u,\bar{\zeta})-\psi(u,\bar{\zeta})| \leqq 0+\limsup_{u\downarrow 0} K(u,\bar{\zeta}) = 0$$

thus implying (i) and (ii) except for uniqueness.

We now embark on the proof of (7). Since for any p.g.f. $f(x)$ with $f'(1-)= m, f(e^{-u/m}) \geqq e^{-u}$ by convexity of $e^{-x}$ for $x \geqq 0$ we see that $g_{n+1}(u, \bar{\zeta}) \geqq g_n(u, \bar{\zeta})$ and so

$$\psi_n(u,\bar{\zeta}) = u^{-1}[g_{n+1}(u,\bar{\zeta})-g_n(u,\bar{\zeta})].$$

By the mean value theorem,

$$\psi_n(u,\bar{\zeta}) \leqq u^{-1}\varphi'_{\zeta_0}(1)\left[g_n\left(\frac{u}{\varphi'_{\zeta_0}(1)}, T\bar{\zeta}\right)-g_{n-1}\left(\frac{u}{\varphi'_{\zeta_0}(1)}, T\bar{\zeta}\right)\right]$$

$$= \psi_{n-1}\left(\frac{u}{\varphi'_{\zeta_0}(1)}, T\bar{\zeta}\right) \qquad \text{for each } n \geqq 1$$

and on iteration we get

$$\psi_n(u,\bar{\zeta}) \leqq \psi_0\left(\frac{u}{P_n}, T^n\bar{\zeta}\right). \tag{8}$$

Now

$$\psi_0\left(\frac{u}{P_n}, T^n\bar{\zeta}\right) = \left(\frac{u}{P_n}\right)^{-1}[\varphi_{\zeta_n}(e^{-u/P_{n+1}})-e^{-u/P_n}]$$

$$= \left(\frac{u}{P_n}\right)^{-1}\left[\varphi_{\zeta_n}(e^{-u/P_{n+1}})-1+\frac{u}{P_n}\right]+\left(\frac{u}{P_n}\right)^{-1}\left[1-\frac{u}{P_n}-e^{-u/P_n}\right].$$

But

$$\left(\frac{u}{P_n}\right)^{-1}\left[1-\frac{u}{P_n}-e^{-u/P_n}\right] \leqq \frac{1}{2}\left(\frac{u}{P_n}\right).$$

Since $\mu = E\log\varphi'_{\zeta_0}(1) > 0$ there exists (by the ergodic theorem) for almost all $\bar{\zeta}$ an integer $N(\bar{\zeta})$ such that $n \geqq N(\bar{\zeta})$ implies $\log P_n \geqq n\mu/2$ or $P_n^{-1} \leqq r^n$ where $r = e^{-\mu/2} < 1$. Thus $\sum_1^\infty P_n^{-1} < \infty$ for almost all $\bar{\zeta}$. Again for any p.g.f. $f(x)$

$$\frac{1-f(e^{-u})}{u} = \frac{1-f(e^{-u})}{1-e^{-u}}\,\frac{1-e^{-u}}{u}$$

and since $(1-e^{-u})/u$ is decreasing in $u$ for small $u > 0$, we can find a $u_0 > 0$ such that for $0 \leqq u \leqq u_0$ and $n \geqq N(\bar{\zeta})$,

$$\left(\frac{u}{P_n}\right)^{-1}\left[\varphi_{\zeta_n}(e^{-u/P_{n+1}})-1+\frac{u}{P_n}\right] \leqq \left[1-\frac{1-\varphi_{\zeta_n}(e^{-ur^{n+1}})}{\varphi'_{\zeta_n}(1)ur^{n+1}}\right]. \tag{9}$$

Now

$$\begin{aligned}
E\sum_{n=0}^{\infty}&\left[1-\frac{1-\varphi_{\zeta_n}(e^{-ur^{n+1}})}{\varphi'_{\zeta_n}(1)ur^{n+1}}\right]\\
&=\sum_{n=0}^{\infty}E\left[1-\frac{1-\varphi_{\zeta_n}(e^{-ur^{n+1}})}{\varphi'_{\zeta_n}(1)ur^{n+1}}\right] \qquad \text{(by nonnegativity of terms)}\\
&=\sum_{n=0}^{\infty}E\left[1-\frac{1-\varphi_{\zeta_0}(e^{-ur^{n+1}})}{\varphi_{\zeta_0}(1)ur^{n+1}}\right] \qquad \text{(by stationarity)}\\
&=E\left(\sum_{n=0}^{\infty}\left[1-\frac{1-\varphi_{\zeta_0}(e^{-ur^{n+1}})}{\varphi'_{\zeta_0}(1)ur^{n+1}}\right]\right) \qquad \text{(by nonnnegativity of terms)}\\
&=E_{\zeta_0}\left\{\sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty}\left[\ -\frac{1-e^{-jur^{n+1}}}{jur^{n+1}}\right]\frac{jp_{\zeta_0}(j)}{\varphi'_{\zeta_0}(1)}\right)\right\}\\
&=E_{\zeta_0}\left\{\sum_{j=1}^{\infty}\frac{jp_{\zeta_0}(j)}{\varphi'_{\zeta_0}(1)}\sum_{n=0}^{\infty}\left[1-\frac{1-e^{-jur^{n+1}}}{jur^{n+1}}\right]\right\}.
\end{aligned}$$

Now for each $u > 0$ and $r < 1$ we claim

$$\sup_{j\geqq 2}(\log j)^{-1}\sum_{n=0}^{\infty}\left[1-\frac{1-e^{-jur^{n+1}}}{jur^{n+1}}\right] < \infty. \tag{10}$$

A quick way to check (10) is to note

(i) $\dfrac{1-e^{-x}}{x}$ is decreasing for $x > 0$ and hence

(ii) $$\sum_{n=0}^{\infty}\left[1-\frac{1-e^{-jur^{n+1}}}{jur^{n+1}}\right] \leqq \int_0^{\infty}\left[1-\frac{1-e^{-jur^x}}{jur^x}\right]dx = -(\log r)^{-1}\int_0^{ju}\left(1-\frac{1-e^{-y}}{y}\right)\frac{dy}{y}$$

and

(iii) $$\lim_{j\to\infty}(\log j)^{-1}\int_0^{ju}\left(1-\frac{1-e^{-y}}{y}\right)\frac{dy}{y} = 1.$$

Invoking our hypothesis we see that

$$E\left\{\sum_{n=0}^{\infty}\left(\frac{u}{P_n}\right)^{-1}\left[\varphi_{\zeta_n}(e^{-u/P_{n+1}})-1+\frac{u}{P_n}\right]\right\} < \infty$$

and hence for almost all $\bar{\zeta}$

$$\sum_{n=0}^{\infty}\left(\frac{u}{P_n}\right)^{-1}\left[\varphi_{\zeta_n}(e^{-u/P_{n+1}})-1+\frac{u}{P_n}\right]<\infty.$$

Clearly the last conclusion also implies by dominated convergence that

$$\lim_{u\downarrow 0}\sum_{n=0}^{\infty}\left(\frac{u}{P_n}\right)^{-1}\left[\varphi_{\zeta_n}(e^{-u/P_n})-1+\frac{u}{P_n}\right]=0.$$

This establishes (7).

We now turn to the proof of the uniqueness part. Let $\psi_i(u,\bar{\zeta})$ for $i = 1, 2$, satisfy for almost all $\bar{\zeta}$

$$\psi_i(u,\bar{\zeta}) = \varphi_{\zeta_0}\left(\psi_i\left(\frac{u}{\varphi'_{\zeta_0}(1)}, T\bar{\zeta}\right)\right) \tag{11}$$

$$\lim_{u\downarrow 0} u^{-1}(1-\psi_i(u,\bar{\zeta})) = 1.$$

Let $f(u,\bar{\zeta}) = u^{-1}|\psi_1(u,\bar{\zeta})-\psi_2(u,\bar{\zeta})|$. Then, it follows from the mean value theorem

$$f(u,\bar{\zeta}) \leqq f\left(\frac{u}{\varphi'_{\zeta_0}(1)}, T\bar{\zeta}\right)$$

and an iteration yields

$$f(u,\bar{\zeta}) \leqq f\left(\frac{u}{P_n}, T^n\bar{\zeta}\right).$$

Fix a $u > 0$. Let $A_n = \{\bar{\zeta}:\log P_k \geqq k\mu_0/2 \text{ for } k \geqq n\}$. Then by the ergodic theorem we have $P(A_n)\uparrow 1$. On $A_n$

$$f(u,\bar{\zeta}) \leqq X_n \equiv \sup_{v\leqq ur^n} f(v, T^n\bar{\zeta}) \tag{12}$$

where $r = e^{-\mu_0/2}$. But by stationarity $X_n$ has the same distribution as $X_n' \equiv \sup_{v\leqq ur^n} f(v,\bar{\zeta})$. Since, $\lim_{u\downarrow 0} u^{-1}(1-\psi_i(u,\bar{\zeta})) = 1$ for $i = 1, 2$ we have $\lim_{u\downarrow 0} f(u,\bar{\zeta}) = 0$ and hence $X_n' \to 0$ with probability one. This implies that $X_n \to 0$ in probability and there exists a fixed nonrandom sequence $\{n_j\}$ of integers $\to\infty$ such that $X_{n_j}\to 0$ with probability one. Now referring to (12) we see that $f(u,\bar{\zeta}) \leqq X_{n_j}$ on $A_{n_j}$, $X_{n_j}\to 0$ for almost all $\bar{\zeta}$, $A_{n_j}\uparrow$ and $PA_{n_j})\uparrow 1$. All these facts together entail $f(u,\bar{\zeta}) = 0$ for almost all $\bar{\zeta}$ and $u > 0$, proving uniqueness.

REMARK 1. If in Theorem 7 we assume the stronger condition that $\eta_j = (\varphi''_{\zeta_j}(1))(\varphi'_{\zeta_j}(1))^{-2}$ satisfies $\sum_{j=1}^{\infty}\eta_j r^j < \infty$ a.s. for some fixed $r$, $0 \leqq r < \delta < 1$, $\delta > 0$ then we can assert the mean square convergence of $W_n \to W$ when conditioned on $\mathbb{F}(\bar{\zeta})$, i.e., $\lim_{n\to\infty} E[(W_n - W)^2 \mid \mathbb{F}(\bar{\zeta})] = 0$ a.s. The condition $E\eta_j < \infty$ is clearly sufficient for this.

REMARK 2. It is tempting to conjecture that the weaker condition $\sum_{j=0}^{\infty}(\varphi'_{\zeta_0}(1))^{-1} p_{\zeta_0}(j)\, j\log j < \infty$ a.s. is both necessary and sufficient for $W(\bar{\zeta})$ to be nondegenerate.

**2. The concepts of critical and subcritical.** In the classical Galton-Watson case the process is said to be supercritical, critical or subcritical according as the mean of the offspring distribution satisfies $m > 1$, $= 1$, or $< 1$. In the supercritical case the process goes to $\infty$ with positive probability. For the subcritical Galton-Watson process, although extinction becomes certain, the distribution of the process conditioned on nonextinction approaches a nondegenerate limit law while for the critical case this limit is a null distribution. We use the corresponding criterion to distinguish the subcritical and critical cases in the B.P.R.E. model. More precisely,

DEFINITION 2.1. Assume $P(q(\bar{\zeta}) = 1) = 1$. If the sequence $\xi_n(\omega)$ of random elements of $l_\infty$ (space of convergent series) where

$$\xi_n(\omega) = \{P(Z_n = k \mid Z_n \neq 0, \zeta_0, \zeta_1, \cdots, \zeta_{n-1}); k = 1, 2, \cdots\} \tag{13}$$

converges in law as $n \to \infty$ to a random element $\xi(\omega) = \{a_k(\bar{\zeta}); k = 1, 2, \cdots\}$ where $a_k(\bar{\zeta}) \geqq 0$, and $\sum_{k=1}^{\infty} a_k(\bar{\zeta}) = 1$ a.s. the B.P.R.E. process is called *subcritical*, while if the same holds with $a_k(\bar{\zeta}) = 0$ for all $k$ a.s. the process is called *critical*.

We now turn to the problem of characterizing more practicably the concept of criticality. For this pupose we need to impose further requirements. *Henceforth throughout this section we assume that for each n the distributions of* $(\zeta_0, \zeta_1, \cdots, \zeta_n)$ *and* $(\zeta_n, \zeta_{n-1}, \cdots, \zeta_0)$ *are identical* (c.f. Definition 1 of Section 0 of *exchangeability*). This property is manifestly satisfied for the independence model of Smith and Wilkinson and for the case of a reversible stationary Markov chain. Of course, we also postulate as previously, that $\zeta_i$ is a stationary ergodic process. From now on unless stated explicitly otherwise, we exclude the supercritical case $E \log \varphi'_{\zeta_0}(1) > 0$.

We need additional notation. Let

$$\pi_n(s, \bar{\zeta}) = \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots(\varphi_{\zeta_{n-1}}(s))\cdots)) \tag{14}$$

$$Y_n(s, \bar{\zeta}) = \frac{\pi_n(s, \bar{\zeta}) - \pi_n(0, \bar{\zeta})}{1 - \pi_n(0, \bar{\zeta})} = E(s^{Z_n} \mid Z_n \neq 0, \mathbb{F}(\bar{\zeta})) \tag{15}$$

$$G_n(s, \bar{\zeta}) = 1 - Y_n(s, \bar{\zeta}) = \frac{1 - \pi_n(s, \bar{\zeta})}{1 - \pi_n(0, \bar{\zeta})} \tag{16}$$

and $\tilde{\pi}_n$, $\tilde{Y}_n$, $\tilde{G}_n$ will denote the analogous quantities defined by the reversed sequence $(\zeta_n, \zeta_{n-1}, \cdots, \zeta_0)$ instead of $(\zeta_0, \zeta_1, \cdots, \zeta_n)$.

We establish first that $\lim_{n\to\infty} \tilde{Y}_n(s, \zeta)$ exists. To this end, note that

$$\begin{aligned}\tilde{G}_{n+1}(s, \zeta) &= \frac{1 - \varphi_{\zeta_n}(\varphi_{\zeta_{n-1}} \cdots \varphi_{\zeta_0}(s)\cdots)}{1 - \varphi_{\zeta_n}(\varphi_{\zeta_{n-1}} \cdots \varphi_{\zeta_0}(0)\cdots)} \\ &= \tilde{G}_n(s, \bar{\zeta})\left[\frac{1 - \varphi_{\zeta_n}(\tilde{\pi}_n(s, \bar{\zeta}))}{1 - \tilde{\pi}_n(s, \bar{\zeta})}\right] \Big/ \left[\frac{1 - \varphi_{\zeta_n}(\tilde{\pi}_n(0, \bar{\zeta}))}{1 - \tilde{\pi}_n(0, \bar{\zeta})}\right] \geqq \tilde{G}_n(s, \bar{\zeta})\end{aligned}$$

because $(1 - \varphi_{\bar{\zeta}_n}(x))/(1 - x)$ is increasing in $x \in (0, 1)$. Thus, $\tilde{G}_n(s, \bar{\zeta})$ is monotonic in $n$ and we may conclude that

(17) $$\tilde{G}(s,\bar{\zeta}) = \lim_{n\to\infty} \tilde{G}_n(s,\bar{\zeta}) \quad \text{and} \quad \tilde{Y}(s,\zeta) = \lim_{n\to\infty} \tilde{Y}_n(s,\bar{\zeta})$$

exists for every $\bar{\zeta}$.

Invoking the property of exchangeability in conjunction with the limit relation (17), we obtain

THEOREM 2. *Let $(\bar{\zeta}_t, t \geqq 0)$ be an exchangeable process. Then for each* $0 \leqq s < 1$

(18) $$Y_n(s,\bar{\zeta}) \to \tilde{Y}(s,\bar{\zeta}) \qquad \textit{in law.}$$

It remains to determine precise conditions assuring that w.p. 1 $\tilde{Y}(s,\bar{\zeta})$ is an honest p.g.f. For this objective, we exhibit first the identity

(19) $$\tilde{G}_n(\varphi_{\zeta_0}(s), T\bar{\zeta}) = \tilde{G}_{n+1}(s,\bar{\zeta})\tilde{G}_{n+1}(\varphi_{\zeta_0}(0), T\bar{\zeta})$$

and this with (17) yields

(20) $$\tilde{G}(\varphi_{\zeta_0}(s), T\bar{\zeta}) = \tilde{G}(s,\bar{\zeta})\tilde{G}(\varphi_{\zeta_0}(0), T\bar{\zeta}).$$

Letting $s$ increase to 1 in (20) gives

(21) $$\tilde{G}(1-, T\bar{\zeta}) = \tilde{G}(1-,\bar{\zeta})m(\bar{\zeta})$$

where

(22) $$m(\bar{\zeta}) = \tilde{G}(\varphi_{\zeta_0}(0), T\bar{\zeta}) = \lim_{n\to\infty} \frac{1-\varphi_{\zeta_{n+1}}(\varphi_{\zeta_n}(\cdots(\varphi_{\zeta_1}(\varphi_{\zeta_0}(0))\cdots))}{1-\varphi_{\zeta_{n+1}}(\varphi_{\zeta_n}(\cdots\varphi_{\zeta_1}(0)\cdots))} \leqq 1.$$

The relation (21) leads to

LEMMA 1. *The following set equivalences hold* w.p. 1.

(23) $$\{\bar{\zeta}; \tilde{G}(1-,\bar{\zeta}) = 0\} = \{\bar{\zeta}; \tilde{G}(1-, T\bar{\zeta}) = 0\},$$

(24) $$\{\bar{\zeta}; \tilde{G}(1-,\bar{\zeta}) = 1\} = \{\bar{\zeta}; \tilde{G}(1-, T\bar{\zeta}) = 1\}.$$

PROOF. By definition

$$m(\bar{\zeta}) = \lim_{n\to\infty} \tilde{G}_n(\varphi_{\zeta_0}(0), T\bar{\zeta}) \geqq \tilde{G}_1(\varphi_{\zeta_0}(0), T\bar{\zeta}) = \frac{1-\varphi_{\zeta_1}(\varphi_{\zeta_0}(0))}{1-\varphi_{\zeta_1}(0)} > 0$$

w.p. 1. Now (21) implies (23).

Again, on the basis of (21) since $m(\bar{\zeta})$ and $\tilde{G}(1-,\bar{\zeta})$ do not exceed one

$$\{\bar{\zeta}; \tilde{G}(1-, T\bar{\zeta}) = 1\} \subset \{\bar{\zeta}; \tilde{G}(1-,\bar{\zeta}) = 1\}$$

and owing to stationarity the two sets have equal probability. These facts clearly imply (24). □

Since $T$ is ergodic we may infer

COROLLARY 1.

$$P\{\bar{\zeta}; \tilde{G}(1-;\bar{\zeta}) = 0\} = 0 \quad \textit{or} \quad 1$$

$$P\{\bar{\zeta}; \tilde{G}(1-;\bar{\zeta}) = 1\} = 0 \quad \textit{or} \quad 1.$$

Our next step is to relate the probabilistic quantities $P(\zeta;\ m(\zeta) < 1)$, $P(\zeta;\ \tilde{G}(1-,\zeta) = 0)$ and $P(\zeta;\ \tilde{G}(1-,\zeta) = 1)$. The following implications hold.

LEMMA 2.

$$P(\zeta; m(\zeta) < 1) > 0 \Rightarrow P(\zeta; \tilde{G}(1-;\zeta) = 0) = 1, \tag{25}$$

$$P(\zeta; m(\zeta) < 1) = 0 \Rightarrow P(\zeta; G(1-;\zeta) = 1) = 1. \tag{26}$$

PROOF. Let $P(\zeta; m(\zeta) < 1) > 0$. Assume to the contrary that $P(\zeta; G(1-,\zeta) = 0) < 1$ then by Corollary 1 its value is 0. According to (21) and the hypothesis, we have

$$\begin{aligned} 0 < C = \int \tilde{G}(1-, T\zeta)\, dP &= \int_{\{\zeta; m(\zeta)<1\}} G(1-, T\zeta)\, dP + \int_{\{\zeta; m(\zeta)=1\}} \tilde{G}(1-, T\zeta)\, dP \\ &< \int_{\{\zeta; m(\zeta)<1\}} \tilde{G}(1-, \zeta)\, dP + \int_{\{\zeta; m(\zeta)=1\}} \tilde{G}(1-, \zeta)\, dP \\ &= \int \tilde{G}(1-, \zeta)\, dP \end{aligned}$$

and $\int \tilde{G}(1-,\zeta) dP = C$ by stationarity thus reaching an absurdity and thereby proving (25).

Next, let $P(\zeta; m(\zeta) < 1) = 0$. Referring to (22) we have

$$P(\zeta; \tilde{G}(\varphi_{\zeta_0}(0), T\zeta) = 1) = 1.$$

Also, the process not being supercritical compels the inequality

$$P(\zeta;\ \varphi_{\zeta_0}(0) > 0) > 0.$$

Therefore

$$P(\zeta; \tilde{G}(\varphi_{\zeta_0}(0), T\zeta) = 1, \varphi_{\zeta_0}(0) > 0) > 0.$$

However, $1-\tilde{G}(s, T\zeta)$ is an analytic function for $|s| < 1$ and nondecreasing in $s$ $(0 \leqq s < 1)$. Therefore $1-\tilde{G}(\varphi_{\zeta_0}(0), T\zeta) = 0$, $\varphi_{\zeta_0}(0) > 0$ requires $1-\tilde{G}(s, T\zeta) = 0$ for $0 \leqq s < 1$ and in particular $1-G(1-, T\zeta) = 0$. Thus,

$$P(\zeta; \tilde{G}(1-, T\zeta) = 1) > 0$$

and again invoking the result of Corollary 1 we deduce

$$P(\zeta; \tilde{G}(1-, T\zeta) = 1) = 1$$

and (26) is established. □

An immediate consequence of Lemma 2 is the assertion that the B.P.R.E. is subcritical if and only if $P(\zeta; m(\zeta) < 1) > 0$. The following result highlights in more recognizable form a necessary and sufficient condition for subcriticality subject to a very mild further hypothesis concerning $\varphi_{\zeta_0}(0)$.

THEOREM 3. *Let* $E(-\log(1-\varphi_{\zeta_0}(0)) < \infty$. *Then*

$$P(\zeta; m(\zeta) < 1) > 0 \tag{27}$$

*if and only if*

$$E(\log \varphi'_{\zeta_0}(1)) < 0. \tag{28}$$

PROOF. Define

$$\tilde{g}_n(\zeta) = -\log\frac{1-\varphi_{\zeta_n}(\cdots\varphi_{\zeta_0}(0)\cdots)}{1-\varphi_{\zeta_n}(\cdots\varphi_{\zeta_1}(0)\cdots)}$$

$$g_n(\zeta) = -\log\frac{1-\varphi_{\zeta_0}(\cdots\varphi_{\zeta_n}(0)\cdots)}{1-\varphi_{\zeta_0}(\cdots\varphi_{\zeta_{n-1}}(0)\cdots)}$$

$$\bar{g}_n(\zeta) = -\log\frac{1-\varphi_{\zeta_0}(\varphi_{\zeta_1}\cdots\varphi_{\zeta_n}(0)\cdots)}{1-\varphi_{\zeta_1}\cdots\varphi_{\zeta_n}(0)\cdots)}.$$

Let $b_n = Eg_n(\zeta)$, $\tilde{b}_n = E\tilde{g}_n(\zeta)$, $\bar{b}_n = E(\bar{g}_n(\zeta))$ all of which exist (are finite) because of $E(-\log(1-\varphi_{\zeta_0}(0)) < \infty$. From (17) we know that $\tilde{g}_n(\zeta)$ decreases to $-\log m(\zeta)$. Invoking the monotone convergence theorem yields

(29) $$\lim_{n\to\infty} E(\tilde{g}_n(\zeta)) = E(-\log m(\zeta)).$$

Let (27) hold. We claim that $P(\zeta; q(\zeta) = 1) = 1$. Otherwise, $P(\zeta; q(\zeta) < 1) = 1$ leads to the result $P(\zeta; \lim_{n\to\infty} g_n(\zeta) = 0) = 1$ which on account of exchangeability implies $P(\zeta; \lim \tilde{g}_n(\zeta) = 0)$ and then by consulting (29) we deduce the equality

$$P(\zeta; m(\zeta) = 1) = 1$$

contradicting (27). Thus, if (27) holds we must have $P(\zeta; q(\zeta) = 1) = 1$ and then *a fortiori* $E \log \varphi'_{\zeta_0}(1) \leqq 0$. By monotone convergence we obtain

(30) $$\bar{b}_n = E(\bar{g}_n(\zeta)) \to -E\log\varphi'_{\zeta_0}(1).$$

But, by exchangeability $\bar{b}_n = \tilde{b}_n$. Then comparing (29) and (30) we may conclude that

(31) $$-E\log\varphi'_{\zeta_0}(1) = -E\log m(\zeta)$$

which is strictly positive. Hence $E \log \varphi_{\zeta_0}(1) < 0$ as was to be shown.

Clearly, (29) implies $P(\zeta; q(\zeta) = 1) = 1$ and hence (31) holds as the above reasoning demonstrates. Thus, $E(-\log m(\zeta)) > 0$ and this inequality is manifestly equivalent to (27). □

Summing up the preceding series of results, we have

THEOREM 4. *Let $(\zeta_t, t \geqq 0)$ be an exchangeable stationary ergodic process. Suppose $E(-\log(1-\varphi_{\zeta_0}(0)) < \infty$. Then*

(i) $+\infty \geqq E\log\varphi'_{\zeta_0}(1)^+ > E\log\varphi'_{\zeta_0}(1)^- \Rightarrow$ *the process is supercritical. That is* $P(\zeta; q(\zeta) < 1) = 1$.

(ii) $\infty > E(\log\varphi'_{\zeta_0}(1))^+ = E(\log\varphi'_{\zeta_0}(1))^- \Rightarrow$ *the process is critical. That is* $P(\zeta; q(\zeta) = 1) = 1$, *but* $\lim_{n\to\infty} P(Z_n = k \mid Z_n \neq 0, \mathbb{F}(\zeta)) \to 0$ *in law for each k.*

(iii) $E(\log\varphi'_{\zeta_0}(1))^+ < E(\log\varphi'_{\zeta_0}(1))^- \Rightarrow$ *the process is subcritical. That is* $P(\zeta; q(\zeta) = 1) = 1$, *and* $P(Z_n = k \mid Z_n \neq 0, \zeta) \to$ *in law to* $a_k(\zeta)$ *for each k and*

$$P(\zeta; \textstyle\sum_{k=1}^{\infty} a_k(\zeta) = 1) = 1.$$

We close this section by proving one further analog of a result on the classical Galton-Watson subcritical case. This concerns the rate of decay of $1-\varphi_{\zeta_n}(\varphi_{\zeta_{n-1}}(\cdots(\varphi_{\zeta_0}(s)\cdots)$ to zero.

THEOREM 5. *Let* $E(\log\varphi'_{\zeta_0}(1))^+ < E(\log\varphi'_{\zeta_0}(1))^-$, $E(-\log(1-\varphi_{\zeta_0}(0))) < \infty$ *and*

$$E((\textstyle\sum_{k=1}^{\infty} p_k(\zeta_0)k\log k)[\varphi'_{\zeta_0}(1)]^{-1}) < \infty.$$

*Then*

$$\lim_{n\to\infty}\frac{1-\varphi_{\zeta_n}(\cdots(\varphi_{\zeta_0}(s))\cdots)}{\prod_{j=0}^{n}\varphi'_{\zeta_j}(1)} = A(s,\zeta) \tag{32}$$

*exists and is strictly* $> 0$ *for* $0 \leqq s < 1$, w.p. 1.

REMARK. We emphasize the fact that in (32) the reverse sequence $\zeta_n, \zeta_{n-1}\cdots, \zeta_0$ appears. Also note that $E(-\log(1-\varphi_{\zeta_0}(0)) < \infty$ implies $E(\log\varphi'_{\zeta_0}(1))^- < \infty$ and with $E(\log\varphi'_{\zeta_0}(1))^+ < E(\log\varphi'_{\zeta_0}(1))^-$ also that $E|\log\varphi'_{\zeta_0}(1)| < \infty$ and $E\log\varphi'_{\zeta_0}(1) < 0$.

We need a lemma to prove Theorem 5.

LEMMA 3. *For any* $c$, $r$ *in* $(0, 1)$,

$$\sup_{c,r}[(-\log r)\sup_{k\geqq 2}((\log k)^{-1}\textstyle\int_0^\infty[1-(1-cr^x)^k]\,dx)] < \infty.$$

PROOF. Record first

$$1-(1-cr^x)^k = cr^x\textstyle\sum_{j=0}^{k-1}(1-cr^x)^j;$$

obviously

$$\begin{aligned}\textstyle\int_0^\infty cr^x(1-cr^x)^j\,dx &= c\textstyle\int_0^\infty e^{-xp}(1-c\,e^{-xp})^j\,dx \quad (\text{where } p = -\log r)\\ &= \frac{1}{p}\int_{1-c}^{1} y^j\,dy \qquad (\text{by setting } y = 1-c\,e^{-xp})\\ &\leqq \frac{1}{p(j+1)} \qquad \text{since } 0 < 1-c < 1.\end{aligned}$$

Thus,

$$(-\log r)(\log k)^{-1}\textstyle\int_0^\infty[1-(1-cr^x)^k]\,dx \leqq (\log k)^{-1}\sum_{j=0}^{k-1}(j+1)^{-1}.$$

But

$$\sup_{k\geqq 2}\left[(\log k)^{-1}\left(\sum_{j=0}^{k-1}\frac{1}{(j+1)}\right)\right] < \infty. \quad \square$$

PROOF OF THEOREM 1. Convexity implies

$$1-\varphi_{\zeta_n}(\cdots(\varphi_{\zeta_0}(s))\cdots) \leqq \textstyle\prod_{j=0}^{n}\varphi'_{\zeta_j}(1)$$

and on taking logarithms we get

$$\log(1-\varphi_{\zeta_n}(\cdots\varphi_{\zeta_0}(s))\cdots) \leqq \sum_{j=0}^{n}\log\varphi'_{\zeta_j}(1). \tag{33}$$

When $E\log\varphi'_{\zeta_0}(1) < 0$ the right side goes to $-\infty$ by the ergodic theorem and thus

$$\lim_{n\to\infty}[1-\varphi_{\zeta_n}(\cdots\varphi_{\zeta_0}(s))\cdots)] = 0 \qquad \text{uniformly for} \quad 0 \leqq s < 1, \quad \text{w.p.1.}$$

Now define

$$A_n(s,\bar{\zeta}) = [1-\varphi_{\zeta_n}(\cdots(\varphi_{\zeta_0}(s))\cdots)]P_{n+1}^{-1} \qquad (\text{recall that} \quad P_{n+1} = \textstyle\prod_{i=0}^{n}\varphi'_{\zeta_i}(1)). \tag{34}$$

Then

$$A_n(s,\bar{\zeta}) = \int_s^1 \prod_{j=0}^{n}(1-a_j(t,\bar{\zeta}))\,dt \tag{35}$$

where $a_j(t,\bar{\zeta}) = 1-\varphi'_{\zeta_j}(1))^{-1}\varphi'_{\zeta_j}(1-b_{j-1}(t,\bar{\zeta}))$

and $b_j(t,\bar{\zeta}) = 1-\varphi_{\zeta_j}(\cdots\varphi_{\zeta_0}(t))\cdots)$.

It suffices to show that under the condition of the theorem

$$\sum_1^\infty a_j(t,\bar{\zeta}) < \infty \qquad \text{uniformly for} \quad 0 \leqq t \leqq 1, \quad \text{w.p.1.} \tag{36}$$

On the basis of (33), we deduce that

$$\limsup_{n\to\infty}\sup_{0\leqq s\leqq 1}\frac{\log(1-\varphi_{\zeta_n}(\cdots\varphi_{\zeta_0}(s))\cdots)}{n\mu} \leqq -1 \qquad \text{w.p.1} \tag{37}$$

where $\mu = -E(\log\varphi'_{\zeta_0}(1))$. Let $\eta > 0$ be arbitrary. By Egoroff's theorem and by virtue of (37) we infer the existence of a set $A$, an integer $N$, and constants $c$, $r$ in (0, 1) with the properties $P(A) > 1-\eta$ and $\zeta \in A$, $n > N$ entails $\sup_{0\leqq s\leqq 1}(1-\varphi_{\zeta_n}(\cdots\varphi_{\zeta_0}(s)) \leqq cr^n$. Hence,

$$\begin{aligned}\zeta\in A, j > N \Rightarrow a_j(t,\bar{\zeta}) &= 1-(\varphi'_{\zeta_j}(1))^{-1}\varphi'_{\zeta_j}(1-b_{j-1}(t,\bar{\zeta}))\\ &\leqq [1-\varphi'_{\zeta_j}(1-cr^j)(\varphi'_{\zeta_j}(1))^{-1}].\end{aligned} \tag{38}$$

Using the estimate in (38) we get

$$\begin{aligned}
&E(\sup_{0\leqq t<1}\textstyle\sum_{j=N}^{\infty}a_j(t,\bar{\zeta});A)\\
&\quad\leqq E(\textstyle\sum_{j=N}^{\infty}[(1-\varphi'_{\zeta_j}(1-cr^j)(\varphi'_{\zeta_j}(1))^{-1}];A)\\
&\quad\leqq E\textstyle\sum_{j=N}^{\infty}[(1-\varphi'_{\zeta_j}(1-cr^j)(\varphi'_{\zeta_j}(1))^{-1}]\\
&\quad= \textstyle\sum_{j=N}^{\infty}E[1-\varphi'_{\zeta_0}(1-cr^j)(\varphi'_{\zeta_0}(1))^{-1}]
\end{aligned}$$

since the $\zeta_j$'s have the same distribution

$$\begin{aligned}
&\quad\leqq E\textstyle\sum_{j=0}^{\infty}[(1-\varphi'_{\zeta_0}(1-cr^j)(\varphi'_{\zeta_0}(1))^{-1}]\\
&\quad= E[\textstyle\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}kp_{\zeta_0}(k)(\varphi'_{\zeta_0}(1))^{-1}(1-(1-cr^j)^{k-1})]
\end{aligned}$$

$$\leqq (-\log r)^{-1} E\left(\sum_{k=2}^{\infty} \frac{k \log k p_{\zeta_0}(k)}{\varphi'_{\zeta_0}(1)} \left[\frac{\sum_{j=0}^{\infty}\{1-(1-cr^j)^{k-1}\}}{(\log k)(-\log r)^{-1}}\right]\right)$$

$$\leqq (-\log r)^{-1} KE\left(\sum_{k=2}^{\infty} \frac{k \log k p_{\zeta_0}(k)}{\varphi'_{\zeta_0}(1)}\right) \qquad \text{(by Lemma 3)}$$

$$< \infty \qquad \text{by hypothesis.}$$

It follows that

$$\sup_{0 \leqq t \leqq 1} \textstyle\sum_{j=N}^{\infty} a_j(t, \bar{\zeta}) < \infty \qquad \text{w.p.1 on } A.$$

But $P(A) > 1-\eta$ and $\eta$ is arbitrary. We now have (36). □

COROLLARY 2. *Under the hypothesis of Theorem* 5 *and exchangeability we have*

$$\frac{1-\varphi_{\zeta_0}(\cdots(\varphi_{\zeta_n}(s))\cdots)}{P_{n+1}} \quad \textit{converges in law to } A(s, \bar{\zeta}).$$

PROOF. Merely note that $(\zeta_0, \zeta_1, \cdots, \zeta_n)$ and $(\zeta_n, \zeta_{n-1}, \cdots, \zeta_0)$ have the same joint distribution. □

**3. Critical case.** This is the case when the extinction probability is one w.p. 1 and the conditional distribution of $Z_n$ conditioned on $Z_n \neq 0$ tends to a defective distribution with all its mass at $\infty$. However, it is anticipated that with proper normalization the distribution may tend to a nondegenerate limit law. This is the situation in the classical Galton-Watson process. We shall now paraphrase that analysis.

To avoid unimportant technical details we impose the following constraints. *Assume that* w.p. 1

$$0 < \alpha \leqq \varphi'_{\zeta_0}(1), 0 < \beta \leqq \varphi''_{\zeta_0}(1) \quad \text{and} \quad \varphi'''_{\zeta_0}(1) \leqq K. \tag{39}$$

These conditions prevail throughout this section unless specified otherwise.

The bounds of (39) yield the following estimates.

LEMMA 4. *Let* $\varphi(x) = \sum_{k=0}^{\infty} a_k x^k$ *be a* p.g.f. *satisfying* (39) *and consider the Taylor expansions,*

$$1-\varphi(x) = \varphi'(1)(1-x) - \frac{\varphi''(\eta)(1-x)^2}{2} \tag{40}$$

$$1-\varphi(x) = \varphi'(1)(1-x) - \frac{\varphi''(1)(1-x)^2}{2} + \frac{\varphi'''(\xi)(1-x)^3}{3!},$$

*where* $\eta$ *and* $\xi$ *are appropriate values in the interval* $(x, 1)$. *Then*

$$\frac{1}{1-\dfrac{\varphi''(\eta)}{2\varphi'(1)}(1-x)} \leqq K_1 < \infty \qquad 0 \leqq x \leqq 1 \tag{41}$$

*and*

$$\frac{\varphi''(1)}{2} - \frac{\varphi'''(\xi)}{3!}(1-x) \geqq \gamma > 0 \qquad 0 \leqq x \leqq 1 \tag{42}$$

*where the bounds $K_1$ and $\gamma$ depend only on the constants $\alpha$, $\beta$ and $K$.*

PROOF. Clearly

$$\frac{1}{1-\frac{\varphi''(\eta)}{2\varphi'(1)}(1-x)} = \frac{\varphi'(1)}{\frac{1-\varphi(x)}{1-x}} \leqq \frac{\varphi'(1)}{1-\varphi(0)}.$$

But the Schwarz inequality tells us that $\varphi'(1) \leqq [2\varphi''(1)]^{\frac{1}{2}}\,[1-\varphi(0)]^{\frac{1}{2}}$ which with the aid of (39) yields (41). In a similar manner we have

$$\frac{\varphi''(1)}{2} - \frac{\varphi'''(\xi)}{3!}(1-x) = \frac{\varphi'(1)(1-x)-(1-\varphi(x))}{(1-x)^2}$$

and the right-hand side is

$$\sum_{k=2}^{\infty} \sum_{v=0}^{k-1} ((k-1-v)x^v)a_k$$

which is uniformly bounded below owing to the conditions (39), and (42) is established. □

Introducing the short hand

$$g_n(s) = \varphi_{\zeta_n}(\varphi_{\zeta_{n-1}}(\cdots(\varphi_{\zeta_0}(s))\cdots)) \tag{43}$$

and referring to the expansions (40) we have

$$\frac{1}{1-g_n(s)} = \frac{1}{[1-g_{n-1}(s)]\varphi'_{\zeta_n}(1)\left[1-\frac{(1-g_{n-1}(s))}{2}\frac{\varphi''_{\zeta_n}(\eta_n)}{\varphi'_{\zeta_n}(1)}\right]}$$

$$= \frac{1}{[1-g_{n-1}(s)]\varphi'_{\zeta_n}(1)}\left[1+\frac{1-g_{n-1}(s)}{2}\frac{\varphi''_{\zeta_n}(\eta_n)}{\varphi'_{\zeta_n}(1)} + \frac{\{[1-g_{n-1}(s)]\varphi''_{\zeta_n}(\eta_n)\}^2}{4(\varphi'_{\zeta_n}(1))^2\left[1-\frac{1-g_{n-1}(s)}{2}\frac{\varphi''_{\zeta_n}(\eta_n)}{\varphi'_{\zeta_n}(1)}\right]}\right]$$

Taking account of (41) and since $P_n = \prod_{i=0}^{n} \varphi'_{\zeta_i}(1)$, we obtain

$$\frac{P_n}{1-g_n(s)} \leqq \frac{P_{n-1}}{1-g_{n-1}(s)} + \frac{1}{2}\frac{P_{n-1}\varphi''_{\zeta_n}(1)}{\varphi'_{\zeta_n}(1)} + K_1\frac{P_{n-1}[1-g_{n-1}(s)](\varphi''_{\zeta_n}(1))^2}{4(\varphi'_{\zeta_n}(1))^2} \tag{44}$$

Iteration of (44) gives

$$\frac{P_n}{1-g_n(s)} - \frac{1}{1-s} \leqq a_n(\zeta) + K_1 \sum_{k=1}^{n} \frac{P_{k-1}}{4}\left[\frac{\varphi''_{\zeta_k}(1)}{\varphi'_{\zeta_k}(1)}\right]^2 (1-g_{k-1}(s)) \tag{45}$$

where

$$a_n(\bar{\zeta}) = \frac{1}{2}\sum_{k=1}^{n} P_{k-1}\frac{\varphi''_{\zeta_k}(1)}{\varphi'_{\zeta_k}(1)}.$$

An analogous argument using the third order Taylor expansion produces the inequality

$$\frac{P_n}{1-g_n(s)} - \frac{1}{1-s} \geqq a_n(\bar{\zeta}) - \sum_{k=1}^{n} \frac{P_{k-1}}{6}\frac{\varphi'''_{\zeta_k}(\xi_k)}{\varphi'_{\zeta_k}(1)}(1-g_{k-1}(s)). \tag{46}$$

The result of (41) applied to (46) implies

$$1-g_n(s) \leqq C\frac{P_n}{\sum_{j=1}^{n} P_j} \tag{47}$$

and by virtue of (45) and (46) that

$$\frac{P_n}{1-g_n(s)} \leqq 1 + C_1 a_n(\bar{\zeta})$$

where $C$ and $C_1$ are positive constants independent of $s$ and $n$.

We now impose further restrictions on the environmental process $\bar{\zeta}$. Specifically, assume $\{\zeta_i, i \geqq 0\}$ is a uniformly mixing stationary process, i.e., $|E(\gamma_k \tilde{\gamma}_{k+n}) - E(\gamma_k) E(\tilde{\gamma}_{k+n})| \leqq c_n$ and $\sum c_n^{\frac{1}{2}} < \infty$ where $\gamma_k$ and $\tilde{\gamma}_l$ are any random variables bounded by 1 measurable $\sigma(\zeta_0, \zeta_1, \cdots, \zeta_k)$ and $\sigma(\zeta_l, \zeta_{l+1}, \cdots)$ respectively. Subject to this condition it is easy to show provided $E \log \varphi'_{\zeta_0}(1) = 0$ that $M_n(\bar{\zeta}) = \max_{1 \leqq r \leqq n} P_r$ tends to $\infty$ w.p. 1 and therefore *a fortiori* $a_n(\bar{\zeta}) \to \infty$ w.p. 1. For the case where $\zeta_i$ are i.i.d. this is a known fact emanating from fluctuation theory.

It is an easy matter to combine (45), (46) and (47) and thereby infer the conclusion

$$\lim_{n\to\infty} \frac{1}{a_n(\zeta)}\left[\frac{P_n}{1-g_n(s)} - \frac{1}{1-s}\right] = 1 \quad \text{uniformly for } 0 \leqq s < 1 \text{ in probability} \tag{48}$$

is equivalent to

$$\lim_{n\to\infty} \frac{1}{\sum_{j=1}^{n} P_j}\left(\sum_{j=1}^{n} \frac{P_j^2}{\sum_{i=1}^{j} P_i}\right) = 0 \quad \text{in probability.} \tag{49}$$

A little care in the analysis reveals also that (49) is equivalent to the assertion

$$P\left\{\left|\frac{1-g_n(e^{-\lambda/a(n)})}{1-g_n(0)} - \frac{1}{1+\lambda}\right| > \varepsilon\right\} \to_{n\to\infty} 0 \tag{50}$$

which in turn under the stipulation of exchangeability (see Definition 1) asserts that $Z_n/a_n(\bar{\zeta}) \mid Z_n \neq 0$ converges in law to an exponentially distributed random variable.

The validation of (49) for the general critical case is unsettled and appears rather delicate. It is interesting to point out that Kaplan [3] has exhibited a rather general example in the critical case where the limit relation (49) does not hold w.p. 1. The equivalence of (48) and (49) w.p. 1 is also correct but in view of Kaplan's example is a vacuous statement.

A special case for which (49) has immediate relevance is where $\varphi'_{\zeta_0}(1) \equiv 1$. Then clearly $P_j = 1$ and the left side of (49) is asymptotically $(\log n)/n \to 0$. In this special circumstance $a_n(\zeta) \sim n/2\, E(\varphi''_{\zeta_0}(1))$ and $1 - g_n(s) \to 0$ as $n \to \infty$ w.p. 1. Of course, the limit relation (50) then prevails.

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