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ON THE EXPECTED VALUES OF THE ELEMENTARY SYMMETRIC FUNCTIONS OF A NONCENTRAL WISHART MATRIX

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A conjecture is made for the expected value of the jth elementary symmetric function (ESF) of the roots of a noncentral Wishart matrix with covariance matrix $\sigma^2 I$. A conjecture is also made if $j = p$ for any covariance matrix Σ. The expected value of the noncentral Wishart matrix is derived for any covariance matrix Σ and therefore also the expected value of the first ESF of the roots.

1. Introduction. If $X(p \times n)$ has independent normally distributed columns with covariance Σ and $E(X) = M$, then $XX' = A$ is distributed as a noncentral Wishart distribution $W'(\Sigma, n, \Omega)$ with n degress of freedom and noncentrality parameter $\Omega = \Sigma^{-1}MM'$. The density is given by (Constantine (1963)).

$$(1.1)\qquad (1/\Gamma_p(\tfrac{1}{2}n)|2\Sigma|^{\frac{1}{2}n})\,\mathrm{etr}(-\tfrac{1}{2}\Omega)|A|^{\frac{1}{2}(n-p-1)} \\ \mathrm{etr}(-\tfrac{1}{2}\Sigma^{-1}A)\,{}_0F_1(\tfrac{1}{2}n;\tfrac{1}{4}\Omega\Sigma^{-1}A)\,, \qquad A > 0$$

where etr $(B) = \exp(\mathrm{tr}\, B)$ and ${}_0F_1$ a hypergeometric function of matrix argument. Denote the jth elementary symmetric function (esf) of the roots of A by $\mathrm{tr}_j\, A$. We are interested in $E(\mathrm{tr}_j\, A)$ and will derive it in Section 2 if $M = 0$, i.e. if A is distributed as a central Wishart distribution $W(\Sigma, n)$. In Section 3 we consider the noncentral Wishart matrix, but can only reach a conjecture for $E(\mathrm{tr}_j\, A)$, $j = 3, \cdots, p$ and $\Sigma = \sigma^2 I_p$. We are able to prove it for $j = 1, 2$.

2. Central Wishart. It has been proved (Constantine (1963)) that

$$(2.1)\qquad E(C_J(A)) = 2^j(\tfrac{1}{2}n)_J C_J(2\Sigma)$$

where $(a)_J = \prod_{i=1}^{p} (a + \frac{1}{2}(1-i))_{j_i}$ and $(a)_j = a(a+1)\cdots(a+j-1)$. For every partition $J = (j_1, j_2, \cdots, j_p)$ of j, $j_1 \geqq j_2 \geqq \cdots \geqq j_p$, the zonal polynomial $C_J(A)$ can be written as (James (1964))

$$(2.2)\qquad C_J(A) = \chi_{(2J)}(1)2^j j!\, Z_J(A)/(2j)!$$

where $Z_J(A)$ has been tabulated by James (1964) in terms of the esf's of the roots of A for $j = 1(1)6$. The constant $\chi_{(2J)}{}^{(1)}$ is also tabulated. Since

$$(2.3)\qquad Z_{1^j}(A) = j!\ \mathrm{tr}_j\, A$$

we have from (2.1)

$$(2.4)\qquad E(\mathrm{tr}_j\, A) = n(n-1)\cdots(n-j+1)\,\mathrm{tr}_j\,\Sigma = (n)^{(j)}\,\mathrm{tr}_j\,\Sigma\,.$$

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3. Noncentral Wishart. It seems difficult to derive a similar result for the noncentral Wishart distribution. From (1.1)

$$E(C_J(A)) = \operatorname{etr}(-\tfrac{1}{2}\Omega) \textstyle\sum_{k=0}^{\infty} \Sigma_K(1/(\tfrac{1}{2}n)_K) \int_{A>0} C_K(\tfrac{1}{4}\Omega\Sigma^{-1}A) \times C_J(A)\, dF(A;\Sigma,n)/k! \tag{3.1}$$

where $dF(A;\Sigma,n)$ is the $W(\Sigma,n)$ probability element and K is a partition on k. We shall find the solution to this integral only for the special case $\Sigma = \sigma^2 I_p$. In this case the integral is symmetric in Ω and we are allowed to transform $\Omega \to H'\Omega H$, $H \in 0(p)$. Since James (1964) $\int_{0(p)} C_K(H'\Omega HA)\,dH = C_K(\Omega)C_K(A)/C_K(I_p)$, the integral can be written as

$$E(C_J(A)) = \operatorname{etr}(-\tfrac{1}{2}\Omega)\Sigma_k\Sigma_K(1/(\tfrac{1}{2}n)_K k!)(C_K(\tfrac{1}{2}\Omega)/C_K(I_p)) \times \int_{A>0} C_K(\tfrac{1}{2}\sigma^{-2}A)C_J(A)\,dF(A;\sigma^2 I_p, n)\,. \tag{3.2}$$

Using the expression (Constantine (1966)) $C_K(A)C_J(A) = \Sigma_D g^D_{K,J} C_D(A)$, $g^D_{K,J}$ being a constant tabulated by Khatri and Pillai (1968) and D a partition of $d = k + j$, and the very useful integral (Constantine (1963))

$$\int_{A>0} |A|^{\frac{1}{2}(n-p-1)} \operatorname{etr}(-\tfrac{1}{2}PA)C_K(TA)\,dA = \Gamma_p(\tfrac{1}{2}n)|2P^{-1}|^{\frac{1}{2}n}(\tfrac{1}{2}n)_K C_K(2TP^{-1})\,,$$

P and T symmetric matrices, it follows that

$$E(C_J(A)) = \operatorname{etr}(-\tfrac{1}{2}\Omega)\Sigma_k\Sigma_K\Sigma_D g^D_{K,J}(\tfrac{1}{2}n)_D(2\sigma^2)^j C_D(I_p) \times C_K(\tfrac{1}{2}\Omega)((\tfrac{1}{2}n)_K C_K(I_p)k!)^{-1}\,. \tag{3.3}$$

Using (2.2), (2.3) and the result $Z_K(I_p) = 2^k(\tfrac{1}{2}p)_K$ we have a result corresponding to (2.4) for the case $\Sigma = \sigma^2 I_p$,

$$E(\operatorname{tr}_j A) = \operatorname{etr}(-\tfrac{1}{2}\Omega)a_0\Sigma_k a_k'\Sigma_K a_K''\Sigma_D a_D''' \tag{3.4}$$

where

$$a_0 = \sigma^{2j}(2j)!/(j!j!\chi_{(2,1^j)}(1))\,,$$
$$a_k' = 2^{2d}d!/(2^k k!\,(2d)!)\,,\quad a_K'' = Z_K(\tfrac{1}{2}\Omega)/((\tfrac{1}{2}n)_K(\tfrac{1}{2}p)_K)\,,$$
$$a_D''' = g^D_{K,1^j}(\tfrac{1}{2}n)_D(\tfrac{1}{2}p)_D\chi_{(2D)}(1)\,.$$

We will come back to this result after deriving $E(\operatorname{tr}_1 A)$ and obtaining a conjecture for $E(\operatorname{tr}_p A)$ using a simpler method. These two moments are considered for the general case $A \sim W'(\Sigma, n, \Omega)$. Since $A \sim XX'$ where $X \sim N(M, \Sigma\otimes I)$, $A \sim (U+M)(U+M)' = UU' + MU' + UM' + MM'$ where $U \sim N(0, \Sigma\otimes I)$. Hence

$$E(A) = n\Sigma + MM' = n\Sigma + \Sigma\Omega\,. \tag{3.5}$$

We have thus

$$E(\operatorname{tr}_1 A) = n\operatorname{tr}_1\Sigma + \operatorname{tr}_1\Sigma\Omega\,. \tag{3.6}$$

It is also easy to see from (1.1) that

$$(3.7)\qquad \begin{aligned} E(\mathrm{tr}_p A) &= \mathrm{etr}(-\tfrac{1}{2}\Omega)(\Gamma_p(\tfrac{1}{2}n+1)/\Gamma_p(\tfrac{1}{2}n))|2\Sigma| \textstyle\sum_{k=0}^{\infty}\sum_K \\ &\quad \times (1/(\tfrac{1}{2}n)_K k!) \textstyle\int_{A>0} C_K(\tfrac{1}{4}\Omega\Sigma^{-1}A)\, dF(A\,;\Sigma, n+2) \\ &= |\Sigma|(n)^{(p)}\,\mathrm{etr}(-\tfrac{1}{2}\Omega)\,{}_1F_1(\tfrac{1}{2}n+1\,;\tfrac{1}{2}n\,;\tfrac{1}{2}\Omega)\,. \end{aligned}$$

It seems that this expression can be written in a much simpler form, but the author has not succeeded in proving it. We shall however consider the two special cases $p = 2, 3$ and make an obvious conjecture for the general case. (3.7) can be written as

$$(3.8)\qquad \begin{aligned} E(\mathrm{tr}_p A) &= |\Sigma|(n)^{(p)}\,\mathrm{etr}(-\tfrac{1}{2}\Omega)\textstyle\sum_k\sum_K \prod_{i=1}^{p}(1+2k_i/(n+1-i)) \\ &\quad \times C_K(\tfrac{1}{2}\Omega)/k! \end{aligned}$$

Case $p = 2$. *If* $p = 2$,

$$\textstyle\prod_{i=1}^{2}(1+2k_i/(n+1-i)) = 1 + 2k/n + 2k_2(1+2k_1)/n(n-1)\,.$$

Therefore, using the fact $\sum_K C_K(S) = (\mathrm{tr}\, S)^k$, and the tables (James (1964)) expressing zonal polynomials in terms of esf's of the roots of a matrix,

$$(3.9)\qquad E(tr_{p=2}A) = |\Sigma|((n)^{(2)} + (n-1)\,\mathrm{tr}_1\Omega + \mathrm{tr}_2\Omega)\,.$$

Case $p = 3$. *If* $p = 3$,

$$\begin{aligned} &\textstyle\prod_{i=1}^{3}(1+2k_i/(n+1-i) \\ &\quad = 1 + 2k/n + (4k_1k_2 + 4k_1k_3 + 4k_2k_3 + 2k_2 + 4k_3)/n(n-1) \\ &\qquad + 4(2k_1k_2k_3 + k_1k_3 + 2k_2k_3 + k_3)/n(n-1)(n-2)\,. \end{aligned}$$

Following the same procedure as for $p = 2$, it follows after substitution

$$(3.10)\qquad E(\mathrm{tr}_{p=3} A) = |\Sigma|((n)^{(3)} + (n-1)^{(2)}\,\mathrm{tr}_1\,\Omega + (n-2)\,\mathrm{tr}_2\,\Omega + \mathrm{tr}_3\,\Omega)\,.$$

Conjecture. An obvious conjecture for the general case follows immediately from (3.9) and (3.10), namely

$$(3.11)\qquad \begin{aligned} E(\mathrm{tr}_p A) &= |\Sigma|((n)^{(p)} + (n-1)^{(p-1)}\,\mathrm{tr}_1\,\Omega + (n-2)^{(p-2)}\,\mathrm{tr}_2\,\Omega \\ &\quad + \cdots + \mathrm{tr}_p\,\Omega)\,. \end{aligned}$$

From (3.6) and (3.11) we now have an intuitive idea that for the special case $\Sigma = \sigma^2 I_p$, $E(\mathrm{tr}_j\, A)$ can be written in terms of the first j esf's of the roots of Ω. By writing down the first few terms of the series (3.4) for $j = 2$, we obtain a conjecture in general. It is only necessary to consider $k = 1(1)4$ and for $k = 4$ we need not have to go further than the partition $K = (3, 1)$ to show that

$$(3.12)\qquad E(\mathrm{tr}_2\, A) = \sigma^4((n)^{(2)}\tbinom{p}{2} + (n-1)\tbinom{p-1}{1}\,\mathrm{tr}_1\,\Omega + \mathrm{tr}_2\,\Omega)\,.$$

This result leads us to the general result.

Conjecture.

$$E(\operatorname{tr}_j A) = \sigma^{2j}((n)^{(j)}\tbinom{p}{j} + (n-1)^{(j-1)}\tbinom{p-1}{j-1}\operatorname{tr}_1 \Omega + (n-2)^{(j-2)}\tbinom{p-j}{j-2}\operatorname{tr}_2 \Omega + \cdots + \operatorname{tr}_j \Omega)\,. \tag{3.13}$$

The first term can be checked with (2.4). Further, if $j = p$ we have (3.11) and if $j = 1$ we have (3.6) for $\Sigma = \sigma^2 I_p$.
Equation (3.13) essentially says:

$$E\,|A - \lambda I| = L(|\Omega - \lambda I|) \tag{3.14}$$

where L is a lower triangular linear operator acting on the vector of coefficients of the polynomial $|\Omega - \lambda I|$. The matrix L has the form

$$L_{jk} = \pm\tbinom{p-k}{j-k}(n-k)^{(j-k)}\,.$$

With suitable normalization on L and defining

$$a_j = \operatorname{tr}_j A/(n)^{(j)}\tbinom{p}{j} \qquad \text{and} \qquad w_j = \operatorname{tr}_j \Omega/(n)^{(j)}\tbinom{p}{j}$$

the matrix L takes on the simple form

$$\tilde{L}_{jk} = \pm\tbinom{j}{k}\,.$$

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