

NOTES ON THE COVARIANCE MATRIX OF A RANDOM, NESTED ANOVA MODEL

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A convenient representation of the covariance matrix for a general k -level random, nested AOV model is obtained, as well as an expression for its inverse and determinant.

1. Introduction. A convenient representation for the covariance matrix of a random, nested analysis of variance model is obtained in Section 2. In Section 3 a representation of V^{-1} is derived and in Section 4 an expression for $|V|$ is derived.

2. Representation of the model. Let J_{ij} be a column vector of 1's having n_{ij} rows. Let

$$(1) \quad K_i = \text{diag}(J_{i1}, J_{i2}, \dots, J_{in_i})$$

be a matrix having n_i columns and $\sum_j n_{ij}$ rows, $i = 1, \dots, k$, and $n_{i+1} = \sum_j n_{ij}$, $n_0 = 0$. A p -way random, nested analysis of variance model may be represented by Y_{p+1} where

$$(2) \quad Y_1 = Z_1$$

and

$$(3) \quad Y_{i+1} = K_i Y_i + Z_{i+1}, \quad i = 1, \dots, p,$$

and

$$(4) \quad V(Z_i) = \sigma_i^2 I$$

and

$$(5) \quad \text{Cov}(Z_i, Z_j) = 0, \quad i \neq j.$$

For example, if $p = 1$, $n_1 = 2$, $n_{11} = 2$, $n_{12} = 3$, $n_2 = 5$,

$$(6) \quad Y_2 = K_1 Y_1 + Z_2$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Z_{11} \\ Z_{12} \end{bmatrix} + \begin{bmatrix} Z_{21} \\ Z_{22} \\ Z_{23} \\ Z_{24} \\ Z_{25} \end{bmatrix} = \begin{bmatrix} Z_{11} + Z_{21} \\ Z_{11} + Z_{22} \\ Z_{12} + Z_{23} \\ Z_{12} + Z_{24} \\ Z_{12} + Z_{25} \end{bmatrix}.$$

which, with a slight change of notation, corresponds to the usual representation $y_{ij} = a_i + b_{ij}$.

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Let $V_i = \text{Cov}(Y_i)$ and $V = V_{p+1}$. Then

$$(7) \quad V_1 = \sigma_1^2 I$$

and

$$(8) \quad V_{i+1} = \sigma_{i+1}^2 I + K_i V_i K_i', \quad i = 1, \dots, p.$$

The dimensions of V_i are $n_i \times n_i$, $i = 1, \dots, p$, and V is $n \times n$ where $n = n_{p+1}$ is the number of components of Y .

3. The inverse of V . Let $\sigma^2 = \sigma_{p+1}^2$, $\gamma_i = \sigma_i^2/\sigma^2$, $B_i = V_i/\sigma^2$, $i = 1, \dots, p+1$. Then

$$(9) \quad V = \sigma^2(I + K_p B_p K_p')$$

and

$$(10) \quad B_i = \gamma_i I + K_{i-1} B_{i-1} K_{i-1}', \quad i = 1, \dots, p,$$

where $B_0 = 0$.

By defining

$$(11) \quad C_{p+1} = I,$$

$$(12) \quad D_{p+1} = I,$$

$$(13) \quad D_i^{-1} = (I + \gamma_i K_i' D_{i+1}^{-1} K_i)^{-1} K_i' D_{i+1}^{-1} K_i, \quad i = 1, \dots, p,$$

$$(14) \quad C_i = (K_i' D_{i+1}^{-1} K_i)^{-1} K_i' D_{i+1}^{-1} C_{i+1}, \quad i = 1, \dots, p,$$

$$(15) \quad C_0 = 0,$$

and

$$(16) \quad A_i = C_i' D_i^{-1} C_i - C_{i-1}' K_{i-1}' D_i^{-1} K_{i-1} C_{i-1}, \quad i = 1, \dots, p+1,$$

it may be shown that

$$(17) \quad (\sigma^{-2} V)^{-1} = A_{p+1} + A_p + \dots + A_1$$

by defining

$$(18) \quad R_i = C_i' (I + K_i' D_{i+1}^{-1} K_i B_i)^{-1} K_i' D_{i+1}^{-1} K_i C_i, \quad i = 1, \dots, p, R_0 = 0,$$

noting that

$$(19) \quad (\sigma^{-2} V)^{-1} = A_{p+1} + R_p$$

and showing that

$$(20) \quad R_i = A_i + R_{i-1}, \quad i = 1, \dots, p.$$

Alternative expressions for A_i are

$$(21a) \quad A_i = (C_i - K_{i-1} C_{i-1})' D_i^{-1} (C_i - K_{i-1} C_{i-1})$$

$$(21b) \quad = C_i' (D_i^{-1} - D_i^{-1} K_{i-1} (K_{i-1}' D_i^{-1} K_{i-1})^{-1} K_{i-1}' D_i^{-1}) C_i.$$

It follows from (21a) that each A_i is nonnegative definite and

$$(22) \quad \text{rank}(A_i) = \text{rank}(D_i) - \text{rank}(D_{i-1}) .$$

It should be noted that D_i is diagonal, so that the only inverses involved in the above expressions are inverses of diagonal matrices. Generally, the inverses used above should be taken as generalized inverses. For definiteness, it will be assumed that if D is diagonal, D^{-1} will be taken as a diagonal array with $d^i = 1/d_i$ if $d_i \neq 0$ and $d^i = 0$ if $d_i = 0$.

Applying (17) results in a partition of $Y'V^{-1}Y$ into $p + 1$ n.n.d. quadratics analogous to the partition appearing in the usual ANOVA table. If $n_{ij} = m_i$, $j = 1, \dots, n_i$, for some m_i , $i = 1, \dots, p$ (that is, if the setup is balanced), then each A_i is the product of a scalar function of $\sigma^2, \sigma_p^2, \dots, \sigma_i^2$, and a matrix which does not depend on $\sigma^2, \sigma_i^2, i = 1, \dots, p$. Finally, (11)–(17) describe an efficient iterative computational procedure for inverting V for specified values of $\sigma^2, \sigma_i^2, i = 1, \dots, p$.

4. The determinant of V . With notation as in Section 3,

$$(23) \quad \begin{aligned} |\sigma^{-2}V| &= |I + K_p B_p K_p'| \\ &= |I + K_p' K_p B_p| \\ &= |I + \gamma_p K_p' K_p + K_p' K_p K_{p-1} B_{p-1} K_{p-1}'| \\ &= |K_p' K_p D_p| |I + D_p^{-1} K_{p-1} B_{p-1} K_{p-1}'| \\ &= |K_p' K_p D_p| |I + K_{p-1}' D_p^{-1} K_{p-1} B_{p-1}| \\ &= |K_p' K_p D_p| |K_{p-1}' D_p^{-1} K_{p-1} D_{p-1}| |I + K_{p-2}' D_{p-1}^{-1} K_{p-2} B_{p-2}| \\ &\quad \vdots \\ &= \prod_{i=1}^p |K_i' D_{i+1}^{-1} K_i D_i| \\ &= \prod_{i=1}^p |I + \gamma_i K_i' D_{i+1}^{-1} K_i| \end{aligned}$$

since

$$(24) \quad K_i' D_{i+1}^{-1} K_i D_i = I + \gamma_i K_i' D_{i+1}^{-1} K_i .$$

5. An example. The results of the previous sections will be applied to the covariance matrix of the balanced two-way nested layout. Let: $p = 2$; $m_0 = n_1$; $n_{1j} = m_1, j = 1, \dots, m_0$; $n_{2j} = m_2, j = 1, \dots, m_0 m_1$. Denote by $A \times B$ the Kronecker product of two matrices A and B . The notation I_m will denote an $m \times m$ identity matrix and J_m an m -vector of 1's. Then

$$(25) \quad K_1 = I_{m_0} \times J_{m_1} ,$$

$$(26) \quad K_2 = I_{m_0 m_1} \times J_{m_2} ,$$

$$(27) \quad D_2^{-1} = \frac{m_2}{m_2 \gamma_2 + 1} I_{m_0 m_1} ,$$

$$(28) \quad D_1^{-1} = \frac{m_1 m_2}{m_1 m_2 \gamma_1 + m_2 \gamma_2 + 1} I_{m_0},$$

$$(29) \quad C_2 = m_2^{-1} I_{m_0 m_1} \times J'_{m_2},$$

$$(30) \quad C_1 = \frac{1}{m_1 m_2} I_{m_0} \times J'_{m_1 m_2},$$

$$(31) \quad C_2 - K_1 C_1 = m_2^{-1} I_{m_0} \times (I_{m_1} - m_1^{-1} J_{m_1} J'_{m_1}) \times J'_{m_2},$$

$$(32) \quad C_3 - K_2 C_2 = I_{m_0} \times I_{m_1} \times (I_{m_2} - m_2^{-1} J_{m_2} J'_{m_2}),$$

so that, using (21a),

$$(33) \quad A_3 = I_{m_0} \times I_{m_1} \times (I_{m_2} - m_2^{-1} J_{m_2} J'_{m_2}),$$

$$(34) \quad A_2 = \frac{1}{m_2(m_2 \gamma_2 + 1)} I_{m_0} \times (I_{m_1} - m_1^{-1} J_{m_1} J'_{m_1}) \times J_{m_2} J'_{m_2},$$

$$(35) \quad A_1 = \frac{1}{m_1 m_2 (m_1 m_2 \gamma_1 + m_2 \gamma_2 + 1)} I_{m_0} \times J_{m_1} J'_{m_1} \times J_{m_2} J'_{m_2}.$$

If the components of Y are denoted by y_{ijk} , $i = 1, \dots, m_0, j = 1, \dots, m_1, k = 1, \dots, m_2$, then

$$(36) \quad Y' A_3 Y = \sum_{i,j,k} (y_{ijk} - \bar{y}_{ij.})^2,$$

$$(37) \quad Y' A_2 Y = \frac{1}{m_2 \gamma_2 + 1} \sum_{i,j,k} (\bar{y}_{ij.} - \bar{y}_{i..})^2,$$

$$(38) \quad Y' A_1 Y = \frac{1}{m_1 m_2 \gamma_1 + m_2 \gamma_2 + 1} \sum_{i,j,k} \bar{y}_{i..}^2.$$

If μ is a scalar then

$$(39) \quad \begin{aligned} &\sigma^2(Y - \mu J)' V^{-1} (Y - \mu J) \\ &= \sum_{i,j,k} (y_{ijk} - \bar{y}_{ij.})^2 + \frac{1}{m_2 \gamma_2 + 1} \sum_{i,j,k} (\bar{y}_{ij.} - \bar{y}_{i..})^2 \\ &\quad + \frac{1}{m_1 m_2 \gamma_1 + m_2 \gamma_2 + 1} [\sum_{i,j,k} (\bar{y}_{i..} - \bar{y}_{...})^2 + (\bar{y}_{...} - \mu)^2]. \end{aligned}$$

To find $|V|$, note that

$$(40) \quad \begin{aligned} |I + \gamma_2 K_2' D_3^{-1} K_2| &= |(m_2 \gamma_2 + 1) I_{m_0 m_1}| \\ &\doteq (m_2 \gamma_2 + 1)^{m_0 m_1}, \end{aligned}$$

$$(41) \quad \begin{aligned} |I + \gamma_1 K_1' D_2^{-1} K_1| &= \left| \frac{m_1 m_2 \gamma_1 + m_2 \gamma_2 + 1}{m_2 \gamma_2 + 1} I_{m_0} \right| \\ &= \left(\frac{m_1 m_2 \gamma_1 + m_2 \gamma_2 + 1}{m_2 \gamma_2 + 1} \right)^{m_0}, \end{aligned}$$

so that, using (23) and simplifying,

$$(42) \quad |V| = (m_1 m_2 \sigma_1^2 + m_2 \sigma_2^2 + \sigma^2)^{m_0} (m_2 \sigma_2^2 + \sigma^2)^{m_0(m_1-1)} (\sigma^2)^{m_0 m_1(m_2-1)}.$$