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EXTREME VALUES IN THE GI/G/1 QUEUE[1]

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Consider a GI/G/1 queue in which W_n is the waiting time of the nth customer, $W(t)$ is the virtual waiting time at time t, and $Q(t)$ is the number of customers in the system at time t. We let the extreme values of these processes be $W_n^* = \max\{W_j: 0 \leqq j \leqq n\}$, $W^*(t) = \sup\{W(s): 0 \leqq s \leqq t\}$, and $Q^*(t) = \sup\{Q(s): 0 \leqq s \leqq t\}$. The asymptotic behavior of the queue is determined by the traffic intensity ρ, the ratio of arrival rate to service rate. When $\rho < 1$ and the service time has an exponential tail, limit theorems are obtained for W_n^* and $W^*(t)$; they grow like $\log n$ or $\log t$. When $\rho \geqq 1$, limit theorems are obtained for W_n^*, $W^*(t)$, and $Q^*(t)$; they grow like $n^{\frac{1}{2}}$ or $t^{\frac{1}{2}}$ if $\rho = 1$ and like n or t when $t > 1$. For the case $\rho < 1$, it is necessary to obtain the tail behavior of the maximum of a random walk with negative drift before it first enters the set $(-\infty, 0]$.

1. Introduction and summary. Our objective in this paper is to study the limiting behavior of the maximum waiting time, maximum virtual waiting time, and the maximum queue length in a GI/G/1 queue for all values of the traffic intensity. This problem has been essentially solved by Cohen (1968), (1969) for the M/G/1 and GI/M/1 queues with traffic intensity less than or equal to one. Limit theorems for the maximum of the embedded queue length process in a GI/M/1 queue are obtained in Heyde (1971). Further related work can be found in Whitt (1971).

In our GI/G/1 queueing system customer number 0 arrives at time $t_0 = 0$, finds a free server, and experiences a service time v_0. The nth customer arrives at time t_n and experiences a service time v_n. Customers are served in their order of arrival and the server is never idle if customers are waiting. Let the interarrival times $t_n - t_{n-1} = u_n$, $n \geqq 1$. We assume the two sequences $\{v_n : n \geqq 0\}$ and $\{u_n : n \geqq 1\}$ each consist of independent, identically distributed (i.i.d.) random variables (rv's) and are themselves independent. Let the $E\{u_n\} = \lambda^{-1}$ and $E\{v_n\} = \mu^{-1}$, where $0 < \lambda$, $\mu < \infty$. The traffic intensity of this system is $\rho = \lambda/\mu$. Each of the three cases $\rho < 1$, $\rho = 1$, and $\rho > 1$ induces a different limiting behavior and they shall be considered separately. The deterministic system in which both the v_n's and u_n's are degenerate is excluded. We let the waiting time of the nth customer be W_n, the virtual waiting time at time t be $W(t)$, and the number of customers in the system at

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time t be $Q(t)$. Now define $W_n^* = \max\{W_j : 0 \leqq j \leqq n\}$, $W^*(t) = \sup\{W(s) : 0 \leqq s \leqq t\}$, and $Q^*(t) = \sup\{Q(s) : 0 \leqq s \leqq t\}$.

For the case $\rho < 1$ we shall assume that $v_0 - u_1$ is nonlattice and that there exists a positive number γ such that $E\{\exp[\gamma(v_0 - u_1)]\} = 1$ and $0 < E\{(v_0 - u_1)\exp[\gamma(v_0 - u_1)]\} < \infty$. The assumption involving γ is tantamount to requiring the distribution function (df) of v_0 to have an exponentially decaying tail. (Cohen also needs this assumption in the M/G/1 and GI/M/1 cases.) This assumption is clearly satisfied if v_0 has a gamma df or is bounded above. With this assumption we show that $W_n^*(W^*(t))$ grows like $\log n^{1/\gamma}(\log t^{1/\gamma})$ and obtain precise nondegenerate limit laws. We have no results for $Q^*(t)$ when $\rho < 1$. It is known, however, that a nondegenerate limit theorem for $Q^*(t)$ does not exists for the M/G/1 queue when $\rho < 1$; cf. Cohen (1969, page 602) and Anderson (1970). This fact is a consequence of the discrete nature of $Q^*(t)$. Tight bounds are available in this case, however, for the $\limsup_{t\to\infty} P\{aQ^*(t) - b(t) \leqq x\}$ and the $\liminf_{t\to\infty} P\{aQ^*(t) - b(t) \leqq x\}$, where a and $b(t)$ are the correct normalizing factors; cf. Cohen (1969, page 602).

The key lemma required to obtain our results for the case $\rho < 1$ is one concerning random walks. Let $X_k = v_{k-1} - u_k$, $k \geqq 1$, and $S_k = X_1 + \cdots + X_k$, $S_0 = 0$. From our independence assumptions we see that $\{S_n : n \geqq 0\}$ is a random walk. We show that the probability that S_n exits the interval $(0, z]$ on the right is asymptotic to $be^{-\gamma z}$ as $z \to \infty$, where b is a constant to be defined later. This result is perhaps of some independent interest.

The analysis of the cases $\rho = 1$ and $\rho > 1$ does not require the additional assumptions made for the case $\rho < 1$. Our results in these cases use previous functional central limit theorems for heavy traffic. We remark in passing that all our limit theorems could be cast in a functional form; cf. Lamperti (1964) for the case $\rho < 1$ and Iglehart and Whitt (1970) for $\rho \geqq 1$.

The organization of this paper is as follows. The random walk result mentioned above is given in Section 2. Results for the cases $\rho < 1$ and $\rho \geqq 1$ are contained in Sections 3 and 4 respectively.

2. A random walk result. Let $\{X_n : n \geqq 1\}$ be a sequence of i.i.d. rv's defined on the probability triple $(\Omega, \mathscr{F}, P)$, where $\Omega = \times_{n=1}^{\infty} \Omega_n$ and each Ω_n is a copy of R^1, $\mathscr{F}$ is the completion of the product Borel field (B.F.), P is the completed product measure constructed from the distribution of X_1, and $\{X_n : n \geqq 1\}$ are coordinate functions. Let $\mathscr{F}_n$ be the completed B.F. generated by $X_1, \cdots, X_n$. An rv α is called *optional* relative to $\{X_n : n \geqq 1\}$ if it takes on strictly positive integer values or $+\infty$ and satisfies the condition $\{\omega : \alpha(\omega) = n\} \in \mathscr{F}_n$, $n = 1, 2, \cdots, \infty$, where $\mathscr{F}_\infty = \mathscr{F}$. Let $S_0 = x$ and $S_n = X_1 + \cdots + X_n$, $n \geqq 1$. For Borel sets A of R^1 we shall be interested in the optional rv's α_A, the first entrance time of the random walk $\{S_n : n \geqq 0\}$ to the set A. For ease of notation we let $\alpha_{(-\infty,0]} = \alpha$ and $\alpha_{(0,z]^c} = \alpha(z)$. We follow the standard convention of letting $P_x\{\cdot\}$ and $E_x\{\cdot\}$ denote the probability and expectation of the random walk under the condition that $S_0 = x \geqq 0$.

Now let the $E\{X_1\} = \mu$ (assumed to exist), $M = \sup\{S_k : k \geqq 0\}$, and $M_+ = \max\{S_k : k = 0, \cdots, \alpha - 1\}$. The following assumption we shall need here and in later sections.

Assumption A. *There exists a number* $\gamma \neq 0$ *such that* $E\{e^{\gamma X_1}\} = 1$, $E\{X_1 e^{\gamma X_1}\} = \mu_\gamma < \infty$, *and* X_1 *is nonlattice.*

We now are in a position to state the following result due essentially to Feller (1966, pages 363, 393).

Lemma 1. *If Assumption A holds and* $-\infty < \mu < 0$ (*hence* $\gamma, \mu_\gamma > 0$), *then for* $x \geqq 0$ *the*

$$P_x\{M > z\} \sim a(x)e^{-\gamma z} \qquad \text{as} \quad z \to \infty\,, \tag{1}$$

where $a(x) = e^{\gamma x}[1 - E_0(e^{\gamma S_\alpha})]/\gamma\mu_\gamma E_0(\alpha)$.

This Lemma is a consequence of the renewal theorem; cf. Feller (1966, page 349). For lattice X_1 a corresponding result would hold, but we shall not pursue that case; cf. Feller (1968, page 331) and Spitzer (1964, page 218). A brief explanation of how $a(x)$ follows from (6.16) of Feller (1966, page 363) is in order. The factor $e^{\gamma x}$ is an easy consequence of starting the random walk at x rather than 0. The term $1/E_0(\alpha)$ is Feller's $1 - L_\infty$; cf. (Chung 1969, page 260). Finally, the term $[1 - E_0(e^{\gamma s_\alpha})]/\mu_\gamma$ is Feller's $1/\mu^\#$ and is calculated as follows using his associated random walk ([8] page 388) and Wald's equation; cf. Chung (1968, page 128). By definition $\mu^\# = E_0\{{}^aS_{{}^a\alpha(0,\infty)}\}$ and hence by Wald $\mu^\# = \mu_\gamma E_0\{{}^a\alpha_{(0,\infty)}\}$, where $\{{}^aS_n : n \geqq 0\}$ is the associated random walk and ${}^a\alpha_{(0,\infty)}$ the hitting time of $(0, \infty)$ for $\{{}^aS_n : n \geqq 0\}$. Recall that from random walk theory

$$E_0\{\alpha_A\} = \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} P_0[S_n \in A^c]\right\}$$

and

$$1 - E_0\{e^{\gamma S_{\alpha_A}}\} = \exp\left\{-\sum_{n=1}^{\infty} \frac{1}{n} E_0[e^{\gamma S_n} : S_n \in A]\right\} > 0\,,$$

where A is one of the four sets $[0, \infty)$, $(0, \infty)$, $(-\infty, 0]$, or $(-\infty, 0)$ and $E_0\{\alpha\} \equiv m < \infty$; cf. Chung (1968, page 260 and 258). Thus

$$E_0\{{}^a\alpha_{[0,\infty)}\} = \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} P_0[{}^aS_n \leqq 0]\right\}$$
$$= \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} E_0[e^{\gamma S_n} : S_n \leqq 0]\right\}$$
$$= [1 - E_0\{e^{\gamma S_\alpha}\}]^{-1} .$$

combining these factors we have $\mu^\# = \mu_\gamma[1 - E_0\{e^{\gamma S_\alpha}\}]^{-1}$.

Using Lemma 1 we easily obtain

THEOREM 1. *If Assumption A holds and* $-\infty < \mu < 0$, *then for* $x \geqq 0$ *the*

$$P_x\{M_+ > z\} \sim b(x)e^{-\gamma z} \qquad \textit{as} \quad z \to \infty , \tag{2}$$

where $b(x) = a(0)[e^{\gamma x} - E_x\{e^{\gamma S_\alpha}\}]$.

PROOF. Decomposing the set $\{M > z\}$ yields

$$P_x\{M > z\} = P_x\{M > z,\, S_{\alpha(z)} > z\} + P_x\{M > z,\, S_{\alpha(z)} \leqq 0\} .$$

Using the fact that $\alpha(z)$ is optional, together with the strong Markov property enjoyed by the random walk, we see that the

$$P_x\{M > z,\, S_{\alpha(z)} \leqq 0\} = \int_{(-\infty,0]} P_0\{M > z - y\}P_x\{S_{\alpha(z)} \in dy\} .$$

Hence

$$e^{\gamma z}P_x\{S_{\alpha(z)} > z,\} = e^{\gamma z}P_x\{M > z\}$$
$$- \int_{(-\infty,0]} e^{\gamma(z-y)}P_0\{M > z - y\}e^{\gamma y}P_x\{S_{\alpha(z)} \in dy\} .$$

Next observe that since $S_n \to -\infty$ a.e. (because $\mu < 0$), $S_{\alpha(z)} \to S_\alpha$ a.e. and hence $P_x\{S_{\alpha(z)} \in \cdot\} \Rightarrow P_x\{S_\alpha \in \cdot\}$, where $\Rightarrow$ denotes weak convergence. Also for all $y \in (-\infty, 0]$, $e^{\gamma(z-y)}P_0\{M > z - y\}$ can be made arbitrarily close to $a(0)$ by selecting z large enough. Finally, since $e^{\gamma y}$ is a bounded continuous function on $(-\infty, 0]$, weak convergence yields

$$\lim_{z\to\infty} e^{\gamma z}P_x\{S_{\alpha(z)} > z\} = a(x) - a(0)E_x\{e^{\gamma S_\alpha}\} . \tag{3}$$

Since the set $\{M_+ > z\} = \{S_{\alpha(z)} > z\}$, we see that (3) is equivalent to (2).

3. Extreme values when $\rho < 1$. We return now to the GI/G/1 queue with $\rho < 1$. Let the probability triple $(\Omega, \mathscr{F}, P) = \prod_{n=1}(R_+^2, \mathscr{R}_+^2, \pi)$, where $R_+^2 = [0, \infty) \times [0, \infty)$, $\mathscr{R}_+^2$ is the B.F. of R_+^2, and π is the common distribution of $\mathbf{X}_n \equiv (v_{n-1}, u_n)$, $n \geqq 1$. Next define $X_n = v_{n-1} - u_n$, $n \geqq 1$, and set $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, $n \geqq 1$. As in Section 2 we take $\alpha = \alpha_{(-\infty,0]}$.

We shall assume without further mention that Assumption A holds throughout this section.

In terms of our queue, $\alpha \equiv \alpha^1$ corresponds to the number of customers served in the first busy period (b.p.). The concept of the α-shift allows one to define further rv's α^k, $k \geqq 2$, which correspond to the number of customers served in the kth b.p. (Consult Iglehart (1971) for full details on these constructions.) Let $\beta_0 = 0$, $\beta_k = \alpha^1 + \cdots + \alpha^k$, and

$$V_k = \{\alpha^k, \mathbf{X}_{\beta_{k-1}+1}, \cdots, \mathbf{X}_{\beta_k}\} .$$

It is well known that the sequence $\{V_k : k \geqq 1\}$ is i.i.d. and that the waiting time of the jth customer $W_j = S_j - S_{\beta_{k-1}}$ on $\{\beta_{k-1} \leqq j < \beta_k\}$.

Next define the maximum waiting time in the kth b.p. as

$$M_+(k) = \max\{W_j : \beta_{k-1} \leqq j < \beta_k\} , \qquad k \geqq 1 .$$

Observe that $M_+(1) \equiv M_+$ of Section 2. Since $M_+(k)$ is defined in terms of V_k, the sequence $\{M_+(k) : k \geqq 1\}$ is i.i.d. Let $\{l(n) : n \geqq 0\}$ be the discrete renewal process associated with the i.i.d. sequence $\{\alpha^k : k \geqq 1\}$. Then the maximum of the first $n + 1$ waiting times, W_n^*, satisfies.

$$\max\{M_+(k) : 1 \leqq k \leqq l(n)\} \leqq W_n^* \leqq \max\{M_+(k) : 1 \leqq k \leqq l(n) + 1\} . \tag{4}$$

From Theorem 1 we derive

LEMMA 2. *If $\rho < 1$, then the*

$$\lim_{n\to\infty} P\{\gamma \max_{1\leqq k\leqq n} M_+(k) - \log bn \leqq x\} = \Lambda(x) , \qquad -\infty < x < \infty , \tag{5}$$

where $\Lambda(x) \equiv \exp\{-e^{-x}\}$ *and* $b \equiv b(0)$.

PROOF. Since the $M_+(k)$'s are i.i.d., well-known extreme value theorems apply; cf. Gnedenko (1943). The method is simply this:

$$\begin{aligned} &P\{\max_{1\leqq k\leqq n} M_+(k) \leqq (x + \log bn)/\gamma\} \\ &\quad = P^n\{M_+(1) \leqq (x + \log bn)/\gamma\} \\ &\quad = [1 - b\exp[-(x + \log bn)] + o(\exp[-(x + \log n)])]^n \end{aligned}$$

using Theorem 1. Letting $n \to \infty$, we obtain (5).

From Lemma 2 it is a small step to find a limit theorem for $\max\{M_+(k) : 1 \leqq k \leqq l(n)\}$ and hence from (4) for W_n^*.

THEOREM 2. *If $\rho < 1$, then the*

$$\lim_{n\to\infty} P\{\gamma W_n^* - \log bn \leqq x\} = \Lambda^{1/m}(x) , \qquad -\infty < x < \infty .$$

PROOF. From renewal theory we know that $l(n)/n \Rightarrow 1/m$ as $n \to \infty$. The result then follows from Lemma 2, (4), and a result of Berman (1962, Theorem 3.2).

COROLLARY 1. *If $\rho < 1$, then*

$$\frac{W_n^*}{\log n^{1/\gamma}} \Rightarrow 1 \qquad \text{as} \quad n \to \infty .$$

PROOF. This follows immediately from Theorem 2 and the continuous mapping theory; cf. Billingsley (1968, Theorem 5.5). Take $h_n : R \to R$ as $h_n(x) = x/\log n$.

Next we turn to the virtual waiting time process, $\{W(t) : t \geqq 0\}$. The maximum virtual waiting time in the kth b.p. is given by

$$M_+^*(k) = \max\{W_j + v_j : \beta_{k-1} \leqq j < \beta_k\}, \qquad k \geqq 1 .$$

Let $M^* = \sup\{S_k + v_k : k \geqq 0\}$ and $M_+^* = \max\{S_k + v_k : k = 0, \cdots, \alpha - 1\}$. Since we have assumed that all v_j's and u_j's are independent, we can write

$$\begin{aligned} M^* &= v_0 + \sup\{S_k + v_k - v_0 : k \geqq 0\} \\ &= v_0 + \sup\{(v_1 - u_1) + \cdots + (v_k - u_k) : k \geqq 0\} \\ &= v_0 + M' \end{aligned}$$

where v_0 and M' are independent and M' has the same distribution as M.

LEMMA 3. *If* $-\infty < E\{v_0 - u_1\} < 0$, *then for* $x \geqq 0$ *the*

$$P_x\{M^* > z\} \sim a^*(x)e^{-\gamma z} \qquad \text{as} \quad z \to \infty, \tag{6}$$

where $a^*(x) = E\{e^{\gamma v_0}\}a(x)$.

PROOF. Let V be the df of v_0. Then the

$$\begin{aligned} P_x\{M^* > z\} &= \textstyle\int_0^\infty P_x\{M' + v_0 > z \mid v_0 = v\}V(dv) \\ &= \textstyle\int_0^\infty P_x\{M > z - v\}V(dv) \end{aligned}$$

and

$$e^{\gamma z} P_x\{M^* > z\} = \textstyle\int_0^\infty e^{\gamma(z-v)} Px\{M > z - v\}e^{\gamma v} V(dv) . \tag{7}$$

Since $E\{e^{\gamma v_0}\} < \infty$ by Assumption A and $e^{\gamma(z-v)} P_x\{M > z - v\} \to a(x)$ by Lemma 1, we can let $z \to \infty$ in (7), apply the Lebesgue dominated convergence theorem, and obtain (6).

Next we use the method of Theorem 1 to find the tail behavior of M_+^*.

LEMMA 4. *If* $-\infty < E\{v_0 - u_1\} < 0$, *then for* $x \geqq 0$ *the*

$$P_x\{M_+^* > z\} \sim b^*(x)e^{-\gamma z} \qquad \textit{as} \quad z \to \infty ,$$

where $b^*(x) = E\{e^{\gamma v_0}\}b(x)$.

PROOF. Decompose $\{M^* > z\}$ and write

$$\begin{aligned} \{M^* > z\} &= \{M^* > z, M_+^* > z\} \cup \{M^* > z, M_+^* \leqq z\} \\ &= \{M_+^* > z\} \cup \{M^* > z, M_+^* \leqq z, S_{\alpha(z)} \leqq 0\} . \end{aligned}$$

Thus the

$$P_x\{M_+^* > z\} = P_x\{M^* > z\} - P_x\{M^* > z, M_+^* \leqq z, S_{\alpha(z)} \leqq 0\} .$$

Using the strong Markov property again we can write

$$P_x\{M^* > z, M_+{}^* \leqq z, S_{\alpha(z)} \leqq 0\}$$
$$= \int_{(-\infty,0]} P_0\{M^* > z - y\} P_x\{M_+{}^* \leqq z, S_{\alpha(z)} \in dy\} .$$

Hence

$$(8) \qquad e^{\gamma z} P_x\{M_+{}^* > z\} = e^{\gamma z} P_x\{M^* > z\}$$
$$- \int_{(-\infty,]} e^{\gamma(z-y)} P_0\{M^* > z - y\} e^{\gamma y} P_x\{M_+{}^* \leqq z, S_{\alpha(z)} \in dy\} .$$

Since $M_+{}^* \leqq M^*$ and M^* is finite a.e., $M_+{}^*$ is also finite a.e. As remarked before $S_{\alpha(z)} \to S_\alpha$ a.e., thus the measure $P_x\{M_+{}^* \leqq z, S_{\alpha(z)} \in \cdot\} \Rightarrow P_x\{S_\alpha \in \cdot\}$. From here the argument is exactly like that of Theorem 1: let $z \to \infty$ in (8), use Lemma 3, and weak convergence.

Using the method employed in Lemma 2 and Theorem 2, we obtain

THEOREM 3. *If* $\rho < 1$, *then the*

$$(9) \qquad \lim_{t\to\infty} P\{\gamma W^*(t) - \log b^* t \leqq x\} = \Lambda^{\lambda/m}(x) ,$$

where $b^* = b^*(0)$.

The only remark needed here is that the renewal process, $\{m(t) : t \geqq 0\}$, associated with the length of the busy cycles, ξ_k, obeys the weak law $m(t)/t \to 1/E\{\xi_1\} = \lambda/m$. This accounts for the exponent on the right-hand side of (9). The next result follows immediately using the method of Corollary 1.

COROLLARY 2. *If* $\rho < 1$, *then*

$$\frac{W^*(t)}{\log t^{1/\gamma}} \Rightarrow 1 \qquad as \quad t \to \infty .$$

4. Extreme values when $\rho \geqq 1$. The results for the case $\rho = 1$ have been obtained previously for much more general systems; see Iglehart and Whitt (1970, Theorem 9.1). We simply quote them here for sake of completeness.

THEOREM 4. *If* $\rho = 1$ *and* $\sigma^2 = \sigma^2\{v_0 - u_1\}(0 < \sigma^2 < \infty)$, *then*

(a) $W_n{}^*/\sigma n^{\frac12} \Rightarrow \sup\{|\xi(t)| : 0 \leqq t \leqq 1\}$,

(b) $Q^*(t)/\mu^{\frac32}\sigma t^{\frac12} \Rightarrow \sup\{|\xi(t)| : 0 \leqq t \leqq 1\}$, *and*

(c) $W^*(t)/\mu^{\frac12}\sigma t^{\frac12} \Rightarrow \sup\{|\xi(t)| : 0 \leqq t \leqq 1\}$,

where $\{\xi(t) : 0 \leqq t \leqq 1\}$ *is a Brownian motion process. The*

$$P\{\sup_{0\leqq t\leqq 1} |\xi(t)| \leqq x\} = 1 - (4/\pi) \sum_{k=1}^{\infty} [(-1)^k/(2k+1)]$$
$$\times \exp\{-[\pi^2(2k+1)^2/8x^2]\} .$$

Now we turn to the case $\rho > 1$. We remark that extreme value limit theorems agree with the ordinary limit theorems since the processes are growing in this case. The result is

THEOREM 5. *If* $\rho > 1$ *and* $0 < \sigma^2 < \infty$, *then*

(a) $[W_n^* - \lambda^{-1}(\rho - 1)n]/\sigma n^{\frac{1}{2}} \Rightarrow \Phi$,
(b) $[W^*(t) - (\rho - 1)t]/\alpha t^{\frac{1}{2}} \Rightarrow \Phi$, *and*
(c) $[Q^*(t) - (\lambda - \mu)t]/\gamma t^{\frac{1}{2}} \Rightarrow \Phi$,

where $\alpha = [\lambda\rho^2\sigma^2\{u_1\} + \lambda\sigma^2\{v_0\}]^{\frac{1}{2}}$, $\gamma = [\lambda^3\sigma^2\{u_1\} + \mu^3\sigma^2\{v_0\}]^{\frac{1}{2}}$, *and* Φ *is the standard normal distribution function.*

PROOF. (a) We know that $W_n = S_n - m_n$, where $m_n = \min(0, S_1, \cdots, S_n)$. Hence

$$S_n \leqq W_n \leqq S_n - m, \tag{10}$$

where $m = \inf(0, S_1, \cdots)$. Taking maxima in (10) yields

$$M_n \leqq W_n^* \leqq M_n - m,$$

where $M_n = \max(0, S_1, \cdots, S_n)$. Since $E\{X_1\} > 0$ (because $\rho > 1$), $m > -\infty$ a.e. by the strong law. Hence

$$\frac{|W_n^* - M_n|}{n^{\frac{1}{2}}} \rightarrow 0 \quad \text{a.e.}$$

In a similar fashion one can show that $|M_n - S_n|/n^{\frac{1}{2}} \rightarrow 0$. Thus the limit behavior of W_n^* is exactly like that of S_n. This completes the proof of (a) since $E\{X_1\} = \lambda^{-1}(\rho - 1)$.

(b) Let $\{A(t) : t \geqq 0\}$ be the renewal process which counts the number of arrivals in $[0, t]$, $L(t) = v_0 + \cdots v_{A(t)-1}$, and $Y(t) = L(t) - t$. Then the following representation for $W(t)$ is well known; cf. Reich (1958).

$$W(t) = Y(t) - \inf\{Y(\tau-) : 0 \leqq \tau \leqq t\}.$$

Since $\rho > 1$, $Y(\mathrm{t}) \rightarrow +\infty$ a.e. by the strong law and using the method employed in Theorem 5(a) we can show that $|W^*(t) - Y(t)|/t^{\frac{1}{2}} \rightarrow 0$. The central limit theorem for $Y(t)$ is well known, cf. Hooke (1970, page 636), and hence completes the proof of (b).

(c) Let $S(t)$ be the renewal process associated with the sequence $\{v_n : n \geqq 0\}$ and set $X(t) = A(t) - S(t)$. In ([10] Theorems 2.2 and 3.1) we showed that $\sup\{|Q(\tau) - X(\tau)| : 0 \leqq \tau \leqq t\}/t^{\frac{1}{2}} \rightarrow 0$ as $t \rightarrow \infty$. This fact allows us to carry thru the analysis of (b) with $X(t)$ playing the role of $Y(t)$. The upshot is that $|Q^*(t) - X(t)|/t^{\frac{1}{2}} \rightarrow 0$ and the result follows from the central limit theorem for $X(t)$; cf. [10] Lemma 2.1).

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