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BOUNDS FOR STOPPED PARTIAL SUMS[1]

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Upper bounds are derived for the expected value of a stopped random sum under each of four sets of assumptions concerning the summands, plus under an additional set describing a related and similar problem. Too complex to abstract, these assumptions in part, typically limit the first two moments of the summands.

The bounds have an interpretation in a stock market timing problem in which the random sum represents the sequence of daily prices of a stock and the positive part of the sum reflects a potential profit a holder of an option in the stock could realize were he to exercise. In only one case are the summands required to be independent and identically distributed, and thus we obtain bounds on the expected profit that do not require the controversial random walk model for stock prices. Of course, the bounds are of interest for other reasons as well. For example, as a related result we show that if (S_n) is a random walk for which the summands (X_n) have a negative mean, then $E[S_T^+] < \infty$ for all stopping times T if and only if $E[(X_1^+)^2] < \infty$.

For the most part, techniques familiar to readers of Dubins and Savage (*How to Gamble if You Must*, McGraw-Hill 1965) are used.

1. Introduction and summary. Our purpose is to derive upper bounds on the expected value of the positive part of a stopped random sum, and this we do under each of four sets of assumptions about the summands, plus under an additional set describing a related and similar problem. In the course of it all, we develop some new and extend some known techniques for verifying and evaluating bounds on expectations in stochastic processes.

To be specific, let $X_1, X_2, \cdots$ be jointly distributed real-valued random variables on some probability space $(\Omega, \mathscr{A}, P)$ having an associated expectation operator E. Let $S_n = X_1 + \cdots + X_n$ for $n \geqq 1$, $S_0 = 0$, and let T be a stopping time relative to (S_n). Let λ be a fixed nonnegative constant.

For random variables X and Y, use the suggestive notation P_X (or, where typographically convenient, $P(X)$) for the probability distribution of X, and $P_{X|Y}$ (or $P(X|Y)$) for the conditional distribution of X given Y. Use E_X and $E_{X|Y}$ for the corresponding expectations. Write Var $[X]$ for $E_X[(X - E_X[X])^2]$. For any set $\mathbf{M}$ of probability distributions, call (X_n) an $\mathbf{M}$-*sequence* if the distributions $P(X_1)$ and $P(X_n | X_1, \cdots, X_{n-1})$ are always in $\mathbf{M}$.

Let

$$(1.1) \qquad f(x, \lambda, \mathbf{M}) = \sup E[e^{-\lambda T}(x + S_T)^+]$$

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where the supremum is over all **M**-sequences (X_n) and stopping times T. Finally, abbreviate $f(x, 0, \mathbf{M})$ to $f(x, \mathbf{M})$ and $f(0, \mathbf{M})$ to $f(\mathbf{M})$.

This paper derives bounds for $f(x, \lambda, \mathbf{M})$ under a variety of **M**'s. Part of our interest in these bounds derives from their interpretation in a stock market timing problem formulated by Samuelson (1965) and Taylor (1967). The random variables $X_1, X_2, \cdots$ represent the daily changes in the price of some stock, and $e^{-\lambda T} S_T^+$ is the discounted return to the holder of a call option in the stock were he to exercise the option (profitably) on day T. This interpretation spurred our interest in finding bounds under a variety of assumptions, including in particular, assumptions that do not require the X_n's to be independent and identically distributed, (which is the controversial "random walk hypothesis" for stock prices), and under the multiplicative assumptions of Result (v). It is probably true that a fairly robust index for evaluating such options can be built around the α in $\operatorname{Var}[X] + 2\alpha E_X[X] \leqq 2\alpha^2 \lambda$, an expression that appears in almost all our formulations.

Of course, as indicated by Ester Samuel (1967), there are other, less pecuniary, reasons for pursuing such bounds, and there is at least one additional reason for burdening ourselves with the factor $e^{-\lambda}$. We have $\int_0^\infty e^{-\lambda(a+T)}(x + S_T)\, d\lambda = (x + S_T)/(a + T)$ so that, using Fubini's Theorem for nonnegative functions,

$$E[(x + S_T)/(a + T)] \leqq \int e^{-\lambda a} f(x, \lambda, \mathbf{M})\, d\lambda$$

whenever (X_n) is an **M**-sequence. Thus our bounds may be used in problems involving the stopping of forms related to S_n/n. While it is doubtful that such bounds are tight, the procedure may sometimes show an expected averaged reward to be finite, often a crucial and difficult part in showing the existence of optimal stopping times.

In addition to these specific bounds, however, we hope that an important contribution of the paper will be to spur interest in the use of the gambling theory (inequalities for stochastic processes) of Dubins and Savage (1965) to obtain bounds in applied probability models.

To state our results more precisely, for a positive number α, set

$$(1.2) \qquad \begin{aligned} f_\alpha(x) &= \alpha e^{x/\alpha - 1} && \text{for } x \leqq \alpha \\ &= x && \text{for } x > \alpha . \end{aligned}$$

Our results include:

(i) *Let $\mathbf{M}_e$ be the set of distributions P_X for which $E_X[e^{X/\alpha}] \leqq e^\lambda$. Then*

$$(1.3) \qquad f(x, \lambda, \mathbf{M}_e) \leqq f_\alpha(x) \quad \textit{for all} \quad x .$$

Equality holds in (1.3) *for all x if $\lambda = 0$ or if $(1 - x/\alpha)/\lambda$ is an integer* (Propositions 1 and 3).

(ii) *As an illustrative application of* (i), *let $\mathbf{M}_N$ be the set of normal distributions* *having a mean* $-\mu$ *and a variance* σ^2 *that together satisfy*

$$-2\alpha\mu + \sigma^2 \leqq 2\alpha^2\lambda \,. \tag{1.4}$$

Then

$$f(x, \lambda, \mathbf{M}_N) \leqq f_\alpha(x) \quad \textit{for all} \quad x \,. \tag{1.5}$$

Again, equality holds in (1.5) *for all* x *if* $\lambda = 0$ *or if* $(1 - x/\alpha)/\lambda$ *is an integer* (Propositions 2 and 3).

For comparison with Result (iii), note that when $\lambda = 0$, (1.4) reduces to $2\alpha\mu \geqq \sigma^2$.

If $\mathbf{M}$ contains exactly one distribution P_X then (X_n) is an $\mathbf{M}$-sequence if and only if $X_1, X_2, \cdots$ are independent and identically distributed with the common distribution P_X.

(iii) *Let* $m = P_X$ *be a fixed probability distribution for which* $-2\alpha E_X[X] \geqq$ Var $[X]$. *Let* $\mathbf{M}_m$ *be the set whose only element is* m. *Then*

$$f(\mathbf{M}_m) \leqq \alpha \,. \tag{1.6}$$

On the lower side,

$$\sup_m f(x, \mathbf{M}_m) \geqq \max\,\{x, \alpha/2\}$$

(Lemma 3).

(iv) *Let* β *be a nonnegative constant and let* $\mathbf{M}_{\alpha,\beta}$ *be the set of distributions* P_X *for which*

$$\text{Var}\,[X] + 2\alpha E_X[X] \leqq 2\alpha^2\beta \,. \tag{1.7}$$

Then

$$f(x, \lambda, \mathbf{M}_{\alpha,\beta}) \leqq \gamma\beta + \max\,\{\gamma/2, x\} \,, \tag{1.8}$$

for all x, *where* $\gamma = \alpha e^{-\lambda}/(1 - e^{-\lambda})$. *Of particular interest are* $\beta = \lambda$, *which compares with* (1.4) *in* (ii) *and* $\beta = 0$ *which compares with* (iii). *Writing* $\mathbf{M}_\alpha$ *for* $\mathbf{M}_{\alpha,0}$

$$f(x, \lambda, \mathbf{M}_\alpha) = \max\,\{\gamma/2, x\} \tag{1.9}$$

$$f(x, \mathbf{M}_\alpha) = \infty \tag{1.10}$$

and, in fact,

$$\sup E[(x + S_T)^+ \,;\, T \leqq n] = \max\,\{x, n\alpha/2\} \tag{1.11}$$

(Propositions 5, 6, and 7).

The difference between Results (iii) and (iv), that is, that the requirement of independence would bound an otherwise infinite expected reward, surprised us. This was especially unexpected since the bound in (ii) is achieved (in the limit) by independent, identically distributed summands.

(v) *Let* $\theta > 1$ *be fixed and let* $\mathbf{M}_G$ *be the set of distributions* P_X *for which* X *is nonnegative and* $E_X[X^\theta] \leqq e^\lambda$. *Suppose* $X_1, X_2, \cdots$ *is an* $\mathbf{M}_G$*-sequence,* $w \geqq 0$, $W_0 = 1$ and $W_{n+1} = W_n \times X_{n+1}$ *for* $n = 0, 1, \cdots$. *Then for all stopping times*

T, $E[e^{-\lambda T}(w \times W_T - 1)^+] \leqq f(w)$ *where*

$$f(w) = [w(\theta - 1)/\theta]^\theta/(\theta - 1) \quad \text{for} \quad w \leqq \theta/(\theta - 1)\,,$$
$$= w - 1 \quad \text{for} \quad w > \theta/(\theta - 1)$$

(Proposition 4).

In addition to these results, we point out that if $X_1, X_2, \cdots$ are independent and identically distributed and $E[(X_1^+)^2] = \infty$, then there exists a stopping time T for which $E[S_T^+] = \infty$, thus completing a theorem begun by Ester Samuel (1967) (Our Theorem 1).

For the most part, our approach is to note that any function f that (a) exceeds a "terminal reward" function r and (b) is expectation decreasing in the sense that $f(S_n) \geqq e^{-\lambda}E[f(S_{n+1}) \mid X_1, \cdots, X_n]$, will also exceed the expected reward $E[e^{-\lambda T}r(x + S_T)]$ for any stopping time T. Several formulations of this theorem have appeared in the literature; we were motivated by (Blackwell; 1954, 1964) and by (Dubins and Freedman; 1965). Our version (Lemma 1) is restricted to nonnegative r but allows the possibility $T = \infty$. We include a finite horizon formulation, used in obtaining (1.11).

2. The basic approach. With one exception, in this paper we will consider only additive processes; that is, processes of the form $S_n = X_1 + \cdots + X_n$. Nevertheless, it is possible and appears potentially useful to establish the basic approach in a more general setting. To this end, let $Z_0, Z_1, \cdots$ be real-valued random variables on the probability space $(\Omega, \mathscr{A}, P)$. For each real z let $\mathbf{M}(z)$ be a set of probability distributions and call (Z_n) an $\mathbf{M}$*-sequence starting at* z if (a) $Z_0 \equiv z$ and (b) given $Z_0, \cdots, Z_n$, the conditional distribution of Z_{n+1} is in $\mathbf{M}(Z_n)$. Thus, if (X_n) is an $\mathbf{M}$-sequence and we set $\mathbf{M}(x) = \{P_{x+X} : P_X \in \mathbf{M}\}$, then $(x + S_n)$ is an $\mathbf{M}$-sequence starting at x. Or, if we set $\mathbf{M}(w) = \{P_{wX} : P_X \in M\}$, and $W_0 = w$, $W_n = wX_1 \cdots X_n$ for $n \geqq 1$, then (W_n) is an $\mathbf{M}$-sequence starting at w.

Relative to (Z_n), a stopping time T is a random variable taking values in $\{0, 1, \cdots, \infty\}$ for which the event $\{T = n\}$ is in the σ-algebra generated by Z_0, $Z_1, \cdots, Z_n$. (If in a particular context, we fail to define a stopping time on part of the sample space, then understand its value to be infinite on that part.) Let r be a nonnegative Baire function, called the *terminal reward*, and let λ be a nonnegative constant. Adopt the convention that $E[e^{-\lambda T} r(Z_T)] = \int_{T<\infty} e^{-\lambda T} r(Z_T)dP$ (which means that a reward of zero is associated with never stopping). Lemma 1 modifies (22) in Dubins and Freedman (1965) to allow extended integer-valued stopping times provided the terminal reward is nonnegative, and to explicitly include discounting. A result appropriate for bounded stopping times is also included.

LEMMA 1. *Let* (Z_k) *be an* $\mathbf{M}$*-sequence starting at* z, *let* r *be a nonnegative terminal reward, and* T *a stopping time. Set* $T(n) = \min\{T, n\}$.

(a) *If for some positive integer* $N, f_0, \cdots, f_N$ *are extended real-valued Baire functions satisfying*

$$f_n \geqq r \quad \text{for} \quad n = 0, \cdots, N \tag{2.1}$$

and

$$f_{n+1}(z') \geqq e^{-\lambda} \int f_n(s) P_Z(ds) \quad \textit{whenever} \quad P_Z \in \mathbf{M}(z') \tag{2.2}$$

then

$$f_N(z) \geqq E[e^{-\lambda T(N)} r(Z_{T(N)})] \,. \tag{2.3}$$

(b) *If* f *is an extended real-valued Baire function satisfying*

$$f \geqq r \tag{2.4}$$

and

$$f(z') \geqq e^{-\lambda} \int f(s) P_Z(ds) \quad \textit{whenever} \quad P_Z \in \mathbf{M}(z') \tag{2.5}$$

then

$$f(z) \geqq E[e^{-\lambda T} r(Z_T)] \,. \tag{2.6}$$

PROOF. For (a), we may suppose $f_N(z) < \infty$, since otherwise the desired conclusion is immediate.

Since (Z_k) is an **M**-sequence starting at z, (2.2) implies that $(e^{-\lambda k} f_{N-k}(Z_k))_{0 \leqq k \leqq N}$ is a nonnegative supermartingale, closed on the right. A martingale systems theorem (Doob, 1953) followed by (2.1) implies $f_N(z) \geqq E[e^{-\lambda T(N)} f_{N-T(N)}(Z_{T(N)})] \geqq E[e^{-\lambda T(N)} r(Z_{T(N)})]$ which completes the proof of (a).

For (b) we may again suppose $f(z) < \infty$. Use (a), note that $r \geqq 0$, and then use monotone convergence to verify that

$$f(z) \geqq E[e^{-\lambda T(N)} r(Z_{T(N)})] \geqq \int_{T<N} e^{-\lambda T} r(Z_T)\, dP \uparrow E[e^{-\lambda T} r(Z_T)]$$

as N increases indefinitely, which completes the proof of (b). □

Lemma 1 provides a means of verifying that a given function is an upper bound on expected rewards. To show that a given bound is a *least* upper bound, we need a procedure for evaluating the expected reward in a specified process under a specified stopping rule. From Blackwell's (1965), (1967) theory of dynamic programming and from results in Markov stopping rule problems (Dynkin, 1963; Taylor, 1968b) we anticipate that, under quite general conditions, when bounds formed in the manner of Lemma 1 are approached, they will be approached by a Markov process (Z_n), stopped upon first hitting a specified set. Thus we motivate Lemma 2, which, in this Markov case, approximately evaluates an expected return.

First, unfortunately, some notation. Let K be a (possibly sub-) Markov transition kernel, so that $K(x, B)$ is a Baire function in x for each linear Borel set B, and a (possibly defective) probability measure on the Borel sets B for each fixed x. For any linear Borel set A, let I_A be its indicator function and $I_{A'}$, the indicator function of the complement of A. Let J_A (respectively, $J_{A'}$) be the kernel corresponding to multiplication by I_A (respectively, $I_{A'}$); i.e., for a nonnegative Baire function f, $J_A f(x) = f(x) \times I_A(x)$ where, of course, $J_A f(x) = \int J_A(x, dy) f(y)$. Let K_A (respectively, $K_{A'}$) be the kernel $J_A K$ (respectively, $J_{A'} K$). Then $K_A(x, dy) = K(x, dy)$ or 0, according as $x \in A$ or $x \notin A$, respectively.

LEMMA 2. *Let $Z_0, Z_1, \cdots$ be a Markov process with stationary transition kernel K. Let $r \geqq 0$ be a terminal reward.*

(a) *Let ε be a nonnegative constant, N be a positive integer, $f_0, \cdots, f_N$ be nonnegative Baire functions, and $\Gamma(0), \cdots, \Gamma(N)$ be Borel sets. Suppose*

$$f_0 \leqq J_{\Gamma(0)} r + \varepsilon \tag{2.8}$$

and

$$f_n \leqq J_{\Gamma(n)} r + K_{\Gamma(n)'} f_{n-1} + \varepsilon \quad \textit{for} \quad n = 1, \cdots, N\,. \tag{2.9}$$

Let T be the first k, if any, for which $Z_k \in \Gamma(N-k)$. Then $E[r(Z_T) \mid Z_0 = z] \geqq f_N(z) - (N+1)\varepsilon$.

(b) *Let ε be a nonnegative constant, f a nonnegative Baire function and Γ a Borel set. Suppose*

$$J_\Gamma f \leqq J_\Gamma r \tag{2.10}$$

and

$$J_{\Gamma'} f \leqq K_{\Gamma'} f + J_{\Gamma'} \varepsilon\,. \tag{2.11}$$

If $J_{\Gamma'} f$ is bounded and $K_{\Gamma'}$ is a contraction in the space **B** *of bounded Baire functions that vanish on Γ, then*

$$f(z) \leqq E[r(Z_T) \mid Z_0 = z] + \varepsilon/(1-\beta)\,,$$

where T is the first n, if any, for which $Z_n \in \Gamma$ and $\beta < 1$ is the modulus of $K_{\Gamma'}$ in **B**.

PROOF. In part (a), for any Borel set B

$$P[T = 0 \quad \text{and} \quad Z_T \in B \mid Z_0 = z] = J_{\Gamma(N)}(z, B)\,,$$
$$P[T = 1 \quad \text{and} \quad Z_T \in B \mid Z_0 = z] = K_{\Gamma(N)'} J_{\Gamma(N-1)}(z, B)\,,$$

and in general,

$$P[T = n \quad \text{and} \quad Z_T \in B \mid Z_0 = z] = K_{\Gamma(N)'} \cdots K_{\Gamma(N-n+1)'} J_{\Gamma(N-n)}(z, B)\,.$$

Thus

$$E[r(Z_T) \mid Z_0 = \cdot\,] = J_{\Gamma(N)} r + K_{\Gamma(N)'} J_{\Gamma(N-1)} r + \cdots + K_{\Gamma(N)'} K_{\Gamma(N-1)'} \cdots K_{\Gamma(1)'} J_{\Gamma(0)} r\,. \tag{2.12}$$

We are given that $f_0 \leqq J_{\Gamma(0)} r + \varepsilon$ and using (2.9) it is easy to induct that

$$f_n \leqq J_{\Gamma(n)} r + K_{\Gamma(n)'} J_{\Gamma(n-1)} r + \cdots + K_{\Gamma(n)'} K_{\Gamma(n-1)'} \cdots K_{\Gamma(1)'} J_{\Gamma(0)} r + (n+1)\varepsilon\,. \tag{2.13}$$

Now compare the right-hand side in (2.12) with that in (2.13) to complete the proof of (a).

Part (b) is similar. First, $E[r(Z_T)\,|\,Z_0 = \cdot] = H_\Gamma r$ where

$$H_\Gamma = J_\Gamma + \sum_{n\geqq 1} (K_{\Gamma'})^n J_\Gamma .$$

Next $J_\Gamma f$ added to both sides of (2.11) will give

$$f \leqq J_\Gamma f + K_{\Gamma'} f + J_{\Gamma'} \varepsilon$$

which iterated, in the second term on the right, n times becomes

$$\begin{aligned} f &\leqq \textstyle\sum_{k=0}^{n} (K_{\Gamma'})^k J_\Gamma f + (K_{\Gamma'})^{n+1} f + \sum_{k=0}^{n} (K_{\Gamma'})^k J_{\Gamma'} \varepsilon \\ &\leqq \textstyle\sum_{k=0}^{n+1} (K_{\Gamma'})^k J_\Gamma f + \beta^{n+1} \|J_{\Gamma'} f\| + (1 - \beta^{n+1})\varepsilon/(1-\beta) . \end{aligned} \tag{2.14}$$

Let n increase indefinitely.
Then

$$f \leqq H_\Gamma f + \varepsilon/(1-\beta) . \tag{2.15}$$

Finally, from (2.10) $H_\Gamma f = H_\Gamma J_\Gamma f \leqq H_\Gamma J_\Gamma r = H_\Gamma r$. Together with (2.15) we have the desired result, $f \leqq H_\Gamma r + \varepsilon/(1-\beta)$. □

3. The exponential case. We proceed to the proof of Results (i), (ii) and (v). Throughout, α is fixed, strictly positive, and λ is fixed and nonnegative. Continue the definition

$$\begin{aligned} f_\alpha(x) &= \alpha e^{x/\alpha - 1} && \text{for} \quad x \leqq \alpha \\ &= x && \text{for} \quad x > \alpha . \end{aligned}$$

PROPOSITION 1. *Let $\mathbf{M}_e$ be the set of distributions P_X for which $E_X[e^{X/\alpha}] \leqq e^\lambda$, and suppose (X_n) is an $\mathbf{M}_e$-sequence. Then*

$$E[e^{-\lambda T}(x + S_T)^+] \leqq f_\alpha(x)$$

for all x and all stopping times T.

PROOF. We apply part (b) of Lemma 1 with $r(x) = x^+$. We need to show (a) $f_\alpha(x) \geqq x^+$ for all x and (b) $f_\alpha(x) \geqq e^{-\lambda} E_X[f_\alpha(x + X)]$ whenever $E_X[e^{X/\alpha}] \leqq e^\lambda$.

Clearly f_α is nonnegative. Also, $f_\alpha(x) - x$ is decreasing in x, vanishes for $x \geqq \alpha$ and hence is nonnegative. Taken together, we have shown (a).

Set $v(a, x) = a \times \exp(x/a - 1)$. Then $v(a, x + y) \geqq f_\alpha(x + y)$ for all $a \geqq \alpha$ and all x, y. We consider two cases, the first, where $x \leqq \alpha$. Then $f_\alpha(x) = v(\alpha, x) \geqq e^{-\lambda} v(\alpha, x) E_X[e^{X/\alpha}] = e^{-\lambda} E_X[v(\alpha, x + X)] \geqq e^{-\lambda} E_X[f_\alpha(x + X)]$. The second case is $x \geqq \alpha$. Then $0 < x^{-1} \leqq \alpha^{-1}$ and, by convexity of $E_X[e^{\theta X}]$ in θ, we have $E_X[e^{X/x}] \leqq e^\lambda$. Thus, $f_\alpha(x) = x = v(x, x) \geqq e^{-\lambda} v(x, x) E_X[e^{X/x}] = e^{-\lambda} E_X[v(x, x + X)] \geqq e^{-\lambda} E_X[f_\alpha(x + X)]$. Lemma 1 now implies the desired conclusion. □

Allow us very quickly to compute, for purposes of later comparison, a bound on the expectation of $M_x = \sup_n (x + S_n)^+$ when (X_n) is an $\mathbf{M}_e$-sequence. Then, as may be found in Section 8.7 of (Dubins and Savage, 1965), for $y \geqq x^+$, we have $P[M_x \geqq y] \leqq e^{-(y-x)/\alpha}$, which with $E[M_x] = \int_0^\infty P[M_x > y]\, dy$ yields

$$E[M_x] \leqq \alpha e^{x/\alpha} \quad \text{for} \quad x \leqq 0 \qquad (3.1)$$
$$\leqq \alpha + x \quad \text{for} \quad x > 0\,.$$

As defined in Proposition 1, $f_\alpha(x)$ exceeds $f(x, \lambda, \mathbf{M}_e)$, the supremum of $E[e^{-\lambda T}(x + S_T)^+]$ over all stopping times T and $\mathbf{M}_e$-sequences (X_n). Before examining in which cases this supremum equals $f_\alpha(x)$, thus completing the proof of Result (i), we will consider the normal process in (ii).

PROPOSITION 2. *Let $\mathbf{M}_N$ be the set of normal distributions that have a mean of $-\mu$ and a variance of σ^2 together satisfying*

$$\sigma^2 - 2\alpha\mu \leqq 2\alpha^2\lambda\,. \qquad (3.2)$$

If (X_n) is an $\mathbf{M}_N$-sequence then $E[e^{-\lambda T}(x + S_T)^+] \leqq f_\alpha(x)$ for all x and stopping times T.

PROOF. The formula for the moment generating function of a normally distributed random variable shows that (3.2) implies $\mathrm{E}_X[e^{X/\alpha}] \leqq e^\lambda$ whenever P_X is in $\mathbf{M}_N$. The result is then immediate from Proposition 1. □

Where the α's are the same, $\mathbf{M}_N \subset \mathbf{M}_e$, and $f(x, \lambda, \mathbf{M}_N) \leqq f(x, \lambda, \mathbf{M}_e)$. This was the gist of Proposition 2. Thus, to examine under what circumstances our bounds are approached, it may be sufficient to consider only the normally distributed case.

PROPOSITION 3. *If $\lambda = 0$, for all x, $f(x, \mathbf{M}_N) = f_\alpha(x)$. If $\lambda > 0$, then $f(x, \lambda, \mathbf{M}_N) = f_\alpha(x)$ whenever $x \geqq \alpha$ or $(1 - x/\alpha)/\lambda$ is an integer.*

PROOF. For $x \geqq \alpha$, easily $f_\alpha(x) = x \leqq \sup E[e^{-\lambda T}(x + S_T)^+]$. Thus, we consider only $x \leqq \alpha$ and first suppose $\lambda > 0$. The distribution that assigns all mass to $\alpha\lambda$ is considered to be in $\mathbf{M}_N$, and under this distribution, $x + S_n = x + n\alpha\lambda$. The first n for which $x + S_n \geqq \alpha$ is $T = (1 - x/\alpha)/\lambda$ and then $e^{-\lambda T}(x + S_T)^+ = \alpha \times e^{x/\alpha - 1} = f_\alpha(x)$.

Now suppose $\lambda = 0$, and let $X_1, X_2, \cdots$ be independent and identically distributed normal random variables with means $-\mu$ and variances σ^2 subject to

$$\sigma^2 = 2\alpha\mu\,. \qquad (3.3)$$

For any $y \geqq x$ let T_y be the first n, if any, for which $x + S_n \geqq y$. Wald (1947, Appendix 2.5) has shown that as $\sigma \to 0$,

$$P[T_y < \infty] = e^{-(y-x)/\alpha} \times (1 - O(\mu/\sigma)) \qquad (3.4)$$

where the remainder term is uniform in x, y. Set $T = T_\alpha$. From (3.4) $E[(x + S_T)^+] \geqq \alpha e^{(x/\alpha - 1)} \times (1 - O(\mu/\sigma))$ which converges to $f_\alpha(x)$ as $\sigma \to 0$ along (3.3). □

In this normally distributed case, subject to $\sigma^2 \leqq 2\alpha\mu$, the bound (3.1) on the expectation of $M_x = \sup_n (x + S_n)^+$ remains. Since (3.4) holds uniformly, this bound is also approached.

The multiplicative random walk, of the form $W_{n+1} = W_n \times X_{n+1}$ with (X_n) a sequence of independent and identically distributed positive random variables, is generally considered a better model for stock prices than are the additive processes we have considered so far. The multiplicative model is more compatible with the exponential growth curves that, in the long run, are typically observed, and it avoids the possibility of negative prices, a distinct unpleasantness in the additive formulation. The appropriate discounted reward in the multiplicative model is given by Samuelson (1965) as $e^{-\lambda T}(w \times W_T - 1)^+$. Thus we are lead to:

PROPOSITION 4. *Let $\theta > 1$ and $\lambda \geqq 0$ be fixed, let $\mathbf{M}_G(w)$ be the set of distributions $P_{w \times X}$ for which X is a nonnegative random variable such that $E_X[X^\theta] \leqq e^\lambda$. Suppose $(w \times W_n)$ is an $\mathbf{M}_G$-sequence starting at $w > 0$. Then $E[e^{-\lambda T}(w \times W_T - 1)^+] \leqq f(w)$ where*

$$\begin{aligned} f(w) &= [w(\theta - 1)/\theta]^\theta/(\theta - 1) \qquad &&\textit{for} \quad 0 \leqq w \leqq \theta/(\theta - 1) \\ &= w - 1 &&\textit{for} \quad w \geqq \theta/(\theta - 1)\,. \end{aligned}$$

PROOF. We apply part (b) of Lemma 1 with $r(x) = (x - 1)^+$. We need to show (a) $f(w) \geqq (w - 1)^+$ for all $w \geqq 0$ and (b) $f(w) \geqq e^{-\lambda} E[f(w \times X)]$ whenever $E_X[X^\theta] \leqq e^\lambda$. Clearly f is nonnegative. Also $f(w) - (w - 1)$ is decreasing in $w \geqq 0$, vanishes for $w \geqq \theta/(\theta - 1)$ and hence is nonnegative. Taken together, we have shown (a).

Define $v(a, w) = (a - 1)(w/a)^{a/(a-1)}$ for $a > 1$, $w \geqq 0$. Then $v(a, w) \geqq f(w)$ for all $a \geqq \theta/(\theta - 1)$. Suppose first that $w \leqq \theta/(\theta - 1)$. Then $f(w) = v(\theta/(\theta - 1), w) \geqq e^{-\lambda} v(\theta/(\theta - 1), w) E_X[X^\theta] = e^{-\lambda} E_X[v(\theta/(\theta - 1), w \times X)] \geqq e^{-\lambda} E_X[f(w \times X)]$. If $w \geqq \theta/(\theta - 1)$ then $f(w) = w - 1 = v(w, w) \geqq e^{-\lambda} E_X[v(w, w \times X)] \geqq e^{-\lambda} E_X[f(w \times X)]$. Thus (b) holds for all w, which completes the proof. □

A special case, parallel to that in Proposition 2, again may be considered. The results are: Let $\alpha \in (0, 1)$ and $\lambda \geqq 0$ be fixed, and let $\mathbf{M}_L$ be the set of distributions P_X for which $\ln X$ has a Gauss distribution with a mean of $-\mu$ and a variance of σ^2 that together satisfy $-2\alpha\mu + \sigma^2 \leqq 2\alpha^2\lambda$. If (X_n) is an $\mathbf{M}_L$-sequence and $W_n = X_1 \times \cdots \times X_n$ $(W_0 = 1)$ then.

$$\begin{aligned} E[e^{-\lambda T}(w W_T - 1)^+] &\leqq \alpha(1 - \alpha)^{(1-\alpha)/\alpha} w^{1/\alpha} \qquad &&\text{for} \quad 0 \leqq w \leqq 1/(1 - \alpha) \\ &\leqq w - 1 &&\text{for} \quad w \geqq 1/(1 - \alpha)\,. \end{aligned}$$

Again, as in Proposition 3, Wald's result may be used to show that these bounds are approached when $\lambda = 0$.

4. The independent and identically distributed case. Proposition 2 bounded $E[S_T{}^+]$ for a fairly general class of Gaussian processes (S_n), and Proposition 3 showed that this bound was approached by independent and identically distributed summands. This raises the question of bounding $E[S_T{}^+]$ in terms of the common mean and variance of independent and identically distributed, but not necessarily Gaussian, summands.

Throughout this section, $\lambda = 0$ and $X, X_1, X_2, \cdots$ are independent, identically distributed random variables having the common distribution P_X, mean $E_X[X] = -\mu < 0$, and variance $\mathrm{Var}[X] = \sigma^2$. Let $M = \sup_n S_n^+$.

For $k > 0$, Kiefer and Wolfowitz (1956) have shown that

$$E_X[(X^+)^{k+1}] < \infty \quad \text{if and only if} \quad E[M^k] < \infty . \tag{4.1}$$

Hence, for $k = 1$, $E_X[(X^+)^2] < \infty$ implies $E[S_T^+] < \infty$ for all stopping times T. If for some positive δ, $E_X[(X^+)^{2-\delta}] = \infty$, then Ester Samuel (1967) has shown there exists a (randomized) stopping time T for which $E[S_T^+] = \infty$. We can draw a slightly stronger conclusion under the weaker hypothesis that $E[(X^+)^2] = \infty$. Let N be the first n, if any, for which $S_n > 0$. Unpublished work of Darling, Erdös and Kakutani shows $E[M] < \infty$ if and only if $E[S_N] < \infty$. To obtain this result begin with Theorem 2, page 576 of Feller (1966) which states for summands having a negative mean and for $\lambda > 0$,

$$\log E[e^{-\lambda M}] = \sum_1^\infty n^{-1} E[(e^{-\lambda S_n} - 1)^+] . \tag{4.2}$$

Then use Lemma 1, page 569 of Feller (1966) to see that for $|s| < 1$ and $\lambda \geqq 0$,

$$E[s^N e^{-\lambda S_N}] = 1 - \exp\{-\textstyle\sum_1^\infty s^n n^{-1} E[e^{-\lambda S_n^+}; S_n > 0]\} \tag{4.3}$$

which shows

$$\textstyle\sum_1^\infty n^{-1} P[S_n > 0] = -\log P[N = \infty] < \infty , \tag{4.4}$$

and

$$E[e^{-\lambda S_N}; N < \infty] = 1 - \exp\{-\textstyle\sum_1^\infty n^{-1} E[e^{-\lambda S_n}; S_n > 0]\} . \tag{4.5}$$

Hence, the derivative with respect to λ of $-\sum_1^\infty n^{-1} E[e^{-\lambda S_n^+}; S_n > 0]$ is finite at $\lambda = 0$ if and only if both expectations $E[M]$ and $E[S_N; N < \infty]$ are finite, which completes the proof.

If we add the Darling, Erdös and Kakutani result to the necessary and sufficient conditions (4.1) the following strengthening of a theorem in Samuel (1967) results.

THEOREM 1. *Let $X, X_1, X_2, \cdots$ be independent and identically distributed random variables having finite means $E_X[X] < 0$. Let $S_0 = 0$ and $S_n = S_{n-1} + X_n$ for $n \geqq 1$. Then $E[S_T^+] < \infty$ for all stopping times T if and only if $E[(X^+)^2] < \infty$.*

Theorem 1 allows stopping times that never stop, whereas the theorem referred to in Samuel (1967) allows only stopping times that are finite almost surely. Clearly, Theorem 1 implies that if $E[(X^+)^2] < \infty$, then $E[S_T^+] < \infty$ for all stopping times T that are almost surely finite. It is not hard to show that if $E[(X^+)^2] = \infty$, then there is a stopping time T that is finite almost surely and has infinite expected reward, provided one allows randomization (which is not allowed under our stopping time definition in Section 2). For suppose $T^\#$ is a positive integer-valued random variable, independent of the (X_n) process, and set $T = \min\{N, T^\#\}$, where N is the first n, if any, for which $S_n > 0$. Then

$$E[S_T^+] = \textstyle\sum_1^\infty P[T^\# > n] \times \int_{N=n} S_N \, dP ,$$

and it is an elementary exercise in analysis to show that the tail probabilities $P[T^\# > n]$ may be chosen so that the sum remains infinite whenever $E[S_N; N < \infty] = \sum_1^\infty \int_{N=n} S_N \, dP = \infty$. We do not know if there exists a finite and non-randomized stopping time with infinite expected reward whenever $E[(X^+)^2] = \infty$.

Incidentally, that $E_X[(X^+)^2] = \infty$ entails $E[S_N{}^+] = \infty$ seems to have interesting economic implications. Of those that believe that sequences of stock prices form a random walk, where price changes are statistically independent of previous price history, there is a sub-school of thought, associated with the work of Mandelbrot (1963), that holds that daily stock price changes have a stable distribution with characteristic exponent less than two. In particular, the variance of the daily price changes is believed to be infinite, and, by our result, a perpetual option in such a stock (and there are such things) would have infinite (nondiscounted) value.

Since $\mathrm{Var}\,[X] < \infty$ implies $E[S_T{}^+] < \infty$, it is natural for Ester Samuel (1967) to raise the question of bounding $E[S_T{}^+]$ in terms of $\mu = -E_X[X]$ and $\sigma^2 = \mathrm{Var}\,[X]$. N.U. Prabhu showed me Kingman's (1962) result, $E[M] \leqq \alpha$ whenever $\sigma^2 \leqq 2\alpha\mu$. However there is a slight lacuna in Kingman's proof which seems to require $E[M^2] < \infty$. In turn, from (4.1), this requires $E[(X_1{}^+)^3] < \infty$, which is not assumed by Kingman. The problem can be circumvented by truncating to $X_n{}^c = \min\{X_n, c\}$ for $c > 0$. Then, using the obvious definitions, $E[\{(X^c)^+\}^3] < \infty$ so that Kingman's analysis does apply, $\alpha^c \geqq E[M^c] \uparrow E[M]$ as c increases indefinitely, by monotone convergence, while $-\frac{1}{2}\{\mathrm{Var}\,[X^c]/E_X[X^c]\} \to \sigma^2/2\mu \leqq \alpha$ by dominated convergence. However, we cannot discover a proof that avoids truncation.

Thus $E[S_T{}^+] \leqq E[M] \leqq \alpha$ which gives Result (iii).

5. Constraints on the mean and variance only. Again, fix a positive α and a nonnegative λ. Proposition 2 bounds $E[e^{-\lambda T}(x + S_T)^+]$ whenever the conditional distribution of each X_k is normal with mean $-\mu$ and variance σ^2 that together satisfy

$$\sigma^2 - 2\alpha\mu \leqq 2\alpha^2\lambda\,. \tag{5.1}$$

The bound is approached by independent and identically distributed summands. In Section 4 we gave a bound on $E[(S_T)^+]$ when the summands were independent and identically distributed, subject only to (5.1). What if we no longer require that the X_k's be normal, nor independent and identically distributed, but insist only that the conditional means and variances satisfy (5.1)? The key to the analysis is the Chebyshev type bound:

LEMMA 3. (a) *Let Y be a random variable with mean θ and variance ν^2. Then*

$$E_Y[Y^+] \leqq [\theta + (\theta^2 + \nu^2)^{\frac{1}{2}}]/2\,. \tag{5.2}$$

(b) *Suppose, in addition, that*

$$\nu^2 + 2\alpha\theta \leqq 2\alpha^2\beta\,, \tag{5.3}$$

for some nonnegative β *and positive* α. *Then for all* y,

$$E_Y[(y + Y)^+] \leqq \max\{\alpha/2, \alpha\beta + y\}\,. \tag{5.4}$$

PROOF. (a) Begin with $E_Y[|Y|] \leqq (E[Y^2])^{\frac{1}{2}} = (\theta^2 + \nu^2)^{\frac{1}{2}}$. Then apply this in the expectation of $Y^+ = (|Y| + Y)/2$ to get (5.2).

(b) $E_Y[y + Y] = y + \theta$ so that (a) and $\nu^2 \leqq 2\alpha^2\beta - 2\alpha\theta$ successively imply $E[(y + Y)^+] \leqq [y + \theta + ((y + \theta)^2 + \nu^2)^{\frac{1}{2}}]/2 \leqq \nu(\theta)$ where

$$\nu(\theta) = [(y + \theta) + ((y + \theta)^2 + 2\alpha^2\beta - 2\alpha\theta)^{\frac{1}{2}}]/2\,. \tag{5.5}$$

By (5.3), the domain of (5.5) is $-\infty < \theta \leqq \alpha\beta$. Let us compute $\sup_\theta \nu(\theta)$. First, for the derivative, we have,

$$\nu'(\theta) = \tfrac{1}{2}\left\{1 + \frac{y + \theta - \alpha}{((y + \theta)^2 + 2\alpha^2\beta - 2\alpha\theta)^{\frac{1}{2}}}\right\} \tag{5.6}$$

and $\nu'(\theta) \leqq 0$ if and only if $0 \leqq ((y + \theta)^2 + 2\alpha^2\beta - 2\alpha\theta)^{\frac{1}{2}} \leqq \alpha - (y + \theta)$ which reduces to $y \leqq \alpha(1 - 2\beta)/2$. Thus, for $y \leqq \alpha(1 - 2\beta)/2$,

$$\begin{aligned}
\sup_\theta \nu(\theta) &= \lim_{\theta\to-\infty} \nu(\theta)\,, \\
&= \lim_{\theta\to-\infty} \frac{|y + \theta|}{2}\left\{-1 + \left(1 + \frac{2\alpha^2\beta - 2\alpha\theta}{(y + \theta)^2}\right)^{\frac{1}{2}}\right\}\,, \\
&= \lim_{\theta\to-\infty} \frac{|y + \theta|}{2}\left\{\frac{\alpha^2\beta - \alpha\theta}{(y + \theta)^2} + o(\theta^{-1})\right\}\,, \\
&= \lim_{\theta\to-\infty} \frac{1}{2}\,\frac{\alpha^2\beta - \alpha\theta}{|y + \theta|} = \alpha/2\,.
\end{aligned}$$

Otherwise, for $y > \alpha(1 - 2\beta)/2$, we have $\nu'(\theta) > 0$ and $\sup_\theta \nu(\theta) = \nu(\alpha\beta) = (y + \alpha\beta)^+ = y + \alpha\beta$. Thus $E_Y[(y + Y)^+] \leqq \sup_\theta \nu(\theta) \leqq \max\{\alpha/2, y + \alpha\beta\}$.

Note that this lemma implies that the supremum over all $\mathbf{M}_m$-sequences (X_n) of $E[S_T{}^+]$ is at least as great as the supremum of $E[S_1{}^+]$, which is $\alpha/2$.

We want to obtain two bounds on $E[e^{-\lambda T}(x + S_T)^+]$, the first when the conditional means $-\mu$ and variances satisfy $\sigma^2 \leqq 2\alpha\mu + 2\alpha^2\lambda$ and the second when $\sigma^2 \leqq 2\alpha\mu$. The first bound, which corresponds to $\beta = \lambda$ in the following Proposition, is comparable to the bound in the normally distributed case studied in Section 3. The second, which corresponds to $\beta = 0$, we use to obtain a result comparable to the bound in the independent and identically distributed case of Section 4.

PROPOSITION 5. *Let* α *and* λ *be positive numbers, let* β *be nonnegative and let* $\mathbf{M}_{\alpha,\beta}$ *be the set of distributions* P_X *for which* $\operatorname{Var}[X] \leqq -2\alpha E_X[X] + 2\alpha^2\beta$. *If* (X_n) *is an* $\mathbf{M}_{\alpha,\beta}$ *sequence, then*

$$E[e^{-\lambda T}(x + S_T)^+] \leqq \gamma\beta + \max\{\gamma/2, x\}\,, \tag{5.7}$$

where $\gamma = \alpha e^{-\lambda}/(1 - e^{-\lambda})$.

PROOF. Set $f(x) = \gamma\beta + \max\{\gamma/2, x\}$. Clearly $f(x) \geqq x^+$ for all x. Also

$E_X[f(x+X)] = \gamma\beta + E_X[\max\{\gamma/2, x+X\}] = \gamma\beta + \gamma/2 + E_X[(x-\gamma/2+X)^+]$. By Lemma 3 and the assumptions on P_X, $E_X[(x-\gamma/2+X)^+] \leqq \max\{\alpha/2, x-\gamma/2+\alpha\beta\}$. Together, $E_X[f(x+X)] \leqq \max\{(\alpha+\gamma)/2, x+\alpha\beta\} + \gamma\beta = \max\{(\alpha+\gamma)/2 - \alpha\beta, x\} + (\alpha+\gamma)\beta \leqq \max\{(\alpha+\gamma)/2, x\} + (\alpha+\gamma)\beta$. Finally, $e^{-\lambda}(\alpha+\gamma) = \gamma$ so that $e^{-\lambda}E_X[f(x+X)] \leqq \max\{\gamma/2, e^{-\lambda}x\} + \gamma\beta \leqq f(x)$. An application of Lemma 1 completes the proof. Let $\mathbf{M}_\alpha = \mathbf{M}_{\alpha,0}$ be the set of distributions P_X for which $\operatorname{Var}[X] \leqq -2\alpha E_X[X]$.

PROPOSITION 6. *Sup* $E[e^{-\lambda T}(x+S_T)^+] = \max\{\gamma/2, x\}$ *for all* x, *where the supremum is over all stopping times* T *and* $\mathbf{M}_\alpha$*-sequences* (X_n).

PROOF. From (4.7), if (X_n) is an $\mathbf{M}_\alpha$-sequence, then $E[e^{-\lambda T}(x+S_T)^+] \leqq \max\{\gamma/2, x\}$ for $\gamma = \alpha e^{-\lambda}/(1-e^{-\lambda})$. We want to show that this bound is approached. For any x, μ and σ^2 let $a = ((x-\mu)^2+\sigma^2)^{\frac{1}{2}}$, $p = \frac{1}{2} + (x-\mu)/2a$ and $q = 1-p$. The distribution that places probability p on $+a$ and q on $-a$ has mean $x-\mu$ and variance σ^2. Furthermore, as $\mu \to \infty$ and $\sigma^2 \to \infty$ along $\sigma^2 = 2\alpha\mu$, $a \times p \to \alpha/2$ for each x. Let $\mu(x)$ be a Baire function whose further requirements will be specified in a moment. Let $a(x) = ((x-\mu(x))^2 + 2\alpha\mu(x))^{\frac{1}{2}}$, $p(x) = \frac{1}{2} - (x-\mu(x))/2a(x)$, and $q(x) = 1-p(x)$. Let K be the sub-Markov kernel which at each x has the distribution $e^{-\lambda}K'(x, \cdot)$, where $K'(x, \cdot)$ assigns probability $p(x)$ to $a(x)$ and $q(x)$ to $-a(x)$. Let $\varepsilon > 0$ be given. We may, and these are our further requirements on $\mu(\cdot)$, specify that, for all x,

$$a(x)p(x) \geqq \alpha/2 - \varepsilon/2\,, \qquad a(x) \geqq \gamma/2\,, \qquad \text{and}$$
$$p(x)\gamma^2/2(\alpha+\gamma) \leqq \varepsilon/2\,,$$

where

$$\gamma = \alpha e^{-\lambda}/(1-e^{-\lambda})\,.$$

Set $r(x) = x^+$, and $\Gamma = [\gamma/2, \infty)$. Define $f(x) = \max\{\gamma/2, x\}$. Then, using the notation introduced prior to Lemma 2,

$$J_\Gamma f \leqq J_\Gamma r$$

and

$$\begin{aligned} J_{\Gamma'}Kf &= e^{-\lambda}J_{\Gamma'}[q\,\gamma/2 + ap]\,,\\ &\geqq e^{-\lambda}J_{\Gamma'}[q\,\gamma/2 + \alpha/2] - \varepsilon/2\,,\\ &\geqq e^{-\lambda}J_{\Gamma'}[\tfrac{1}{2}(\gamma+\alpha)] - \varepsilon\,,\\ &= J_{\Gamma'}f - \varepsilon\,. \end{aligned}$$

Hence, applying Lemma 2 with the kernel K we conclude

$$f(x) \leqq E[e^{-\lambda T}(Z_T)^+ \mid Z_0 = z] + \varepsilon/(1-e^{-\lambda})\,.$$

Since ε is arbitrary, we have shown that the bound is approached. □

Writing $\mathbf{M}_\alpha$ for $\mathbf{M}_{\alpha,0}$, we have shown that

$$f(x, \lambda, \mathbf{M}_\alpha) = \max\{\gamma/2, x\}$$

where $\gamma = \alpha e^{-\lambda}/(1-e^{-\lambda})$. If we let $\lambda \to 0$ we get $f(x, \mathbf{M}_\alpha) = \infty$, a surprising contrast to (1.5) and (1.6).

It is possible in this particular case to derive explicit bounds on the truncated stopping problem.

PROPOSITION 7. *Let N be a positive integer and for any stopping time T let $T(N) = \min\{T, N\}$. Then*

$$\sup E[(x + S_{T(N)})^+] = \max\{N\alpha/2, x\}$$

where the supremum is over all stopping times T and $\mathbf{M}_\alpha$-sequences (X_n).

PROOF. For $n = 0, 1, \cdots, N$ let $f_n(x) = \max\{n\alpha/2, x\}$. First we show $f_N(x)$ bounds the expected return. Clearly $f_1(x) \geqq f_0(x) = x^+$ for all x. To apply Lemma 1 we need to show $f_{n+1}(x) \geqq E_X[f_n(x + X)]$ whenever $P_X \in \mathbf{M}_\alpha$. But $E_X[f_n(x + X)] = E_X[\max\{n\alpha/2, x + X\}] = n\alpha/2 + E_X[\max\{0, x + X - n\alpha/2\}] \leqq n\alpha/2 + \max\{\alpha/2, x - n\alpha/2\} = \max\{(n + 1)\alpha/2, x\} = f_{n+1}(x)$, where the inequality derives from Lemma 3. Lemma 1 now implies $f_n(x) \geqq E[(x + S_{T(N)})^+]$ for all stopping times T.

To show that this bound is tight, introduce the Markov kernel K where for any x, $K(x, \cdot)$ assigns probability $p(x)$ to $a(x)$ and $1 - p(x)$ to $-a(x)$ with

$$a(x) = ((x - \mu(x))^2 + 2\alpha\mu(x))^{\frac{1}{2}}$$

and

$$p(x) = \tfrac{1}{2} - (x - \mu(x))/2a(x) .$$

For any given $\varepsilon > 0$, choose $\mu(\cdot)$ to be Baire and such that

$$a(x)p(x) \geqq \alpha/2 - \varepsilon/2 , \qquad a(x) \geqq N\alpha$$

and

$$p(x)N\alpha/2 \leqq \varepsilon/2 ,$$

for all x. That this can be done was shown in Proposition 7.

Set $\Gamma(n) = [n\alpha/2, \infty)$ for $n = 0, \cdots, N$. Then using the notation of Lemma 2, with $r(x) = x^+$, $f_0 \leqq J_{\Gamma(0)} r$
while $Kf_{n-1}(x) = a(x)p(x) + [(n - 1)\alpha/2]q(x) \geqq n\alpha/2 - \varepsilon$,
so that

$$J_{\Gamma(n)'} f_n \leqq K_{\Gamma(n)'} f_{n-1} + \varepsilon ,$$

while

$$J_{\Gamma(n)} f_n \leqq J_{\Gamma(n)} r$$

which added gives (2.9) of Lemma 2, which then applies. Thus, for T the first n, if any, for which $Z_n \in \Gamma(N - n)$ we have $E[r(Z_T) \mid Z_0 = z] \geqq f_N(z) - (N + 1)\varepsilon$. Since ε is arbitrary, we are through.

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Added in proof. Following Theorem 1 we ask if there exists a finite and nonrandomized stopping time with infinite expected reward whenever $(*)E[(X^+)^2] = \infty$. The referee suggests we choose a function $\tau(x)$ and stop at $T = \min\{N, \tau(X_1)\}$, where N is the first n, if any, for which $S_n > 0$. Using $(*)$, we may choose τ large enough so that the expected reward is infinite, but T does not involve additional randomness. The manner of choosing τ parallels that given for choosing $T^\#$ in the paper.

On the same point, Larry Shepp has shown me how to choose a hitting time to a set that is finite and has infinite expected reward. The set is a union of intervals $[a_n, b]$ where $0 > a_1 > b_1 > a_2 > b_2 \cdots$. The intervals are sparse enough to give infinite expected reward, yet dense enough to ensure stopping.