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# CONVERGENCE IN DISTRIBUTION OF RANDOM MEASURES

BY MILOSLAV JIŘINA

*Flinders University*

Let $(S, \mathscr{S})$ be a measurable space, $M$ the set of all finite measures on $\mathscr{S}$, $\mathscr{F}_M$ the $\sigma$-algebra generated by the family of all measurable cylindrical sets $\bigcap_{i=1}^k \{\mu \in M: \mu(A_i) \leqq a_i\}$. With each probability measure $P$ on $\mathscr{F}_M$ the family $\{P_{A_1,\cdots,A_k}\}$ of all finite-dimensional probability measures of the cylidrical sets is associated. The following problem is considered: Given a sequence $P^{(n)}$ of probability measures on $\mathscr{F}_M$ such that each sequence $P^{(n)}_{A_1,\cdots,A_k}$ converges weakly to a $k$-dimensional probability measure $P_{A_1,\cdots,A_k}$, does the family $\{P_{A_1,\cdots,A_k}\}$ generate a probability measure $P$ on $\mathscr{F}_M$? It is proved that the answer is affirmative if $(S, \mathscr{S})$ is the Euclidean $n$-space with the $\sigma$-algebra of Borel sets.

**1. Introduction.** In the whole paper, $R = (-\infty, \infty)$, $\mathscr{B}^{(k)} =$ the $\sigma$-algebra of all Borel sets in $R^k$, $R_+ = [0, \infty)$. Let $T$ be an arbitrary index set, $F$ a subset of $R^T =$ the set of all real functions on $T$, $\mathscr{F}_F$ the least $\sigma$-algebra containing all sets $\{f \in F: f(t) \in E\}$, $t \in T$, $E \in \mathscr{B}^{(1)}$. If $P$ is a probability measure on $\mathscr{F}_F$, then the probability measures $P_{t_1,\cdots,t_k}$ on $\mathscr{B}^{(k)}$ ($k = 1, 2, \cdots, t_i \in T$) defined by $P_{t_1,\cdots,t_k}(E_1 \times \cdots \times E_k) = P(\bigcap_{i=1}^k \{f \in F: f(t_i) \in E_i\})$, $E_i \in \mathscr{B}^{(1)}$, will be called the finite-dimensional probability distributions (f.d.p.d.'s) of $P$. We shall adopt the following two definitions:

DEFINITION 1. Let $P^{(n)}$, $P$ be probability measures on $\mathscr{F}_F$. We shall say that $P^{(n)}$ converge in distribution (or $D$-converge) to $P$, if each f.d.p.d. of $P^{(n)}$ converges weakly to the corresponding f.d.p.d. of $P$; $P$ will be called the $D$-limit of $P^{(n)}$.

DEFINITION 2. A sequence $P^{(n)}$ of probability measures on $\mathscr{F}_F$ will be called fundamental in distribution (or $D$-fundamental), if each f.d.p.d. of $P^{(n)}$ converges weakly to a finite-dimensional probability measure. It is clear that the $D$-limit $P$, if it exists, is unique. Using the well-known Kolmogorov theorem we can see easily that each $D$-fundamental sequence is $D$-convergent if $F = R^T$. This need not be true if $F$ is a proper subspace of $R^T$; e.g., take $T = [0, 1]$, $F =$ the set of all continuous functions on $T$, $P^{(n)} =$ the probability measure concentrated on the one-point set $\{f_n\}$, where $f_n(t) = t^n$. It is therefore rather a surprising fact that each $D$-fundamental sequence $P^{(n)}$ is $D$-convergent, if $F$ is the set of all finite measures on $(R^m, \mathscr{B}^{(m)})$ or, more generally, on any measurable space $(S, \mathscr{S})$ satisfying the conditions C listed below. This is the main result of this paper—see Theorem 2.

**2. Random measures.** Let $S$ be an arbitrary set, $\mathscr{S}$ a $\sigma$-algebra of subsets of $S$, $M$ the set of all finite measures on $\mathscr{S}$. Since $M$ is a subset of $R^{\mathscr{S}}$, $\mathscr{F}_M$ is the

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least $\sigma$-algebra containing all sets $\{\mu \in M: \mu(A) \in E\}$, $A \in \mathscr{S}$, $E \in \mathscr{B}^{(1)}$. If $P$ is a probability measure on $\mathscr{F}_M$, then the probability field $\{M, \mathscr{F}_M, P\}$ is one of the possible models for a random measure. It is easy to see that the corresponding f.d.p.d.'s $P_{A_1,\cdots,A_k}$ (with $A_i \in \mathscr{S}$) satisfy the following *conditions* M:

(M1) $$P_{A_1,\cdots,A_k}(E_1 \times \cdots \times E_k) = P_{A_1,\cdots,A_k,A_{k+1}}(E_1 \times \cdots \times E_k \times R);$$

(M2) $$P_{A_1,\cdots,A_k}(R_+{}^k) = 1;$$

(M3) If $A_1 \cap A_2 = \emptyset$ and $A_3 = A_1 \cup A_2$, then $P_{A_1,A_2,A_3}$ is concentrated on the plane $x_3 = x_1 + x_2$ in $R^3$.

(M4) If $A_j \supset A_{j+1}$, $\bigcap_{j=1}^\infty A_j = \emptyset$, then for each $x > 0$

$$P_{A_j}((x, \infty)) \to_{j\to\infty} 0 .$$

We shall say that the measurable space $\{S, \mathscr{S}\}$ satisfies *conditons* C if there exists a *countable* algebra $\mathscr{A}$ and a class $\mathscr{C}$ of subsets of $S$ such that

(C1) $\mathscr{S}$ is generated by $\mathscr{A}$;

(C2) $$C_1, C_2 \in \mathscr{C} \Rightarrow C_1 \cap C_2 \in \mathscr{C};$$

(C3) $$C_j \in \mathscr{C}, \quad C_j \supset C_{j+1}, \quad C_j \neq \emptyset, \quad k = 1, 2, \cdots \Rightarrow \bigcap_{j=1}^\infty C_j \neq \emptyset ;$$

(C4) To each $A \in \mathscr{A}$ there exist sequences $B_{j,A} \in \mathscr{A}$ and $C_{j,A} \in \mathscr{C}$ such that $B_{j,A} \subset B_{j+1,A}$, $B_{j,A} \subset C_{j,A} \subset A$ and $\bigcap_{j=1}^\infty B_{j,A} = A$.

$\{R^m, \mathscr{B}^{(m)}\}$ and some other metric or topological spaces satisfy conditions C.

The following theorem is a generalization of Theorem 3.1 Chapter III of [2]. The Appendix 1 to Chapter III of [2] contains the basic ideas of its proof and some details are implicitly contained in the proof of slightly different assertion in [3], Theorem 1.4. We shall therefore present the main steps of the proof only and omit the details.

THEOREM 1. *Let measurable space* $\{S, \mathscr{S}\}$ *satisfy the condition* C *and let a family* $\{P_{A_1,\cdots,A_k}\}$ *of probability measures on* $\mathscr{B}^{(k)}$ $(k = 1, 2, \cdots, A_i \in \mathscr{S})$ *satisfy the conditions M. Then there exists exactly one probabilty measure P on* $\mathscr{F}_M$ *such that* $P_{A_1,\cdots,A_k}$ *are its* f.d.p.d.'*s*.

PROOF. The uniqueness follows from the fact that the family of all sets $\bigcap_{i=1}^k \{\mu \in M: \mu(A_i) \in E_i\}$ is a half-ring generating $\mathscr{F}_M$. The existence follows from the following construction:

*Step* 1. Let $F_0 = R^{\mathscr{A}} =$ the set of all set functions on the algebra $\mathscr{A}$ of (C1). Using (M1) and the well-known Kolmogorov theorem we can construct a probability measure $P^{(0)}$ on $\mathscr{F}_{F_0}$ such that $P_{A_1,\cdots,A_k}$ (with $A_i \in \mathscr{A}$) are its f.d.p.d.'s.

*Step* 2. Let $F_1$ be the set of all finite, nonnegative and two-additive set functions on $\mathscr{A}$. Since $\mathscr{A}$ is countable, $F_1 \in \mathscr{F}_{F_0}$ and (M2) and (M3) imply that $P^{(0)}(F_1) = 1$. Hence, the restriction $P^{(1)}$ of $P^{(0)}$ to $\mathscr{F}_{F_1} = F_1 \cap \mathscr{F}_{F_0}$ is a probability measure on $\mathscr{F}_{F_1}$ such that $P_{A_1,\cdots,A_k}$ are its f.d.p.d.'s.

*Step* 3. Let $F_2$ be the set of all $\nu \in F_1$ such that for each $A \in \mathscr{A}$

(1) $$\nu(A - B_{k,A}) \to_{k\to\infty} 0\,,$$

where $B_{k,A}$ are the sets mentioned in (C4). Again $F_2 \in \mathscr{F}_{F_1}$ and $P^{(1)}(F_2) = 1$ because of (M4) and (C4). Hence, the restriction $P^{(2)}$ of $P^{(1)}$ to $\mathscr{F}_{F_2} = F_2 \cap \mathscr{F}_{F_1}$ is a probability measure on $\mathscr{F}_{F_2}$ such that $P_{A_1,\cdots,A_k}$ are its f.d.p.d.'s. Each $\nu \in F_2$ is finite, nonnegative and finitely additive set function on $\mathscr{A}$ satisfying (1) with $B_{k,A}$ satisfying (C4). Using this and the properties (C2) and (C3) we can prove that $\nu$ is countably additive on $\mathscr{A}$ and can be therefore extended uniquely to a finite measure $\bar{\nu}$ on $\mathscr{S}$. For each fixed $A \in \mathscr{S}$, $\bar{\nu}(A)$ is (as a function of $\nu \in F_2$) $\mathscr{F}_{F_2}$-measurable. This is trivial for $A \in \mathscr{A}$ and it can be proved for all $A \in \mathscr{S}$ by means of the monotone-class theorem. ([1], Chapter I, Section 6, Theorem B).

*Step* 4. Let $h$ be the transformation (of $F_2$ into $M$) assigning to each $\nu \in F_2$ its extension $\bar{\nu}$. The transformation $h$ is $\mathscr{F}_{F_2} - \mathscr{F}_M$-measurable and the set function $P$ on $\mathscr{F}_M$ defined by $P(D) = P_2(h^{-1}(D))$ is a probability measure on $\mathscr{F}_M$. It remains to show that $P_{A_1,\cdots,A_k}$ are its f.d.p.d.'s. This is trivial for $A_i \in \mathscr{A}$. To prove it generally, the monotone-class theorem is to be applied successively to each of the indices $A_1, \cdots, A_k$. The following four relations are essential in that part of the proof.

If $A_{k,j} \subset A_{k,j+1}$, $A_k = \bigcup_{j=1}^{\infty} A_{k,j}$, then

(2) $$P(\{\mu \in M\colon \mu(A_i) \in E_i,\, i = 1, 2, \cdots, k-1,\, \mu(A_k) \leqq x\}) = \lim_{j\to\infty} P(\{\mu \in M\colon \mu(A_i) \in E_i,\, i = 1, 2, \cdots, k-1,\, \mu(A_{k,j}) \leqq x\})$$

and

(3) $$P_{A_1,\cdots,A_{k-1},A_k}(E_1 \times \cdots \times E_{k-1} \times [0, x]) = \lim_{j\to\infty} P_{A_1,\cdots,A_{k-1},A_{k,j}}(E_1 \times \cdots \times E_{k-1} \times [0, x])\,.$$

If $A_{k,j} \supset A_{k,j+1}$, $A_k = \bigcap_{j=1}^{\infty} A_{k,j}$, then

(4) $$P(\{\mu \in M\colon \mu(A_i) \in E_i,\, i = 1, 2, \cdots, k-1,\, \mu(A_k) < x\}) = \lim_{j\to\infty} P(\{\mu \in M\colon \mu(A_i) \in E_i,\, i = 1, 2, \cdots, k-1,\, \mu(A_{k,j}) < x\})$$

and

(5) $$P_{A_1,\cdots,A_{k-1},A_k}(E_1 \times \cdots \times E_{k-1} \times [0, x)) = \lim_{j\to\infty} P_{A_1,\cdots,A_{k-1},A_{k,j}}(E_1 \times \cdots \times E_{k-1} \times [0, x))\,.$$

The relations (2) and (4) are easy; (3) and (5) must be derived directly from the conditions M. Let us do that for (3). To simplify writing, we shall omit the indices $A_1, \cdots, A_{k-1}$ (as if $E_1 = \cdots = E_{k-1} = R$) and we shall write $B_j$, $B$ instead of $A_{k,j}$, $A_k$. Using the 3-dimensional distribution $P_{B_j, B-B_j, B}$ and (M1)—(M3), we see that for any $x > 0$ and $\varepsilon > 0$

$$P_{B_j}([0, x]) \geqq P_{B_{j+1}}([0, x]) \geqq P_B([0, x])$$

and

$$P_{B_j}([0, x]) \leqq P_B([0, x + \varepsilon]) + P_{B-B_j}((\varepsilon, \infty))\,.$$

Hence, by (M4),

$$P_B([0, x]) \leqq \lim P_{B_j}([0, x]) \leqq P_B([0, x + \varepsilon]) .$$

Since $\varepsilon > 0$ was arbitrary, $P_B([0, x]) = \lim P_{B_j}([0, x])$, which proves (3). The proof of (5) is similar.

### 3. *D*-convergence of random measures.

THEOREM 2. *Let the measurable space $\{S, \mathscr{S}\}$ satisfy the condition* C. *Then each D-fundamental sequence of probability measures $P^{(n)}$ on $\{M, \mathscr{F}_M\}$ is D-convergent.*

PROOF. Let us denote the f.d.p.d.'s of $P^{(n)}$ by $P^{(n)}_{A_1,\cdots,A_k}$ ($n \geqq 1$). By assumption, to each sequence $A_1, \cdots, A_k$ ($A_i \in \mathscr{S}$) there exists a probability measure $P^{(0)}_{A_1,\cdots,A_k}$ on $\mathscr{B}^{(k)}$ such that

$$P^{(n)}_{A_1,\cdots,A_k} \rightarrow_{n\rightarrow\infty} P^{(0)}_{A_1,\cdots,A_k} \quad \text{(weakly)}. \tag{6}$$

By Theorem 1, it is sufficient to show that the family $P^{(0)}_{A_1,\cdots,A_k}$ satisfies the conditions M. Since, for each $n \geqq 1$, the family $P^{(n)}_{A_1,\cdots,A_k}$ satisfies M, it is sufficient to show that the conditions M are preserved under (6). This is trivial for (M1), (M2) and (M3) and only (M4) remains to be proved. To an arbitrary $\varepsilon > 0$ there exists $a > 0$ such that

$$P_S((a, \infty)) < \tfrac{1}{2}\varepsilon \quad \text{and} \quad P_S(\{a\}) = 0 . \tag{7}$$

Put

$$\begin{aligned} f(x) &= 0 \quad \text{if } x \leqq 0 \\ &= x \quad \text{if } 0 \leqq x \leqq a \\ &= a \quad \text{if } x \geqq a \end{aligned} \qquad \text{and}$$

$$M_n(A) = \int_{R\times(-\infty,a)} f(x)P^{(n)}_{A,S}(d(x, y))$$

for all $A \in \mathscr{S}$ and $n \geqq 0$. The function $f$, as a function of $(x, y)$, is bounded and continuous on the domain $R \times (-\infty, a)$ and the boundary of this domain has $P^{(0)}_{A,S}$-measure zero by (M1) and (7). Hence the weak convergence (6) implies

$$M_n(A) \rightarrow M_0(A) \qquad \text{for each } A \in \mathscr{S}. \tag{8}$$

Since $\mu(A) \leqq \mu(S)$ for all $\mu \in M$,

$$P^{(n)}_{A,S}((a, \infty) \times (-\infty, a)) = 0 \tag{9}$$

for all $n \geqq 1$. Hence

$$M_n(A) = \int_{R_+(-\infty,a)} xP^{(n)}_{A,S}(d(x, y)) = \int_{\{\mu\in M:\, \mu(S)<a\}} \mu(A)P^{(n)}(d\mu)$$

for all $n \geqq 1$. The second integral shows that $M_n$ is, for each $n \geqq 1$, a measure on $\mathscr{S}$ and that the sequence $M_n$ is uniformly bounded (by $a$). Hence, by (8) and a well-known theorem, $M_0$ is also a finite measure on $\mathscr{S}$. (9) is clearly preserved under (6) and, therefore

$$M_0(A) = \int_{R_+\times(-\infty,a)} xP^{(0)}_{A,S}(d(x, y)) \geqq xP^{(0)}_{A,S}((x, \infty) \times (-\infty, a)) \tag{10}$$

for all $A \in \mathscr{S}$ and an arbitrary $x > 0$.

Consider now a sequence $A_j \in \mathscr{S}$, $A_j \supset A_{j+1}$, $\bigcap_{j=1}^{\infty} A_j = \emptyset$. Since $M_0$ is a finite measure, $M_0(A_j) \to_{j\to\infty} 0$ and therefore, by (10), there exists $j_0$ such that

$$P^{(0)}_{A_j,S}((x, \infty)) \times (-\infty, a)) < \tfrac{1}{2}\varepsilon \qquad \text{for all } j \geqq j_0 . \tag{11}$$

Finally, by (7), (11) and (M1)

$$P^{(0)}_{A_j}((x, \infty)) = P^{(0)}_{A_j,S}((x, \infty) \times (-\infty, a)) + P^{(0)}_{A_j,S}((x, \infty) \times [a, \infty)) < \varepsilon$$

for all $j \geqq j_0$, which proves that (M4) holds for $P^{(0)}_{A_1,\cdots,A_k}$.

## REFERENCES

[1] Halmos, P. R. (1950). *Measure Theory*. Van Nostrand, New York.
[2] Harris, T. E. (1963). *The Theory of Branching Processes*. Springer-Verlag, Berlin.
[3] Jiřina, M. (1964). Branching processes with measure-valued states. *Trans. Third Prague Conf. Inform. Theor.* 333-357.

School of Mathematical Sciences
Flinders University
Bedford Park, South Australia 5042