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ON ESTIMATION OF TAIL END PROBABILITIES OF THE SAMPLE MEAN FOR LINEAR STOCHASTIC PROCESSES

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Let $\{X_j, j \geqq 1\}$ be a linear process defined by the relation $X_j = \sum_{r=0}^{\infty} g_v Y_{j-v}$, where $\{Y_j; j = 0, \pm 1, \cdots\}$ is a sequence of i.i.d. random variables which possess a mgf $M_Y(t)$ over an open interval $I = (-c, c)$ $(c > 0)$. Let a be a fixed positive constant and denote the mgf of $Z_1 = Y_1 - a$ by $M_z(t)$. Assume that $E\{Y_1\} = 0$, $|g_v| \leqq C\rho_0{}^v$ $(v \geqq 0)$ for some finite positive constants ρ_0 (< 1), C and $\sum_{v=0}^{\infty} g_v \neq 0$ (we can take $\sum_{v=0}^{\infty} g_v = 1$ without loss of generality). Further assume that there exists a constant $\tau \in I_0$, $I_0 = (-cA^{-1}, cA^{-1})$, $A = \sum_{v=0}^{\infty} |g_v|$, such that $\rho = M_Z(\tau) = \inf_{t \in I} M_Z(t) < 1$ and $M_Z'(\tau) = 0$. Then it is proved that for each $u = 0, 1, 2, \cdots$ we can find a bounded sequence $\{\beta_{u,n}\}$ of constants such that for any integer $r \geqq 3$ $P(\sum_{j=1}^{n} X_j/n \geqq a) = (\rho^n \lambda_n/\tau\sigma_n(2\pi)^{\frac{1}{2}})[\sum_{u=0}^{r-3} \beta_{u,n}\sigma_n{}^{-u} + O(\sigma_n{}^{-(r-2)})]$ as $n \to \infty$, where $\{\lambda_n\}$ and $\{\sigma_n\}$ are sequences of positive constants, and, as $n \to \infty$, λ_n is bounded away from 0 and ∞ and $n^{-1}\sigma_n{}^2$ approaches a finite positive constant.

1. Introduction. Let $\{Y_j; -\infty < j < \infty\}$ be a doubly infinite sequence of independent and identically distributed (i.i.d.) random variables (rv) which possess a finite moment generating function (mgf) $M_Y(t)$ over an open interval $I = (-c, c)$, $(c > 0)$ with $E\{Y_1\} = 0$. Let $\{X_j; j \geqq 1\}$ be another sequence of rv's defined by

$$X_j = \textstyle\sum_{v=0}^{\infty} g_v Y_{j-v}, \qquad \sum_0^{\infty} g_v \neq 0, \tag{1}$$

where we assume that there exists a positive constant $\rho_0 < 1$ such that $|g_v| < C\rho_0{}^v$, C being a generic symbol which denotes a finite positive constant. For a fixed number $a > 0$ consider the probability

$$P_n(a) = P\{\textstyle\sum_{j=1}^{n} X_j/n \geqq a\}. \tag{2}$$

The object of this article is to obtain an estimate $Q_n(a)$ of $P_n(a)$ which is precise in the sense that

$$P_n(a)/Q_n(a) = 1 + O(1) \qquad \text{as } n \to \infty. \tag{3}$$

The above formulation has been largely motivated by the rather ingenious result derived by Bahadur and Ranga Rao [1] a few years ago about $P_n(a)$ when the X_j's are i.i.d.; i.e., $g_v = 0$ for all $v > 0$. These authors have used the results concerning the asymptotic expansion of the characteristic function of $\sum_{j=1}^{n} X_j$ derived by Cramér [4] and Esséen [5] for the special case when the mgf of X_1 exists and satisfies certain regularity conditions. One important result due to Koopmans [6] regarding the validity of the strong law of large numbers for linear processes (1) and an extension due to Chanda [3] of the results by Berry [2] and

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others so as to apply to linear processes have suggested the possibility of extending Bahadur and Rango Rao's results to linear processes.

In order not to make the discussion too complicated mathematically, we assume throughout that the cumulative distribution function (cdf) of Y_1 satisfies Cramér's condition (C) [4, page 81]. Also, without loss of generality we assume that $\sum_{v=0}^{\infty} g_v = 1$.

Write $Z_j = Y_j - a$. Then $E\{Z_1\} = -a < 0$. Denote the mgf of Z_1 by $M_Z(t)$ and let us assume that there exists a finite positive constant τ such that

$$\rho = M_Z(\tau) = \inf_{t \in I} M_Z(t) < 1 \tag{4}$$

and $M_Z'(\tau) = 0$.
Condition (4) is satisfied if, for example, $M_Y(t) < \infty$ for all finite t and $P(Y_1 > a + \delta) > 0$ for some $\delta > 0$. Since $M_Z(t) < \infty$ for all $t \in I$, it follows that $M_Z^{(r)}(t) < \infty$ for all finite r and $t \in I$. Write $I_0 = (-cA^{-1}, cA^{-1})$ where $A = \sum_{v=0}^{\infty} |g_v| \geqq 1$. Then $I_0 \subset I$.

The main result of this article can now be stated in the form of the following

THEOREM. *Assume that $\tau \in I_0$. Then for each $u = 0, 1, 2, \cdots$, there exists a bounded sequence of constants $\beta_{u,1}, \beta_{u,2}, \cdots$ such that for any positive integer $r \geqq 3$*

$$P_n(a) = (\rho^n \lambda_n / \tau \sigma_n (2\pi)^{\frac{1}{2}})[\sum_{u=0}^{r-3} \beta_{u,n} \sigma_n^{-u} + O(\sigma_n^{-(r-2)})] , \tag{5}$$

as $n \to \infty$, where $\{\lambda_n\}$ and $\{\sigma_n\}$ are sequences of positive constants such that, as $n \to \infty$, λ_n is bounded away from 0 and ∞, and $n^{-1}\sigma_n^2$ approaches a finite positive constant.

2. A few lemmas. Let $W_j = X_j - a$. Then

$$W_j = \sum_{v=0}^{\infty} g_v Z_{j-v} ,$$

where $Z_j = Y_j - a$. Now define for every n, the cdf G_n by

$$dG_n(w) = \exp(\tau w)\, dF_n(w)/\Psi_n(\tau) , \tag{6}$$

where $F_n(w)$ is the cdf of $\sum_{j=1}^{n} W_j$ and $\Psi_n(t)$ is the mgf of $\sum_1^n W_j$. Let V_n be a rv with the cdf $G_n(v)$. Then

$$\text{mgf of } V_n = M_{V_n}(t) = \Psi_n(t + \tau)/\Psi_n(\tau) . \tag{7}$$

LEMMA 1. *There exists a function $\lambda_n(t)$ (> 0) of n and t bounded in n and $t \in I_0$ such that*

$$\Psi_n(t) = [M_Z(t)]^n \lambda_n(t)$$

for all $t \in I_0$.

PROOF. It is easy to see that

$$\Psi_n(t) = \prod_{u=-\infty}^{n} M_Z(c_{n,u} t) ,$$

where

$$c_{n,u} = \sum_{\max(0,1-u)}^{n-u} g_v .$$

For every n, $\Psi_n(t)$ exists for all $t \in I_0$ (see Koopmans [6]). Also

$$\log \Psi_n(t) = \sum_{u=1}^{n} K_Z(c_{n,u} t) + \sum_{u=0}^{\infty} K_Z(d_{n,u} t) ,$$

where $d_{n,u} = c_{n,-u}$, $u \geqq 0$ and $K_Z(t) = \log M_Z(t)$. Again

$$(8) \qquad |\sum_0^\infty K_Z(d_{nu}t)| \leqq |t| \sum_0^\infty |d_{n,u}| \, |M_Z'(t_{n,u})|/M_Z(t_{n,u}) ,$$

where $t_{n,u}$ lies between 0 and $d_{n,u}t$. But for $u \geqq 0$

$$|d_{n,u}| \leqq \sum_{1+u}^{n+u} |g_v| \leqq C \sum_{v=1+u}^{n+u} \rho_0^{\,v} \leqq C\rho_0^{\,1+u} .$$

Also $|d_{n,u}| < A$. Hence for all $t \in I_0$, $t_{n,u} \in I$ and consequently $|M_z'(t_{n,u})| < C$. We can, therefore, write

$$(9) \qquad \sum_{u=0}^\infty K_Z(d_{n,u}t) = \log \lambda_n^{(1)}(t) ,$$

where $\lambda_n^{(1)}(t) > 0$ and is bounded in n and $t \in I_0$. Again since $|M_Z'(t)| < \infty$ for all $t \in I$ implies that $|K_Z'(t)| < \infty$ for all $t \in I$, we have

$$\begin{aligned} |\sum_1^n K_Z(c_{n,u}t) - nK_Z(t)| &\leqq C \sum_{u=1}^n |c_{n,u}t - t| \\ &\leqq C|t| \sum_{u=1}^n \sum_{v=n-u+1}^n |g_v| \\ &\leqq C|t| \sum_{u=1}^n \sum_{v=u}^\infty \rho_0^{\,v} < C|t| , \end{aligned}$$

whenever $t \in I_0$.

Thus

$$(10) \qquad \sum_1^n K_z(c_{n,u}t) = nK_z(t) + \log \lambda_n^{(2)}(t) ,$$

where again $\lambda_n^{(2)}(t) > 0$ and is bounded in n and $t \in I_0$. Combining (9) and (10) and writing $\lambda_n(t) = \lambda_n^{(1)}(t)\lambda_n^{(2)}(t)$, we easily have the result of the lemma.

COROLLARY. *Assume that I is such that $\tau \in I_0$. Then*

$$(11) \qquad \Psi_n(\tau) = \rho^n\lambda_n$$

where $\rho = M_Z(\tau) < 1$ and $\lambda_n = \lambda_n(\tau)$.

PROOF. The result follows easily from Lemma 1, and condition (4).

LEMMA 2. *Let for any fixed number $a > 0$*

$$P_n(a) = P\{\sum_{j=1}^n X_j/n \geqq a\} .$$

Then

$$(12) \qquad P_n(a) = \rho^n\lambda_n \int_0^\infty \exp(-\tau w)\, dG_n(w) .$$

PROOF. By definition,

$$\begin{aligned} P_n(a) &= P\{\sum_{j=1}^n W_j \geqq 0\} \\ &= \int_0^\infty dF_n(w) \\ &= \Psi_n(\tau) \int_0^\infty \exp(-\tau w)\, dG_n(w) \\ &= \rho^n\lambda_n \int_0^\infty \exp(-\tau w)\, dG_n(w) , \qquad \text{(by corollary to Lemma 1).} \end{aligned}$$

LEMMA 3. *If $\gamma_{r,n}$ denotes the* rth *cumulant of V_n defined in Section 2 then $\gamma_{r,n}$ exist for all finite r and $|\gamma_{r,n}| \leqq Cn$.*

PROOF. By (8)

$$\log M_{V_n}(t) = \log \Psi_n(t + \tau) - \log \Psi_n(\tau)$$

so that

$$\gamma_{r,n} = [d^r \log M_{V_n}(t)/dt^r]_{t=0} \tag{13}$$
$$= [d^r \log \Psi_n(t+\tau)/dt^r]_{t=0}$$
$$= n[d^r K_Z(t+\tau)/dt^r]_{t=0} + [d^r w_n(t+\tau)/dt^r]_{t=0}$$
$$= nK_Z^{(r)}(\tau) + w_n^{(r)}(\tau) ,$$

where $w_n(t) = \log \lambda_n(t)$. It is easy to see that both $K_Z^{(r)}(\tau)$ and $w_n^{(r)}(\tau)$ exist for all finite r and are of order unity. The result of the lemma, therefore, follows immediately.

Note that if $\mu_n = \gamma_{1,n} = E\{V_n\}$ then $\mu_n = w_n'(\tau)$ (since $K_Z'(\tau) = 0$) is of order unity. Write $\sigma_n^2 = \gamma_{2,n}$. Then $n^{-1}\sigma_n^2 \to K_Z''(\tau)$ as $n \to \infty$. Note that $K_Z''(\tau) \geqq 0$; equality cannot hold for otherwise Z_1 is a constant a.e., which is impossible under our assumptions.

3. Proof of the theorem. Let $H_n(u)$ denote the cdf of $U_n = (V_n - \mu_n)/\sigma_n$. Then

$$\int_0^\infty \exp(-\tau y)\, dG_n(y) = \exp(-\tau\mu_n) \int_{\alpha_n}^\infty \exp(-\tau\sigma_n y)\, dH_n(y) \tag{14}$$
$$= \exp(-\tau\mu_n)\tau\sigma_n \int_{\alpha_n}^\infty \exp(-\tau\sigma_n y)\{H_n(y) - H_n(\alpha_n)\}\, dy$$
$$= \exp(-\tau\mu_n) I_n \quad \text{say,}$$

where $\alpha_n = -\mu_n/\sigma_n$. Again since $\gamma_{r,n}$ is of order n we can use the arguments leading to the theorem in Chanda [3] to prove that for any finite y

$$H_n(y) = K_{n,r}(y) + R_{n,r}(y) , \tag{15}$$

where

$$K_{n,r}(y) = \Phi(y) + G_{n,r}(y) ,$$
$$\Phi(y) = \int_{-\infty}^{y} \phi(x)\, dx , \qquad \phi(x) = \exp(-\tfrac{1}{2}x^2)/(2\pi)^{\frac{1}{2}} .$$
$$(2\pi)^{\frac{1}{2}}\phi(\theta)\chi_{n,r}(i\theta) = \int_{-\infty}^{\infty} \exp(i\theta y)\, dG_{n,r}(y) ,$$

$\chi_{n,r}(i\theta)$ is a polynomial in $i\theta$ obtained by expanding $\sum_{j=1}^{r-3} s^j/j!$ $(s \equiv s(i\theta) = \sum_{\nu=3}^{r-1} (i\theta)^\nu \gamma_{\nu,n}/(\nu!\,\sigma_n^{\nu}))$ and retaining terms of order $\sigma_n^{-(r-3)}$ and higher, and

$$|R_{n,r}(y)| \leqq C/\sigma_n^{r-2} .$$

If we denote by $P_j \equiv P_j(i\theta)$ the polynomial in $i\theta$ in $\chi_{n,r}(i\theta)$ of order σ_n^{-j} $(1 \leqq j \leqq r-3)$ then we can write

$$\chi_{n,r}(i\theta) = \sum_{j=1}^{r-3} P_j .$$

Now define

$$f_n(y) = \exp(-\tau\sigma_n y) \quad \text{if} \quad y \geqq \alpha_n \tag{16}$$
$$= 0 \quad \text{if} \quad y < \alpha_n .$$

Let for all real θ

$$g_n(\theta) = \int_{-\infty}^{\infty} \exp(i\theta y) f_n(y)\, dy \tag{17}$$
$$= \int_{\alpha_n}^{\infty} \exp(i\theta y - \tau\sigma_n y)\, dy$$
$$= \exp(\tau\mu_n + i\alpha_n\theta)/(\tau\sigma_n - i\theta) ,$$

and

$$\pi_{n,r}(\theta) = \int_{-\infty}^{\infty} \exp(i\theta y) K'_{n,r}(y)\, dy$$
$$= (2\pi)^{\frac{1}{2}} \phi(\theta) \sum_{j=0}^{r-3} P_j , \qquad (P_0 = 1) .$$

Then to order $\sigma_n^{-(r-2)}$

$$I_n = \tau\sigma_n \int_{\alpha_n}^{\infty} \exp(-\tau\sigma_n y)\{K_{n,r}(y) - K_{n,r}(\alpha_n)\}\, dy$$
$$= \int_{\alpha_n}^{\infty} \exp(-\tau\sigma_n y) K'_{n,r}(y)\, dy$$
$$= \int_{-\infty}^{\infty} f_n(y) K'_{n,r}(y)\, dy$$

(18) $$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{g_n(\theta)} \pi_{n,r}(\theta)\, d\theta \qquad \text{(by Parseval's Theorem)}$$

$$= (\exp(\tau\sigma_n)/\tau\sigma_n(2\pi)^{\frac{1}{2}}) \int_{-\infty}^{\infty} \exp(-i\alpha_n\theta)\left(1 + \frac{i\theta}{\tau\sigma_n}\right)^{-1} \phi(\theta) \sum_0^{r-3} P_j\, d\theta$$
$$= (\exp(\tau\mu_n)/\tau\sigma_n(2\pi)^{\frac{1}{2}}) \sum_{u=0}^{r-3} \sigma_n^{-u} \int_{-\infty}^{\infty} \exp(-i\alpha_n\theta)\phi(\theta) h_{u,n}(i\theta)\, d\theta ,$$

where

$$h_{u,n}(i\theta) = \sum_{j=0}^{u} (-1)^j (i\theta)^j \sigma_n^{u-j} P_{u-j}/\tau^j .$$

Note that since P_j is of order σ_n^{-j}, $h_{u,n}(i\theta)$ must be of order unity. Finally, therefore, combing (12), (14) and (18) we obtain (5), where $\lambda_n = \lambda_n(\tau)$ and

(19) $$\beta_{u,n} = \int_{-\infty}^{\infty} \exp(-i\alpha_n\theta)\phi(\theta) h_{u,n}(i\theta)\, d\theta ,$$

$\beta_{u,n}$ being uniformly bounded in n for all finite u. Again,

$$\int_{-\infty}^{\infty} \exp(-i\alpha_n\theta)(i\theta)^j \phi(\theta)\, d\theta = (2\pi)^{\frac{1}{2}}(-1)^j d^j\phi(\alpha_n)/d\alpha_n^j$$
$$= (2\pi)^{\frac{1}{2}} \phi(\alpha_n) H_j(\alpha_n) ,$$

where $H_j(x)$ is the Hermite polynomial of degree j in x. Hence if we write

$$h_{u,n}(i\theta) = \sum_{j=0}^{ru} h_{u,n}^{(j)} (i\theta)^j ,$$

where $h_{u,n}^{(j)}$ are constants, then

(20) $$\beta_{u,n} = (2\pi)^{\frac{1}{2}} \phi(\alpha_n) \sum_{j=0}^{ru} h_{u,n}^{(j)} H_j(\alpha_n) .$$

(a) *Special cases.*

$r = 3$:

$$h_{0,n}(i\theta) = P_0 = 1 , \qquad \beta_{0,n} = (2\pi)^{\frac{1}{2}}\phi(\alpha_n)$$
$$P_n(a) = (\rho^n \lambda_n \phi(\alpha_n)/\tau\sigma_n)[1 + O(\sigma_n^{-1})]$$
$$= (\rho^n \lambda_n/\tau\sigma_n(2\pi)^{\frac{1}{2}})[1 + O(\sigma_n^{-1})] \qquad (\because\ \alpha_n = -\mu_n/\sigma_n = O(\sigma_n^{-1})) .$$

$r = 4$:

$$P_1 = (i\theta)^3 \gamma_{3,n}/6\sigma_n^3 ,$$
$$h_{1,n}(i\theta) = \sigma_n P_1 - i\theta/\tau = (i\theta)^3\gamma_{3,n}/6\sigma_n^2 - i\theta/\tau .$$

Hence

$$\beta_{1,n} = (2\pi)^{\frac{1}{2}}\phi(\alpha_n)[\gamma_{3,n} H_3(\alpha_n)/6\sigma_n^2 - H_1(\alpha_n)/\tau] .$$

Recall that $\alpha_n = O(\sigma_n^{-1})$. Using the appropriate polynomial expression for $H_r(x)$ and retaining terms to order σ_n^{-1}, we have then

$$P_n(a) = (\rho^n \lambda_n/\tau\sigma_n(2\pi)^{\frac{1}{2}})[1 + O(\sigma_n^{-2})] .$$

$r = 5$:

$$P_2 = (i\theta)^4\gamma_{4,n}/24\sigma_n^{\ 4} + (i\theta)^6\gamma_{3,n}^2/72\sigma_n^{\ 6}$$

$$h_{2,n}(i\theta) = \sigma_n^{\ 2}P_2 - (i\theta/\tau)\sigma_n P_1 + (i\theta)^2/\tau^2$$
$$= (i\theta)^6\gamma_{3,n}^2/72\sigma_n^{\ 4} + ((i\theta)^4/24\tau\sigma_n^{\ 2})(\tau\gamma_{4,n} - 4\gamma_{3,n}) + (i\theta)^2/\tau^2 .$$

Hence

$$\beta_{2,n} = (2\pi)^{\frac{1}{2}}\phi(\alpha_n)[\gamma_{3,n}^2 H_6(\alpha_n)/72\sigma_n^{\ 4} + (\tau\gamma_{4,n} - 4\gamma_{3,n})H_4(\alpha_n)/24\tau\sigma_n^{\ 2} + H_2(\alpha_n)/\tau^2] .$$

On simplification we have

$$P_n(a) = (\rho^n\lambda_n/\tau\sigma_n(2\pi)^{\frac{1}{2}})[1 - \alpha_n^{\ 2}/2 - \alpha_n/\tau\sigma_n - \gamma_{3,n}\alpha_n/2\sigma_n^{\ 3} - 1/\tau^2\sigma_n^{\ 2} + (\tau\gamma_{4,n} - 4\gamma_{3,n})/8\tau\sigma_n^{\ 4} - 5\gamma_{3,n}^2/24\sigma_n^{\ 6} + O(\sigma_n^{\ -3})] .$$

4. Two examples. (i) Let Y_1 be distributed normally with mean zero and variance unity. Then $M_Z(t) = \exp(-at + \frac{1}{2}t^2)$, $I = I_0 = (-\infty, \infty)$, $\tau = a$ and $\rho = \exp(-\frac{1}{2}a^2) < 1$. Also $\log \Psi_n(t) = -at \sum_{u=-\infty}^n c_{n,u} + \frac{1}{2}t^2 \sum_{u=-\infty}^n c_{n,u}^2$. Note that

$$\textstyle\sum_{u=-\infty}^n c_{n,u} = n \sum_{v=0}^\infty g_v = n$$

and

$$\textstyle\sum_{u=-\infty}^n c_{n,u}^2 = \mathrm{Var}(\sum_{j=1}^n W_j) = n \sum_{u=-\infty}^\infty \rho_u - \sum_{u=-\infty}^\infty |u|\rho_u + O(\rho_0^{\ n}) ,$$

where

$$\textstyle\rho_u = \sum_{v=0}^\infty g_v g_{v+|u|} .$$

It is easy to see that

$$\textstyle\sum_{u=-\infty}^\infty \rho_u = (\sum_0^\infty g_v)^2 = 1 .$$

Hence

$$\log \lambda_n(\tau) = -\tfrac{1}{2}\delta a^2 + O(\rho_0^{\ n}) , \qquad \text{where } \delta = 2 \textstyle\sum_0^\infty u\rho_u ,$$
$$\mu_n = \gamma_{1,n} = -\delta a + O(\rho_0^{\ n}) \quad \text{and} \quad \gamma_{2,n} = \sigma_n^{\ 2} = n - \delta + O(\rho_0^{\ n}) ,$$
$$\gamma_{r,n} = 0 \qquad r \geqq 3 ,$$

$n^{-1}\sigma_n^{\ 2} \to 1$ as $n \to \infty$. In particular, if $\{X_j; j \geqq 1\}$ is linear Markov with $g_v = (1 - \rho_0)\rho_0^v$, $(v \geqq 0)$, $\delta = 2\rho_0(1 - \rho_0^2)^{-1}$.

(ii) Let $Y_1 = T_1 - T_2$ where T_1, T_2 are i.i.d. rv's with a common gamma distribution with parameter $\theta > 0$. Then $M_Z(t) = \exp(-at)(1 - t^2)^{-\theta}$, $I = (-1, 1)$, $I_0 = (-A^{-1}, A^{-1})$, $A = \sum_0^\infty |g_v|$. $M_Z'(\tau) = 0$ has only one solution in I viz., $\tau = [(\theta^2 + a^2)^{\frac{1}{2}} - \theta]a^{-1} < 1$. $\tau \in I_0$ if $a < 2\,\theta A(A^2 - 1)^{-1}$. $\log \Psi_n(t) = -nat - \theta \sum_{u=-\infty}^n \log(1 - c_{n,u}^2 t)$, $\log \lambda_n(\tau) = \theta(-\sum_{u=-\infty}^n \log(1 - c_{n,u}^2\tau^2) + n\log(1 - \tau^2)) = \theta \sum_{r=1}^\infty r^{-1}\tau^{2r}h_{n,r}$, where $h_{n,r} = \sum_{u=-\infty}^n c_{n,u}^{2r} - n$. $|h_{n,r}| < (2A^{2r} - A^{2r-1} - 1)(A - 1)^{-1} \sum_{v=0}^\infty v|g_v|$, $\gamma_{\nu,n} = nK_Z^{(\nu)}(\tau) + \theta \sum_{r=1}^\infty r^{-1}(2r)_\nu \tau^{2r-\nu}h_{n,r}$. In particular, if $\{X_j;$ $j \geqq 1\}$ is linear Markov with $g_v = (1 - \rho_0)\rho_0^{\ v}$, $(v \geqq 0)$, then

$$h_{n,r} = \textstyle\sum_{s=1}^{r-1} \binom{r}{s}(-1)^s\rho_0^{\ s}(1 - \rho_0^{\ s})^{-1} + [1 + (-1)^r]\rho_0^{\ r}(1 - \rho_0^{\ r})^{-1} + O(\rho_0^{\ n}) .$$

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