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A REPRESENTATION OF INDEPENDENT INCREMENT PROCESSES WITHOUT GAUSSIAN COMPONENTS[1]

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1. Introduction and summary. It is the purpose of this paper to describe a simple way of representing processes with independent increments having no Gaussian components and no fixed points of discontinuity. As is well known, the only random part of such processes are the jump discontinuities occurring at random points with random heights. The representation appearing in this paper describes the joint distribution of the ordered heights of the jumps and of the points at which these jumps occur. In fact, such a process in represented as a countable sum of functions each with one random point of discontinuity at a random height (Formula (7)). There is an analogy to the way that Wiener [8] described the Brownian motion process W_t on the interval $[0, \pi]$ as a countable sum,

$$W_t = tY_0 + 2^{\frac{1}{2}} \sum_{m=1}^{\infty} Y_m \frac{\sin mt}{m}, \tag{1}$$

where $Y_0, Y_1, \cdots$ are independent normal random variables with zero means and unit variances. In the same way that certain almost sure properties of the sample paths of the Brownian motion process can be read from (1) (see, for example, Itô and McKean [4] page 21), so also may certain almost sure properties of the general process with independent increments be read from (7).

We use the notation $\mathscr{P}(\lambda)$ to represent the Poisson distribution with parameter λ, $\mathscr{G}(\alpha, \beta)$ to represent the gamma distribution with shape parameter α and scale parameter β, $\mathscr{U}(\alpha, \beta)$ to represent the uniform distribution on the interval (α, β), and $\mathscr{N}(\mu, \sigma)$ to represent the normal distribution with mean μ and variance σ^2. (See [2] Section 3.1 for this notation.) $I_S(x)$ denotes the indicator function of the set S: one if $x \in S$, and zero if $x \notin S$. Expectations with subscripts always represent conditional expectations given the subscripted variables. $\mathbb{R}$ represents the real line and $\mathbb{R}^m$ Euclidean m-dimensional space.

Let X_t denote a process with independent increments and no fixed points of discontinuity. For the purposes of this paper, we restrict the domain of t to be the interval $[0, 1]$, and assume that $X_0 \equiv 0$. As is well known the increments of such processes have infinitely divisible distributions. Let $\psi_t(u)$ denote the logarithm of the characteristic function of X_t. The Lévy representation [5] of ψ_t may be written as

$$\psi_t(u) = ium(t) - \lambda(t)u^2 + \int_{-\infty}^{0} \left(e^{iuz} - 1 - \frac{iuz}{1+z^2}\right) dM_t(z) + \int_{0}^{\infty} \left(e^{iuz} - 1 - \frac{iuz}{1+z^2}\right) dN_t(z)$$

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where $m(t)$ is continuous where $\lambda(t)$ is non-decreasing and continuous, where M_t and N_t are measures on the Borel subsets of $(-\infty, 0)$ and $(0, \infty)$ respectively, such that $M_t(A)$ and $N_t(B)$ are non-decreasing and continuous in t for fixed Borel sets A and B, and where

$$\int_{-\infty}^{0} \frac{z^2}{1+z^2}\, dM_1(z) < \infty \qquad \text{and} \qquad \int_0^\infty \frac{z^2}{1+z^2}\, dN_1(z) < \infty .$$

The Gaussian component of this distribution is found in the term $-\lambda(t)u^2$. We do not treat this component; we assume that $\lambda(t) \equiv 0$. The first term, $ium(t)$, is a component degenerate at $m(t)$. This component is easily treated, so we assume that $m(t) \equiv 0$, also. Of the last two components, it is sufficient to consider just one, since the other may be treated by symmetry. Thus, we intend to represent X_t as a sum of a countable number of terms when X_t is the process with independent increments and log characteristic function

$$\psi_t(u) = \int_0^\infty \left(e^{iuz} - 1 - \frac{iuz}{1+z^2}\right) dN_t(z) . \tag{2}$$

We relax one condition on N_t because it is not needed; namely continuity in t. Thus we allow the process X_t to have some fixed points of discontinuity provided the lengths of the jumps at these points from the left and the right have infinitely divisible distributions. There is at least one application that requires such a generalization [3].

Thus, we assume that the Lévy function N_t is, for each $t \in [0, 1]$, a measure on the Borel subsets of $(0, \infty)$ that satisfies the conditions

CONDITION 1. $N_0 \equiv 0$.

CONDITION 2. For every Borel set B, $N_{t_1}(B) \leqq N_{t_2}(B)$ whenever $0 \leqq t_1 < t_2 \leqq 1$, and

CONDITION 3. $\int_0^\infty (z^2/(1+z^2))\, dN_1(z) < \infty$.

It is convenient to use the distribution function form of the measure N_1. Condition 3 implies that $N_1[z, \infty) < \infty$, for all $z > 0$, so we define

$$N_1(z) = -N_1[z, \infty) .$$

Then, $N_1(z)$ is a non-decreasing function on $(0, \infty)$ such that $\lim_{z\to\infty} N_1(z) = 0$. In this case, Condition 3 becomes simply

$$\int_0^1 z^2\, dN_1(z) < \infty . \tag{3}$$

The jumps of the independent increment process X_t with log characteristic (2) are all positive. We intend to describe X_t as the sum of a countable number of functions of the form, $J_j I_{(t,1)}(T_j) - c_j(t)$, where $J_j > 0$ represents the height of a jump, T_j represents its position, and $c_j(t)$ is a given function (nonrandom). We order the heights of the jumps $J_1 \geqq J_2 \geqq \cdots$.

The main theorem states that the distribution of the ordered heights of the jumps, $J_1, J_2, \cdots$ depends only on N_1 and not otherwise on N_t, as follows. *The distribution of $J_1, J_2, \cdots$ is the same as the distribution of $N_1^{-1}(-S_1), N_1^{-1}(-S_2), \cdots$, where $S_1, S_2, \cdots$ is a Poisson point process at unit rate*—that is, S_1, $S_2 - S_1$, $S_3 - S_2, \cdots$ are independent identically distributed with negative exponential distribution $\mathscr{G}(1, 1)$. The inverse function $N_1^{-1}(y) = \inf\{z: N_1(z) \geqq y\}$ is well-defined except at an at most countable number of points $y < 0$ (the images of the intervals measure zero under N_1) so that the random variables $J_1, J_2, \cdots$ are well defined almost surely.

The actual distribution of the ordered jumps may easily be obtained from this, *provided $N_1(z)$ is continuous in z* as follows. The largest jump, J_1, has distribution function, for $x > 0$,

$$P(J_1 \leqq x) = P(N_1^{-1}(-S_1) \leqq x) = P(S_1 \geqq -N_1(x)) = e^{N_1(x)} .$$

If $\lim_{z\to 0} N_1(z) = N_1(0) > -\infty$, then the distribution of J_1 has mass $\exp N_1(0)$ at the origin and is otherwise continuous. To find the conditional distribution of J_2 given $J_1 = x_1$, we compute, for $0 < x_2 < x_1$,

$$\begin{aligned} P(J_2 \leqq x_2 \mid J_1 = x_1) &= P(N_1^{-1}(-S_2) \leqq x_2 \mid N_1^{-1}(-S_1) = x_1) \\ &= P(S_2 \geqq -N_1(x_2) \mid S_1 = -N_1(x_1)) \\ &= \exp[N_1(x_2) - N_1(x_1)] . \end{aligned}$$

Thus, the conditional distribution of J_2, given $J_1 = x_1$, is the same as the distribution of J_1 truncated above at x_1. This procedure is easily continued. Thus, the distribution of J_j, given $J_{j-1} = x_{j-1}, \cdots, J_1 = x_1$, is the same as the distribution of J_1 truncated above at x_{j-1}.

Condition 2 implies that N_{t_1} is absolutely continuous with respect to N_{t_2} whenever $0 \leqq t_1 \leqq t_2 \leqq 1$. Hence, the Radon-Nikodym derivative of N_t with respect to N_1, call it $n_t(z)$,

$$n_t(z) = \frac{dN_t}{dN_1}(z) ,$$

exists and is determined up to equivalence dN_1. It is shown in Lemma 3 that there is a determination of $n_t(z)$ such that for all $z \in (0, \infty)$, $n_t(z)$ is a non-decreasing function of t on $[0, 1]$ with $n_0(z) \equiv 0$ and $n_1(z) \equiv 1$. One is tempted to describe such a function of t as a distribution function on $[0, 1]$; however, the specific values assumed by $n_t(z)$ at points of discontinuity in t are important—they play a role in determining the distribution of the left and right hand jumps of X_t at the fixed points of discontinuity.

The conditional distribution of the points $T_1, T_2, \cdots$ at which the respective jumps $J_1, J_2, \cdots$ occur, given $J_1, J_2, \cdots$, is, essentially, as independent random variables with respective distribution functions, $n_t(J_1), n_t(J_2), \cdots$. This description is valid if all the n_t are continuous in t (more generally, right-continuous in t). However, a more precise description is needed at points of discontinuity of n_t because these are the fixed points of discontinuity of the process and part of the jump at such points may be due to a discontinuity on the left and the rest of the jump due to a discontinuity on the right. This more precise description, as found in the main theorem, is as follows. Let $U_1, U_2, \cdots$ be independent identically distributed $\mathscr{U}(0, 1)$ random variables, independent of $J_1, J_2, \cdots$. *The* jth *jump*, J_j, *occurs at the point* t *at which the jump in*

$$I_{[0, n_t(J_j))}(U_j)$$

occurs. If this occurs at a point t_0 *of discontinuity of* $n_t(J_j)$, *then* J_j *is part of the left discontinuity at* t_0 *if* $U_j < n_{t_0}(J_j)$, *and part of the right discontinuity if* $U_j \geqq n_{t_0}(J_j)$.

If $\int_0^1 z\, dN_1(z) < \infty$, then the sum of the jumps $\sum_1^\infty J_j$ is finite with probability one. In such a case,

$$X_t = \sum_{j=1}^\infty J_j I_{[0, n_t(J_j))}(U_j) \tag{4}$$

already represents a process with independent increments having log characteristic function

$$\psi_t(u) = \int_0^\infty (e^{iuz} - 1)\, dN_t(z)\,. \tag{5}$$

However, when $\int_0^1 z\, dN_1(z) = \infty$, the sum of the jumps $\sum_1^\infty J_j$ is infinite so that the series (4) is infinite also (at least at $t = 1$). As is well known, it is sometimes possible to center such divergent series (each term centered at its mean, say), and obtain series convergent with probability one, even though the series is not absolutely convergent. This is possible under Condition 3. One possible centering constant for the jth term is

$$c_j(t) = \int_{j-1}^j \frac{N_1^{-1}(-v)}{1 + N_1^{-1}(-v)^2}\, n_t(N_1^{-1}(-v))\, dv\,. \tag{6}$$

With these centering constants, the main theorem states that

$$X_t = \sum_{j=1}^\infty (J_j I_{[0, n_t(J_j))}(U_j) - c_j(t)) \tag{7}$$

converges with probability one for each $t \in [0, 1]$, and represents a process with independent increments and log characteristic function (2).

This paper is incomplete in one aspect. We would like to be able to state that the set of measure zero outside of which the series (7) converges may be taken to be independent of t. Under the condition

$$\int_0^1 z\, dN_1(z) < \infty\,, \tag{8}$$

this is the case because (4) itself converges for $t = 1$ with probability 1, and is obviously non-decreasing. (In addition, $\sum_1^\infty c_j(t) < \infty$ under (8).) Thus, when (8) is satisfied, the following well-known property of the sample paths of the process (7) is visible: almost every sample path of $X_t - \sum_1^\infty c_j(t)$ is a pure jump function (see Breiman [1] page 314). However, when condition (8) is not satisfied, we do not know whether or not the series (7) converges for all t almost surely, although we suspect it does.

The process with independent increments and log characteristic function (2) is said to be *homogeneous* if its Lévy function is linear in t, $N_t(z) = tN_1(z)$. There is a simplification in the representation (7) when the process is homogeneous or, more generally, when

$$N_t(z) = G(t)N_1(z)$$

for some non-decreasing function $G(t)$ on $[0, 1]$ such that $G(0) = 0$ and $G(1) = 1$. In this case, $n_t(z) = dN_t(z)/dN_1(z) = G(t)$ independent of z. Hence the points at which the jumps $J_1, J_2, \cdots$ occur are independent identically distributed random variables, independent of $J_1, J_2, \cdots$, having distribution function $G(t)$ (with the difficulties previously noted at points of discontinuity of $G(t)$). In addition, $c_j(t) = G(t)c_j$ where

$$c_j = \int_{j-1}^{j} \frac{N_1^{-1}(-v)}{1 + N_1^{-1}(-v)^2}\, dv .$$

Hence, the representation (7) becomes

$$X_t = \sum_{j=1}^{\infty} (J_j I_{[0,G(t))}(U_j) - G(t)c_j) .$$

2. The proof. We precede the proof of the main theorem by four lemmas. We have defined a Poisson point process at unit rate as a sequence of random variables, $S_1, S_2, S_3, \cdots$ such that $S_1, S_2 - S_1, S_3 - S_2, \cdots$ are independent identically distributed with the negative exponential distribution, $\mathscr{G}(1, 1)$. An alternative definition of the Poisson point process at unit rate is as the times of the jumps in a Poisson process at unit intensity. The following well-known lemma is based in part upon this fact, and its proof is omitted (see Parzen [6] Section 4–4).

LEMMA 1. *Let $S_1, S_2, \cdots$ be a Poisson point process at unit rate.*

(a) *The conditional distribution of $S_1, \cdots, S_k$ given S_{k+1}, is as the order statistics of a sample of size k from $\mathscr{U}(0, S_{k+1})$.*

(b) *Let K be the largest integer k such that $S_k < \lambda$, where $\lambda > 0$. Then $K \in \mathscr{P}(\lambda)$, and the conditional distribution of $S_1, \cdots, S_k$ given $K = k$ is as the order statistics of a sample of size k from $\mathscr{U}(0, \lambda)$.*

The next lemma appears to be new and interesting in its own right. It is used to show that the process X_t defined in (7) has independent increments.

Let $\Theta_m = \{\boldsymbol{\theta} \in \mathbb{R}^m : \theta_j \geqq 0, \sum_1^m \theta_j \doteq 1\}$. A random m-dimensional vector $\mathbf{M}$ is said to be multinomial with probability vector $\boldsymbol{\theta} \in \Theta_m$ if $P_\theta(\mathbf{M} = \mathbf{e}_j) = \theta_j$ where $\mathbf{e}_j$ is the unit vector with jth coordinate one and the remaining coordinates zero (or, equivalently, if $E \exp i\mathbf{u}'\mathbf{M} = \sum_1^m \theta_j e^{iu_j}$).

LEMMA 2. *Let $K, Y_1, Y_2, \cdots$ be random variables, $K \in \mathscr{P}((\lambda)$, and given $K = k$ let $Y_1, Y_2, \cdots, Y_k$ be independent, identically distributed with common distribution function $F(y)$. Let $\boldsymbol{\theta}(y)$ be a measurable map of $\mathbb{R}$ into Θ_m. Let $\mathbf{M}_1, \mathbf{M}_2, \cdots$ be a sequence of m-dimensional random vectors whose conditional distribution given K, $Y_1, Y_2, \cdots$ is as independent multinomials with respective probability vectors $\boldsymbol{\theta}(Y_1)$, $\boldsymbol{\theta}(Y_2), \cdots$. Let $\mathbf{Z} = \sum_1^K Y_j \mathbf{M}_j$. Then $Z_1, \cdots, Z_m$ are stochastically independent.*

PROOF. We show that the characteristic function of $\mathbf{Z}$ factors:

$$\begin{aligned}\varphi_{\mathbf{Z}}(\mathbf{u}) &= E\exp[i\mathbf{u}'\mathbf{Z}] = E\prod_{j=1}^{K} E_{K,Y_1,Y_2,\cdots}\exp[iY_j\mathbf{u}'\mathbf{M}_j] \\ &= E\prod_{j=1}^{K} E_K(\sum_{l=1}^{m}\theta_l(Y_j)\exp[iY_ju_l]) \\ &= E(\sum_{l=1}^{m}\int\theta_l(y)e^{iyu_l}\,dF(y))^K \\ &= \exp\{\lambda\sum_{l=1}^{m}\int\theta_l(y)e^{iyu_l}\,dF(y)-\lambda\} \\ &= \prod_{l=1}^{m}\exp\{\lambda\int\theta_l(y)(e^{iyu_l}-1)\,dF(y)\}\,,\end{aligned}$$

completing the proof.

Condition 2 on the Lévy function N_t implies that $N_t \ll N_1$, so that by the Radon-Nikodym theorem there is a measurable function $n_t = dN_t/dN_1$ determined up to an equivalence dN_1 such that

$$\int_A n_t(z)\,dN_1(z) = N_t(A)$$

for all Borel sets $A \subset (0, \infty)$. For $t_1 < t_2$, then,

$$\int_A n_{t_1}(z)\,dN_1(z) \leqq \int_A n_{t_2}(z)\,dN_1(z)$$

for all Borel sets $A \subset (0, \infty)$, so that $n_{t_1}(z) \leqq n_{t_2}(z)$ for almost all $z(dN_1)$. It is important to establish that $n_t(z)$ can be chosen so that it is non-decreasing in t for $t \in [0, 1]$ and for all $z \in (0, \infty)$ i.e., that the null set on which $n_{t_1}(z) > n_{t_2}(z)$ can be assumed independent of t_1 and t_2. That this is the case is the content of the following lemma.

LEMMA 3. *There exists a determination of the Radon-Nikodym derivative* $n_t = dN_t/dN_1$ *such that for all* $x \in [0, 1]$, $n_t(x)$ *is non-decreasing in* t, $n_0(x) \equiv 0$, *and* $n_1(x) \equiv 1$.

PROOF. Let D be a denumerable dense set in $[0, 1]$. Include in D the points 0 and 1, and all fixed discontinuity points of the process (i.e. points t_0 for which there exist a Borel set A such that

$$\lim_{t'\nearrow t_0} N_{t'}(A) \neq \lim_{t'\searrow t_0} N_{t'}(A))\,.$$

There are only a countable number of such points. To see this, let T_m be the set of all t for which there exists a Borel set $A_t \subset (1/m, \infty)$ such that

$$\lim_{t'\searrow t} N_{t'}(A_t) - \lim_{t'\nearrow t} N_{t'}(A_t) > \frac{1}{m}\,.$$

Then T_m is a finite set. (If not, then $N_1(1/m, \infty)$ being at least the sum of these jumps would be equal to infinity, contradicting Condition 3.) The sets T_m are non-decreasing in m, and the limit as $m \to \infty$ is exactly the set of all fixed points of discontinuity, which must therefore be countable.

Find, for each $t \in D$, $n_t(x)$ such that $\int_A n_t\,dN_1 = N_t(A)$ for all Borel sets. A. Then, for $t_1 < t_2$, $n_{t_1}(x) \leqq n_{t_2}(x)$ a.e. (dN_1). Redefine $n_t(x)$ for $t \in D$ if necessary so that $n_{t_1}(x) \leqq n_{t_2}(x)$ for all $x \in [0, \infty)$ and all $t_1 \in D$, $t_2 \in D$, $t_1 < t_2$, and so that $n_0(x) \equiv 0$ and $n_1(x) \equiv 1$. Define for $t \notin D$

$$n_t(x) = \lim_{t'\nearrow t,\,t'\in D} n_{t'}(x)\,.$$

Then, for all $x \in [0, \infty)$, $n_t(x)$ is non-decreasing in t. Furthermore, for $t \notin D$

$$\int_A n_t dN_1 = \lim_{t' \nearrow t, t' \in D} \int_A n_{t'} dN_1 = \lim_{t' \nearrow t, t' \in D} N_{t'}(A) = N_t(A)$$

for all Borel sets A, showing that $n_t(x)$ is a Radon-Nikodym derivative dN_t/dN_1, and completing the proof.

LEMMA 4. *Let g be a non-increasing square-integrable real-valued function defined on $[0, \infty)$. Let $X_1, X_2, \cdots$ be a sequence of independent random variables with $EX_i = 0$ and $\operatorname{Var} X_i = \sigma_i^2 < \infty$. Let $S_n = \sum_1^n X_i$, let $a_n = \sum_1^n \sigma_i^2$, and assume $a_n \to \infty$ as $n \to \infty$. Then*

$$\int_{a_n}^{(a_n+S_n)^+} g(x)\, dx \to_{\text{a.s.}} 0 \qquad \textit{as } n \to \infty .$$

PROOF. Let $\varepsilon > 0$. Let $t_n = \sup\{t : \int_{a_n}^{a_n+t} g(x)\, dx < \varepsilon\}$, and let $\hat{t}_n = \sup\{t : \int_{a_n-t}^{a_n} g(x)\, dx < \varepsilon\}$. We are to show $P\{S_n \geqq t_n \text{ i.o.}\} = 0$ and $P\{S_n \leqq -\hat{t}_n \text{ i.o.}\} = 0$. Since g is non-increasing, we have $\hat{t}_n \leqq t_n$, so by the symmetry of the problem with respect to S_n, it is sufficient to show, say, $P\{S_n \geqq \hat{t}_n \text{ i.o.}\} = 0$. Let $t_n' = \min(\hat{t}_n, a_n/2)$. It is sufficient to show $P\{S_n \geqq t_n' \text{ i.o.}\} = 0$. Let $b(n) = \inf\{b : a_b \geqq 2^n\}$. Then, for all k

$$\begin{aligned}
P\{S_n \geqq t_n' \text{ i.o.}\} &\leqq P\{\textstyle\bigcup_{j \geqq b(k)} \{S_j \geqq t_j'\}\} \\
&\leqq \textstyle\sum_{n \geqq k} P\{\bigcup_{b(n) \leqq j < b(n+1)} \{S_j \geqq t_j'\}\} \\
&\leqq \textstyle\sum_{n \geqq k} P\{\bigcup_{b(n) \leqq j < b(n+1)} \{S_j \geqq t'_{b(n)}\}\} \\
&\leqq \textstyle\sum_{n \geqq k} P\{\max_{j < b(n+1)} S_j \geqq t'_{b(n)}\} \qquad (9)\\
&\leqq \textstyle\sum_{n \geqq k} a_{b(n+1)-1}(t'_{b(n)})^{-2} \qquad \text{(Kolmogorov's inequality)} \\
&\leqq \textstyle\sum_{n \geqq k} 2^{n+1}(t'_{b(n)})^{-2} \\
&= \textstyle\sum_{n \geqq k, n \in E} 2^{n+1}(t'_{b(n)})^{-2} + \sum_{n \geqq k, n \in E^c} 2^{n+1}(t'_{b(n)})^{-2}
\end{aligned}$$

where $E = \{n : \hat{t}_{b(n)} \leqq a_{b(n)}/2\}$. If $n \in E^c$, $t'_{b(n)} = a_{b(n)}/2 \geqq 2^{n-1}$, so that

$$\textstyle\sum_{n \geqq k, n \in E^c} 2^{n+1}(t'_{b(n)})^{-2} \leqq \sum_{n \geqq k} 2^{n+1}(2^{n-1})^{-2} < \infty .$$

If $n \in E$, $t'_{b(n)} = \hat{t}_{b(n)} \leqq a_{b(n)}/2$, and

$$\varepsilon = \int_{a_n-\hat{t}_n}^{a_n} g(x)\, dx \leqq g(a_n - \hat{t}_n)\hat{t}_n .$$

Thus, $t'_{b(n)} = \hat{t}_{b(n)} \geqq \varepsilon/g(a_{b(n)} - \hat{t}_{b(n)}) \geqq \varepsilon/g(a_{b(n)}/2) \geqq \varepsilon/g(2^{n-1})$, so that

$$\textstyle\sum_{n \geqq k, n \in E} 2^{n+1}(t'_{b(n)})^{-2} \leqq \sum_{n \geqq k} 2^{n+1} \dfrac{g(2^{n-1})^2}{\varepsilon^2} .$$

But since g is non-increasing,

$$\int_0^\infty g(x)^2\, dx < \infty \Rightarrow \textstyle\sum_1^\infty g(n)^2 < \infty \Rightarrow \sum_1^\infty 2^n g(2^n)^2 < \infty .$$

Thus both summations in (9) are finite and hence converge to zero as $k \to \infty$.

THEOREM. *Let N_t satisfy Conditions* 1, 2, *and* 3. *Let $J_j = N_1^{-1}(-S_j)$ $j = 1, 2, \cdots$ where $S_1, S_2, \cdots$ is a Poisson process at unit rate. Let $U_1, U_2, \cdots$ be independent* *identically distributed* $\mathscr{U}(0, 1)$, *independent of* $S_1, S_2 \cdots$. *Let*

$$c_j(t) = \int_{j-1}^{j} \frac{N_1^{-1}(-v)}{1 + N_1^{-1}(-v)^2} n_t(N_1^{-1}(-v))\, dv$$

where n_t *is as in Lemma* 3. *Then for each* $t \in [0, 1]$, *the series*

$$X_t = \sum_{j=1}^{\infty} (J_j I_{[0, n_t(J_j))}(U_j) - c_j(t)) \tag{10}$$

converges with probability 1, *and* X_t *is a process with independent increments and log characteristic function*

$$\psi_t(u) = \int_0^\infty \left(e^{iux} - 1 - \frac{iux}{1 + x^2}\right) dN_t(x) . \tag{11}$$

PROOF. Let $a_0 = 0$ and $a_n = \sum_1^n 1/j$. Let K_n be the largest integer k for which $S_k \leqq a_n$. With probability one $K_1 \leqq K_2 \leqq \cdots$ and $K_n \to \infty$. Since $K_n - K_{n-1} \in \mathscr{P}(1/n)$ $P\{K_n - K_{n-1} \geqq 2\} = O(1/n^2)$ so that $\sum_1^\infty P\{K_n - K_{n-1} \geqq 2\} < \infty$. The Borel-Cantelli Lemma implies that with probability one $K_n - K_{n-1} \geqq 2$ only finitely often. In other words, with probability one the sequence $K_1, K_2, \cdots$ contains all the integers from some integer on.

Let

$$X_t^{(n)} = \sum_{j=1}^{K_n} (J_j I_{[0, n_t(J_j))}(U_j) - c_j(t)) .$$

We will show that $X_t^{(n)}$ converges almost surely for each $t \in [0, 1]$. Then by the above paragraph the series (10) converges almost surely as well. Let

$$V_t^{(n)} = \sum_{j=1}^{K_n} J_j I_{[0, n_t(J_j))}(U_j) - \int_0^{a_n} g_t(v)\, dv$$

where for $0 < v < \infty$

$$g_t(v) = \frac{N_1^{-1}(-v)}{1 + N_1^{-1}(-v)^2} n_t(N_1^{-1}(-v)) .$$

We shall complete the proof by showing,

(i°) $X_t^{(n)} - V_t^{(n)} \to_{\text{a.s.}} 0$ as $n \to \infty$ for each $t \in [0, 1]$

(ii°) for each n, $V_t^{(n)}$ is a process with independent increments, and

(iii°) $V_t^{(n)}$ converges with probability one for each $t \in [0, 1]$ to a random variable with characteristic function (11).

(i°) $$|X_t^{(n)} - V_t^{(n)}| = |\int_{a_n}^{K_n} g_t(v)\, dv| \leqq |\int_{a_n}^{K_n} g_1(v)\, dv| .$$

The function $g_1(v)$ is square-integrable since

$$\begin{aligned}\int_0^\infty g_1(v)^2\, dv &= \int_0^\infty \left(\frac{N_1^{-1}(-v)}{1 + N_1^{-1}(-v)^2}\right)^2 dv \\ &= \int_0^\infty \left(\frac{z}{1 + z^2}\right)^2 dN_1(z) \\ &\leqq 2 \int_0^\infty \frac{z^2}{1 + z^2} dN_1(z) < \infty .\end{aligned}$$

Also $g_1(v)$ is eventually non-increasing and so is bounded by a square integrable non-increasing function g say $g_1 \leqq g$. Hence by Lemma 4

$$|X_t^{(n)} - V_t^{(n)}| \leqq |\int_{a_n}^{K_n} g(v)\,dv| \to_{\text{a.s.}} 0 \qquad \text{as } n \to \infty .$$

(ii°) Let $t(0) = 0 < t(1) < t(2) < \cdots < t(m) = 1$. Then $V_{t(1)}^{(n)}, V_{t(2)}^{(n)} - V_{t(1)}^{(n)}, \cdots, V_{t(m)}^{(n)} - V_{t(m-1)}^{n}$ are independent if $Z_1, Z_2, \cdots, Z_m$ are independent where

$$Z_\alpha = \sum_{j=1}^{K_n} J_j I_{[n_{t(\alpha-1)}(J_j), n_{t(\alpha)}(J_j))}(U_j) \qquad \alpha = 1, \cdots, m .$$

From Lemma 1, $K_n \in \mathscr{P}(a_n)$ and the conditional distribution of $S_1, \cdots, S_k$ given $K_n = k$ is as the order statistics of a sample of size k from $\mathscr{U}(0, a_n)$. Thus, Lemma 2 applies and $Z_1, Z_2, \cdots, Z_m$ are independent.

(iii°) Let

$$V_n(t) = \sum_{j=K_{n-1}+1}^{K_n} J_j I_{[0, n_t(J_j))}(U_j) - \int_{a_{n-1}}^{a_n} g_t(v)\,dv .$$

Then, for each $t \in [0, 1]$, $V_1(t), V_2(t), \cdots$ are independent since $V_n(t)$ is determined by those S_j that fall in the interval $[a_{n-1}, a_n]$. Therefore, $V_t^{(n)} = \sum_{j=1}^{n} V_j(t)$ converges almost surely if it converges in law. (See Neveu [5] page 155.) The characteristic function of the first term of $V_t^{(n)}$ is

$$\begin{aligned}
&E\exp\{iu \textstyle\sum_1^{K_n} J_j I_{[0, n_t(J_j))}(U_j)\} \\
&\quad = E E_{K_n} E_{K_n, J_1, \cdots, J_{K_n}} \exp\{\textstyle\sum_{j=1}^{K_n} iu\, J_j I_{[0, n_t(J_j))}(U_j)\} \\
&\quad = E E_{K_n} \textstyle\prod_1^{K_n} (1 + (e^{iuJ_j} - 1) n_t(J_j)) \\
&\quad = E \textstyle\prod_1^{K_n} \int_0^{a_n} (1 + (\exp[iuN_1^{-1}(-v)] - 1) n_t(N_1^{-1}(-v)) \dfrac{1}{a_n}\,dv \\
&\quad = E\left(1 + \dfrac{1}{a_n} \int_0^{a_n} (\exp[iuN_1^{-1}(-v)] - 1) n_t(N_1^{-1}(-v))\,dv\right)^{K_n} \\
&\quad = \exp\{\textstyle\int_0^{a_n} (\exp[iuN_1^{-1}(-v)] - 1) n_t(N_1^{-1}(-v))\,dv\} .
\end{aligned}$$

The third equality follows since the distribution of the J_j given K_n is as the order statistics of a sample of size K_n from $\mathscr{U}(0, a_n)$, and the product involves the J_j symmetrically, so that the expectation given K_n may be computed as if J_j were independent $\mathscr{U}(0, a_n)$. The characteristic function of $V_t^{(n)}$ is therefore

$$\begin{aligned}
&E\exp\{iuV_t^{(n)}\} \\
&\quad = \exp\left\{\int_0^{a_n} \left(\exp[iuN_1^{-1}(-v)] - 1 - \frac{iuN_1^{-1}(-v)}{1 + N_1^{-1}(-v)^2}\right) n_t(N_1^{-1}(-v))\,dv\right\}
\end{aligned}$$

which converges as $n \to \infty$ to

$$\begin{aligned}
&\exp\left\{\int_0^\infty \left(\exp[iuN_1^{-1}(-v)] - 1 - \frac{iuN_1^{-1}(-v)}{1 + N_1^{-1}(-v)^2}\right) n_t(N_1^{-1}(-v))\,dv\right\} \\
&\qquad\qquad = \exp\left\{\int_0^\infty \left(e^{iux} - 1 - \frac{iux}{1 + x^2}\right) n_t(x)\,dN_1(x)\right\} .
\end{aligned}$$

Since n_t is the Radon-Nikodym derivative of N_t with respect to N_1, this is the characteristic function (11).

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