

## A CRITICAL AGE-DEPENDENT BRANCHING PROCESS WITH IMMIGRATION

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It is shown that the number of cells alive at time  $t$ , denoted by  $Z_0(t)$ , of a critical age dependent branching process with immigration satisfies  $t^{-1}Z_0(t)$  approaches a gamma law with specified constants in distribution.

**1. Introduction and summary.** Jagers [4] has given the following model for an age-dependent branching process with immigration. Starting at time  $t = 0$ , let a group of  $k$  cells of age zero all arrive with probability  $p_{k0}$ , with generating function  $h_0(s) \equiv \sum_{k=0}^{\infty} p_{k0} s^k$ , according to a renewal process with i.i.d. interarrival times  $\{X_i\}$  each distributed as  $P[X \leq t] = G_0(t)$ ,  $G_0(0+) = 0$  and non-lattice, independent of the number of cells which arrive and of past history. Each arriving new cell initiates a standard Bellman-Harris ([3], Chapter 6) process with offspring distribution  $h(s) = \sum_{i=0}^{\infty} p_i s^i$  and offspring lifetime distribution  $G(t)$ ,  $G(0+) = 0$ , and also non-lattice.

Let  $Z_0(t)$  = total number of cells in existence at time  $t$  in the overall, or immigration process, and let  $Z_1(t)$  = total number of cells alive at time  $t$  starting with one newborn cell at time  $t = 0$  in a Bellman-Harris process with offspring distribution  $h(s)$  and lifetime distribution  $G(t)$ .

The object of this paper is to show that in the critical case, where  $h^{(1)}(1) = 1$ , that  $t^{-1}Z_0(t)$  converges in distribution to a suitable gamma law. A special case of this was done for Markov branching processes in continuous time by Sevast'yanov [6] using differential equations for certain generating functions. Durham [1] also obtained a gamma limit law for immigration of cells arriving one at a time in accord with a non-homogeneous Poisson process  $X(t)$  with  $E(X(t)) = \theta(t)$ . His critical age-dependent process started by each arrival is more general than the one treated here, as cells may be born all during the life of the parent cell. His moment methods differ from those used here. Also, his basic equation (2) page 1122 of [1] can be obtained from (1) by substituting  $1 - G_0(t) = \exp(-\theta(t))$ ,  $h_0(s) \equiv s$  and differentiating this  $F(s, t)$  with respect to  $\theta$ -measure after a change of variables to obtain an easily solved differential equation.

Seneta [5] has obtained a gamma law for a similar immigration process in discrete time. Foster [2] has also considered such discrete time processes with these and further results, and has considered multitype processes. Generating function expansions are used for these results.

### 2. Limit law.

LEMMA. Assume  $\{h_0^{(n)}(1)\}$  and  $\{h^{(n)}(1)\}$  exist for  $n \geq 1$ . Then  $E(Z_0(t))^n$  exists for all  $t$ , all  $n \geq 1$ , and is non-decreasing in  $t$  for each fixed  $n$ , when  $h^{(1)}(1) = 1$ .

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PROOF. The existence of moments follows from arguments in Jagers [4] and standard results in Harris ([3], Chapters 3 and 6).

Now  $Z_0(t)$  can be expressed as a random sum of standard independent Bellman-Harris critical age-dependent branching processes (i.e. with  $h^{(1)}(1) = 1$ ). It was shown in [7] (page 1565) that  $E(Z_1(t))^n$  is non-decreasing in  $t$  for each  $n$ . Hence the result follows.

THEOREM 1. Let  $h_0(s)$  be the immigration generating function with all moments  $\{h_0^{(n)}(1)\}$ ,  $n \geq 1$  assumed to exist, with  $h_0^{(1)}(1) \equiv m_0$ . Let the interarrival time distribution function be  $G_0(t)$  with

$$0 < m_{0G} \equiv \int_0^\infty u dG_0(u) < \infty .$$

Let  $h(s)$ , the offspring distribution, have  $\{h^{(n)}(1)\}$ ,  $n \geq 1$  all finite, and  $h^{(1)}(1) \equiv 1$ ,  $0 < h^{(2)}(1)$ . Assume that the lifetime distribution function has

$$0 < m_G \equiv \int_0^\infty u dG(u) .$$

Denote  $b = 2m_G/h^{(2)}(1)$ .

Denote  $c = bm_0/m_{0G}$ .

Then

$$\lim_{t \rightarrow \infty} E(t^{-1}Z_0(t))^n = b^{-n} \frac{\Gamma(c+n)}{\Gamma(c)} .$$

PROOF. Defining the generating functions  $F(s, t) = E(\exp(-sZ_0(t)))$  and  $\Phi(s, t) = E(\exp(-sZ_1(t)))$ , Jagers [4] obtains the integral equation

$$(2.1) \quad F(s, t) = 1 - G_0(t) + \int_0^t h_0(\Phi(s, t-u))F(s, t-u) dG_0(u) .$$

$$(2.2) \quad \text{Let } D(s, t) = 1 - \Phi(s, t) .$$

Putting (2) into the right side of (1) and expanding  $h_0(1 - D(s, t))$  in a Taylor series about 1, we obtain

$$(2.3) \quad F(s, t) = 1 - G_0(t) + \int_0^t F(s, t-u) dG_0(u) \\ + \sum_{r=1}^\infty h_0^{(r)}(1)(r!)^{-1} \int_0^t (-D(s, t-u))^r F(s, t-u) dG_0(u) .$$

Taking Laplace-Stieltjes transforms of (2.3), rearranging and reinverting, we obtain

$$(2.4) \quad F(s, t) = 1 + \sum_{r=1}^\infty h_0^{(r)}(1)(r!)^{-1} \int_0^t (-D(s, t-u))^r F(s, t-u) dH_0(u) ,$$

where

$$(2.5) \quad H_0(t) = \sum_{n=0}^\infty G_0^{(n)}(t) ,$$

with  $G_0^{(n)}(t)$  the  $n$ th convolution of  $G_0(t)$ , and  $G_0^{(0)}(t)$  the unit step function.

We now obtain asymptotic formulas for the  $E(Z_0(t))^n$  by induction. Denote

$$(2.6) \quad E(Z_0(t))^n \equiv M_{0n}(t) , \quad \text{with } M_{01}(t) \equiv m_0(t)$$

and

$$(2.7) \quad E(Z_1(t))^n \equiv M_n(t) , \quad \text{with } M_1(t) \equiv m(t) .$$

On differentiating (2.4) once with respect to  $s$  and setting  $s = 0$ , one obtains, since  $D(0, t) \equiv 0$ ,

$$(2.8) \quad m_0(t) = m_0 \int_0^t m(t - u) dH_0(u) .$$

Note that

$$(2.9) \quad \left. \frac{\partial^n F}{\partial s^n}(s, t) \right|_{s=0} = M_{0n}(t) \quad \text{and} \quad \left. \frac{\partial^n}{\partial s^n} (-D(s, t)) \right|_{s=0} = M_n(t) .$$

From ([7] page 1566),  $m(t) \equiv 1$ , so that

$$(2.10) \quad m_0(t) = m_0 H_0(t) ,$$

and standard renewal theory yields that

$$(2.11) \quad \lim_{t \rightarrow \infty} t^{-1} m_0(t) = m_0 / m_{0G} .$$

Taking second derivatives with respect to  $s$  in (2.4) and setting  $s = 0$ , one obtains

$$(2.12) \quad M_{02}(t) = m_0 \int_0^t (M_2(t - u) + 2m_0(t - u)) dH_0(u) .$$

Taking now Laplace-Stieltjes transforms of (2.12) expanding around  $s \downarrow 0$ , and denoting the transform of a function  $K(t)$  by  $K(s)$ , one obtains, using  $sM_2(s) \sim 2/b$ , ([7] page 1567),

$$(2.13) \quad \lim_{s \downarrow 0} s^2 M_{02}(s) = 2m_0 / m_{0G} [m_0 / m_{0G} + 1/b] .$$

Reinverting, using Abelian and Tauberian theorems in Widder [8], which apply by the lemma, one obtains

$$(2.14) \quad \lim_{t \rightarrow \infty} t^{-2} M_{02}(t) = m_0 / m_{0G} [m_0 / m_{0G} + 1/b] .$$

Similarly, the result of Theorem 1 is easily checked for the cases  $n = 3, 4$ . Following now the argument in [7] (page 1566), a further induction appealing to standard Abelian and Tauberian theorems yields that the limit behavior of  $M_{0n}(t)$  is obtained solely from the behavior of the  $n$ th derivative  $(D(0, t)F(0, t))^{(n)}$  in the right hand side of (2.4).

Using the Leibnitz rule for differentiation, using  $D(0, t) = 0$ ,

$$(2.15) \quad \begin{aligned} (-D(0, t)F(0, t))^{(n)} &= \sum_{k=0}^n \binom{n}{k} F^{(k)}(0, t) (-D(0, t))^{(n-k)} \\ &= \sum_{k=0}^{n-1} \binom{n}{k} F^{(k)}(0, t) (-D(0, t))^{(n-k)} \end{aligned}$$

using  $D(0, t) = 0$ .

Assume now that the theorem holds by induction up to  $n - 1$ .

Hence, for  $k \leq n - 1$ ,

$$(2.16) \quad t^{-k} M_{0k}(t) \sim b^{-k} \Gamma(c + k) / \Gamma(c) .$$

From [7] (page 1567),

$$(2.17) \quad t^{-(n-k-1)} M_{n-k}(t) = t^{-(n-k-1)} (-D(0, t))^{(n-k)} \sim (n - k)! b^{-(n-k-1)} .$$

From (2.15) to (2.17)

$$(2.18) \quad t^{-(n-1)}[-D(0, t)F(0, t)]^{(n)} \sim \sum_{k=0}^{n-1} \binom{n}{k} (n-k)! b^{-(n-k-1)} b^{-k} \Gamma(c+k) / \Gamma(c) \\ = n! b^{-(n-1)} \sum_{k=0}^{n-1} \frac{\Gamma(c+k)}{\Gamma(c)k!}.$$

A straightforward induction repeatedly using  $\Gamma(a+1) = a\Gamma(a)$  yields that

$$(2.19) \quad \sum_{k=0}^{n-1} \frac{\Gamma(c+k)}{\Gamma(c)k!} = \frac{\Gamma(c+n)}{\Gamma(c+1)\Gamma(n)}.$$

Substituting (2.19) into (2.18) yields

$$(2.20) \quad t^{-(n-1)}[-D(0, t)F(0, t)]^{(n)} \sim n! b^{-(n-1)} \frac{\Gamma(c+n)}{\Gamma(c+1)\Gamma(n)}.$$

Taking Laplace transforms in (2.20) it follows that for  $s \downarrow 0$ ,

$$(2.21) \quad s^{n-1}(-D \cdot F)^{(n)}(s) \sim \Gamma(n)n! b^{-(n-1)} \frac{\Gamma(c+n)}{\Gamma(c+1)\Gamma(n)}.$$

Putting (2.21) into (2.4), after taking  $n$ th derivatives and setting  $s \downarrow 0$ , using the fact that only  $(D \cdot F)^{(n)}$  terms count in the asymptotic expansion, yields that

$$(2.22) \quad s^n M_{0n}(s) \sim \frac{m_0}{m_{0G}} \Gamma(n) b^{-(n-1)} n! \frac{\Gamma(c+n)}{\Gamma(c+1)\Gamma(n)} \\ = b^{-n} n! \Gamma(c+n) / \Gamma(c).$$

Again by Abelian and Tauberian theorems, (2.22) implies

$$(2.23) \quad t^{-n} M_{0n}(t) \sim b^{-n} \frac{\Gamma(c+n)}{\Gamma(c)}$$

as  $t \rightarrow \infty$ , and the induction is complete.

**THEOREM 2.** *Under the hypotheses of Theorem 1,  $Z_{0t}/t$  converges in law to a gamma distribution  $\Gamma(c, b)$ .*

**PROOF.** Theorem 1 yields that for all  $n \geq 1$ ,

$$\lim_{t \rightarrow \infty} E(Z_{0t}/t)^n = b^{-n} \Gamma(c+n) / \Gamma(c),$$

the moments of a  $\Gamma(c, b)$  law, and which uniquely characterize it. This suffices for the proof.

**3. Remarks.** Theorems 1 and 2 may be immediately generalized to the case of this general immigration of cells undergoing a general critical branching process as treated for a special case of immigration in [1]. This is due to the facts that the asymptotic moment structure for the general critical process is the same as for the critical branching process, that (2.1) holds formally, that the monotonicity of moments follows using the same argument as [7] (page 1565) and the lemma given here, and finally since the results depend only on asymptotic moment computations. The formal definitions of modified parameters and explicit limiting gamma law will not be given here.

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