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ON A THEOREM OF DE FINETTI, ODDSMAKING, AND GAME THEORY

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A theorem of de Finetti states that if odds are posted on each set in a finite partition of a probability space, then either the odds are consistent with a finitely additive probability measure or a sure win is possible. A generalization of this result is proved which in turn implies a generalization of Von Neumann's theorem on the existence of the value of a game. Also, two horse race examples are considered.

0. Introduction and summary. Suppose odds are posted on some collection $\mathscr{E}$ of subsets of the set S of all possible outcomes of some experiment of chance, and bets are accepted on or against the sets in any finite subcollection of $\mathscr{E}$. A typical result of this note is that there is either a betting scheme which guarantees a positive return or the odds posted are consistent with some finitely additive probability measure defined on all subsets of S. This theorem specializes to give a theorem of Bruno de Finetti if $\mathscr{E}$ is a finite partition of S.

In Section 1, a separating hyperplane argument is used to prove a generalization of the above result to the case of an arbitrary collection of bounded payoff functions, and a connection with game theory is pointed out. Section 2 is an interpretation of the theorem of Section 1 for the special case when the payoff functions are those available when bets are accepted on certain events at given odds. Section 3 is a study of two examples from horse racing.

1. Basic results. Let S and T be sets and let $\{f_t : t \in T\}$ be a family of bounded, real-valued functions on S. We regard the f_t as payoff functions available.

By a *probability* P on a set S is meant a finitely additive probability measure defined on all subsets of S. If f is a bounded function on S, $E_p(f)$ or $E_p(f(s))$ denotes the expectation of f under P.

THEOREM 1. *Either* (i) *there exist* $t_1, \cdots, t_n \in T$ *and* $c_1, \cdots, c_n \in R^+$ *such that* $\sum_{i=1}^n c_i f_{t_i}(s) > 0$ *for all* $s \in S$, *or* (ii) *there is a probability* P *on* S *such that* $E_p(f_t) \leqq 0$ *for all* $t \in T$, *or both.*

PROOF. In the space of bounded functions on S (with supremum norm) consider the sets $K_1 = \{f: f = \sum_{i=1}^n c_i f_{t_i},\ c_i \in R^+,\ t_i \in T,\ i = 1, \cdots, n\}$ and $K_2 = \{f: f(s) > 0 \text{ for all } s \in S\}$. Then, if (i) is false, $K_1 \cap K_2 = \emptyset$. Clearly K_1 and K_2 are convex and the constant function 1 belongs to the interior of K_2. Hence (see Dunford and Schwartz, [3] page 417, Theorem 8) there exists a nonzero, continuous linear functional π separating K_1 and K_2. Without loss of generality

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we may assume $\pi \leqq c$ on K_1 and $\pi \geqq c$ on K_2. Since 0 is a limit point of K_1 and of K_2, we must have $c = 0$. Since π is not identically zero, $\pi(1) > 0$. Normalize π so that $\pi(1) = 1$. But then, as is easily seen, there is a probability P on S such that $E_p(f) = \pi(f)$ for all bounded functions f on S. Therefore, $E_p(f_t) \leqq 0$ for all $t \in T$, since $f_t \in K_1$ for all $t \in T$. □

An argument similar to the above was used by Purves and Freedman to prove Theorem 4 of [5], which contains interesting extensions of de Finetti's theorem different from those given here.

The following example shows that (i) and (ii) can occur simultaneously.

EXAMPLE. Let $S = \{1, 2, \cdots\}$, $T = \{1\}$, and $f_1(n) = 1/n$ for all $n \in S$. Certainly (i) holds and any P such that $P(\{n\}) = 0$, for all n, satisfies (ii).

COROLLARY 1. *For every $b \in R$, either* (i) *there exist $t_1, \cdots, t_n \in T$ and $c_1, \cdots, c_n \in R^+$ such that $\sum_{i=1}^n c_i = 1$ and $\sum_{i=1}^n c_i f_{t_i}(s) > b$ for all $s \in S$, or* (ii) *there is a probability P on S such that $E_p(f_t) \leqq b$ for all $t \in T$, or both.*

PROOF. Apply the previous theorem to the family $\{g_t: t \in T\}$, where $g_t(s) = f_t(s) - b$. □

COROLLARY 2. *Either* (i) *there exist $t_1, \cdots, t_n \in T$ and $c_1, \cdots, c_n \in R$ such that $\sum_{i=1}^n c_i f_{t_i}(s) > 0$ for all $s \in S$, or* (ii) *there is a probability P on S such that $E_P(f_t) = 0$ for all $t \in T$, or both.*

PROOF. Let $T' = T \times \{+, -\}$ and define $g_{t'}(s) = +f_t(s)$ if $t' = (t, +)$ and $-f_t(s)$ if $t' = (t, -)$. Then apply Theorem 1 to the family $\{g_{t'}: t' \in T'\}$. □

Now write $f(s, t)$ for $f_t(s)$, and consider a game in which Player 1 has pure strategies T and Player 2 has pure strategies S, with payoff function f. Theorem 1 asserts that either Player 1 has a mixed strategy which achieves a positive expected payoff or Player 2 has a mixed strategy which assures a non-positive expected payoff. The next two theorems, which are not used in the remainder of this paper, show that if finitely additive mixed strategies are allowed, every zero-sum two person game with a suitably bounded payoff function has a value (but this value depends, in general, upon the order in which the expectations are taken).

Let $\mathscr{S}$ and $\mathscr{T}$ be the collections of probabilities on S and T respectively, and let $\mathscr{T}^\circ = \{Q \in \mathscr{T}: Q(F) = 1 \text{ for some finite set } F \subseteqq T\}$.

THEOREM 2. $\inf_{P \in \mathscr{S}} \sup_{Q \in \mathscr{T}^\circ} E_Q(E_P(f(s, t))) = \sup_{Q \in \mathscr{T}^\circ} \inf_{P \in \mathscr{S}} E_Q(E_P(f(s, t)))$.

PROOF. Clearly we have $\geqq$. Suppose that the right-hand side is $<b$. Then, for every $Q \in \mathscr{T}^\circ$, there is a $P \in \mathscr{S}$ such that $E_Q(E_P(f(s, t))) < b$. But then there must be an $s \in S$ for which $E_Q(f(s, \cdot)) < b$. So (i) of Corollary 1 is false. Hence, by (ii), there is a $P_0 \in \mathscr{S}$ with $E_{P_0}(f(\cdot, t)) \leqq b$ for all $t \in T$. Thus $E_Q(E_{P_0}(f(s, t))) \leqq b$ for all $Q \in \mathscr{T}^\circ$ and so $\inf_{P \in \mathscr{S}} \sup_{Q \in \mathscr{T}^\circ} E_Q(E_P(f(s, t))) \leqq b$. □

For the next theorem, we suppose that, for every $P \in \mathscr{S}$, $E_P(f(\cdot, t))$ is a bounded function of t.

THEOREM 3. $\inf_{P\in\mathscr{S}}\sup_{Q\in\mathscr{T}} E_Q(E_P(f(s,t))) = \sup_{Q\in\mathscr{T}}\inf_{P\in\mathscr{S}} E_Q(E_P(f(s,t)))$.

PROOF. As before, $\geqq$ is clear. To show $\leqq$, suppose the right-hand side is $<b$. But then $\sup_{Q\in\mathscr{T}^\circ}\inf_{P\in\mathscr{S}} E_Q(E_P(f(s,t))) < b$. So, as in the proof of Theorem 2, we can find $P_0\in\mathscr{S}$ with $E_{P_0}(f(\cdot,t)) \leqq b$ for all $t\in T$. Hence, $E_Q(E_{P_0}(f(s,t))) \leqq b$ for every $Q\in\mathscr{T}$ and the result follows. □

For $P\in\mathscr{S}$, $Q\in\mathscr{T}^\circ$, we have $E_PE_Q = E_QE_P$ so that in Theorem 2 the order of the expectations may be interchanged. In general, however, $E_PE_Q \neq E_QE_P$ for $P\in\mathscr{S}$, $Q\in\mathscr{T}$. Nevertheless, the result of Theorem 3 is correct if expectations are reversed on both sides (provided that, for every $Q\in\mathscr{T}$, $E_Q(f(s,\cdot))$ is bounded as a function of s). This can be derived by applying Theorem 3 with the roles of P and Q (i.e., $\mathscr{S}$ and $\mathscr{T}$) reversed and using the functions $-f(s,t)$.

Further applications of finitely additive probabilities to game theory are in [7].

2. Oddsmaking. Let $\mathscr{E}$ be a collection of subsets of S. In this section, we assume that a bookie posts odds on each event in $\mathscr{E}$. More formally, we assume there is a function μ from $\mathscr{E}$ to the unit interval. If $E\in\mathscr{E}$, then $\mu(E): 1-\mu(E)$ are the *odds posted* on E. A gambler may *bet* a nonnegative amount b on E and his net return is $b[1_E(s)-\mu(E)]$ if s occurs. (Our terminology differs from popular gambling language, where $b\mu(E)$ would be called the stake or amount bet.) A *betting scheme* is a finite collection of nonnegative bets $b_1,\cdots,b_n$ placed on events $E_1,\cdots,E_n$ respectively. Such a betting scheme is called a *sure win* iff $\sum_{i=1}^n b_i(1_{E_i}(s)-\mu(E_i)) > 0$ for all $s\in S$.

THEOREM 4. *Either* (i) *there is a sure win or* (ii) *there is a probability P on S such that $P(E)\leqq\mu(E)$ for all $E\in\mathscr{E}$, or both.*

PROOF. Apply Theorem 1 to the family $\{f_E: E\in\mathscr{E}\}$, where $f_E(s) = 1_E(s) - \mu(E)$. □

In his original result ([2] pages 102–104), de Finetti allowed the gambler to make negative as well as positive bets. Now, if $\mu(E) = 1-\mu(E^c)$, then a bet of $-b$ on E has the same return as a bet of b on E^c. Also, if $\mu(E)\neq 1-\mu(E^c)$, a gambler can easily construct a sure win by placing postive or negative bets on E and E^c. Thus the next theorem extends de Finetti's theorem.

THEOREM 5. *Assume that if $E\in\mathscr{E}$, then $E^c\in\mathscr{E}$ and $\mu(E^c) = 1-\mu(E)$. Then either* (i) *there is a sure win or* (ii) *there is a probability P on S such that $P(E)=\mu(E)$ for all $E\in\mathscr{E}$, or both.*

PROOF. Apply the previous theorem. □

The final result of this section is a countably additive analogue of de Finetti's theorem.

THEOREM 6. *Assume $\mathscr{E}$ is an algebra and $\mu(E) = 1-\mu(E^c)$ for all $E\in\mathscr{E}$. Then either* (i) *there exist $b_i\in R^+$, $E_i\in\mathscr{E}$, for $i = 1, 2,\cdots$ with total stake $\sum_{i=1}^\infty b_i\mu(E_i) <$*

$+\infty$ and payoff $\sum_{i=1}^{\infty} b_i(1_{E_i}(s) - \mu(E_i)) > 0$ for all $s \in S$, or (ii) *μ is a countably additive probability measure on $\mathscr{E}$. It is not possible that both* (i) *and* (ii) *occur.*

PROOF. Assume (i) is false. Then, certainly, there is no sure win. So, by the previous theorem, μ is a finitely additive probability on $\mathscr{E}$. If μ is not countably additive, then there is a sequence E_i of disjoint sets in $\mathscr{E}$ such that $UE_i = S$ and $\sum \mu(E_i) < 1$. Take $b_i = 1$, for all i. Then (i) holds, a contradiction.

Now assume (i) and (ii) both hold. By (i), there exist $b_i \in R^+$ and $E_i \in \mathscr{E}$, such that

$$\sum b_i 1_{E_i}(s) = \sum b_i(1_{E_i}(s) - \mu(E_i)) + \sum b_i \mu(E_i) > \sum b_i \mu(E_i) , \tag{1}$$

for all $s \in S$. But, by the monotone convergence theorem,

$$\int (\sum b_i 1_{E_i}) \, d\mu = \sum b_i \mu(E_i) ,$$

which, together with (1) and (ii), gives a contradiction. □

3. Two examples from horse racing. Consider a race of n horses and suppose $n \geqq 3$. Since only the positions of the first three horses are of interest in the sequel, we set

$$S = \{s = (s_1, s_2, s_3) : s_1, s_2, s_3 \text{ are distinct integers between } 1 \text{ and } n\} .$$

The events A_i, B_i, or C_i that horse i wins, places (i.e., finishes first or second), or shows (i.e., finishes first, second, or third) are given by

$$\begin{aligned} A_i &= \{s : s_1 = i\} , \\ B_i &= \{s : s_1 = i \text{ or } s_2 = i\} , \\ C_i &= \{s : s_1 = i \text{ or } s_2 = i \text{ or } s_3 = i\} , \end{aligned}$$

for $i = 1, 2, \cdots, n$. (Outside of North America, "place" means to finish among the first three or what is meant here by "show.")

Our first example deals with the standard pari-mutuel system outside of North America (cf. [4] page 723). Bets are accepted on the events A_i and C_i for $i = 1, \cdots, n$. Money in the win pool, after the track's fee is deducted, is divided among those who backed the winner, and money in the show pool, after deduction of the fee, is divided into three equal parts and each third is divided among backers of the horses which showed. For our analysis, we make the simplifying assumptions that no fee is deducted and that we know ahead of time the amounts to be bet on each event. (The second assumption is almost true if we place our bets at the last moment.) Suppose a_i and c_i are the total amounts wagered on horse i to win and show, respectively. Let

$$p_i = a_i / \sum_j a_j \qquad \text{and} \qquad q_i = 3c_i / \sum_j c_j ,$$

where both denominators are assumed positive. Then, under our assumptions, the track is effectively posting odds of $p_i : 1 - p_i$ on A_i and $q_i : 1 - q_i$ on C_i for every i. Also, $0 \leqq p_i \leqq 1$, $0 \leqq q_i \leqq 3$, $\sum p_i = 1$, $\sum q_i = 3$.

THEOREM 7. *Either* (i) *there is a sure win, or* (ii) $p_i \leqq q_i \leqq 1$ *for* $i = 1, \cdots, n$, *but not both.*

PROOF. If (ii) is false, then either $p_i > q_i$ for some i or $q_i > 1$ for some i. If $p_i > q_i$, then the betting scheme which bets 1 on each A_k, $k \neq i$, and 1 on C_i has return

$$\textstyle\sum_{k\neq i}[1_{A_k} - p_k] + [1_{C_i} - q_i] > \sum_k [1_{A_k} - p_k] = 0\,.$$

Thus the scheme is a sure win. Similarly, if $q_i > 1$, then the scheme which bets 1 on each C_k, $k \neq i$ is a sure win.

Now assume (ii) is true. To prove (i) is false, it is enough to show there is a probability P on S such that $P(A_i) = p_i$ and $P(C_i) = q_i$ for all i. Let $D_i = C_i - A_i$ and $r_i = q_i - p_i$. It then suffices to find a probability P on S such that $P(A_i) = p_i$ and $P(D_i) = r_i$ for all i.

To this end, consider the convex set

$$C = \{x = (x_1, \cdots, x_{2n}) \in R^{2n} : x_i \geqq 0, \text{ all } i;\; x_i + x_{i+n} \leqq 1,\ i = 1, \cdots, n;\ \textstyle\sum_{i=1}^{n} x_i = 1,\ \sum_{i=n+1}^{2n} x_i = 2\}\,.$$

For i, j, k distinct integers between 1 and n, let e_{ijk} be that point in C which has its ith, $j + n$th, and $k + n$th coordinates equal to 1 and all other coordinates equal to 0. It can be seen by a straightforward, but tedious argument that the collection of e_{ijk} is the set of extreme points of C. Now the point $(p, r) = (p_1, \cdots, p_n, r_1, \cdots, r_n)$ is in C. Therefore, there exist numbers α_{ijk} for i, j, k distinct integers between 1 and n such that $0 \leqq \alpha_{ijk} \leqq 1$, $\sum \alpha_{ijk} = 1$, and $(p, r) = \sum \alpha_{ijk} e_{ijk}$.

If we define $P(\{(i, j, k)\}) = \alpha_{ijk}$, then P is the desired probability. □

The pari-mutuel systems at U.S. tracks are more complicated (cf. [4] pages 723–724). Let a_i, b_i, and c_i be the total amounts wagered on horse i to win, place, and show respectively. The payoff functions corresponding to a \$1 ticket on horse i to win, place, or show are (assuming no cut for the track) respectively:

$$f_i(s) = \left(\frac{1}{a_i} \textstyle\sum_{j=1}^{n} 1_{A_j^c}(s) a_j + 1\right) 1_{A_i}(s) - 1$$

$$g_i(s) = \left(\frac{1}{2b_i} \textstyle\sum_{j=1}^{n} 1_{B_j^c}(s) b_j + 1\right) 1_{B_i}(s) - 1$$

$$h_i(s) = \left(\frac{1}{3c_i} \textstyle\sum_{j=1}^{n} 1_{C_j^c}(s) c_j + 1\right) 1_{C_i}(s) - 1\,,$$

for $s \in S$.

The latter two payoff functions do not correspond to simple oddsmaking. Of course, Theorem 1 still applies and, in fact, the value of the game corresponding to Theorem 2 and the optimal bets can be computed by the simplex method for given values of the a_i's, b_i's, and c_i's. Willis [6] presents a linear programming method for obtaining "nearly optimal" bets in a similar situation.

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