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PLAY-THE-WINNER RULE AND INVERSE SAMPLING FOR SELECTING THE BEST OF $k \geqq 3$ BINOMIAL POPULATIONS

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1. Introduction.

Given. k independent binomial populations with unknown probabilities p_i' of success and q_i' of failure ($p_i' + q_i' = 1$) on a single trial ($i = 1, 2, \cdots, k$).

Problem. to select the (or a) best population (i.e., the one with the largest p', say p_1).

Main emphasis. the comparison of procedures (all using inverse-sampling stopping rules) that differ only in the sampling method.

The first procedure, R_I, uses play-the-winner, cyclic variation (PWC) sampling rule. It puts the populations in a random order at the outset say $\pi_1, \pi_2, \cdots, \pi_k$. Population π_j is sampled until a failure is observed and then π_{j+1} is sampled ($j = 1, 2, \cdots, k$); π_{k+1} is identified with π_1. Sampling terminates as soon as any one population has r successes; that population is selected as best. We determine r so that the probability of a correct selection (CS) satisfies

$$P\{\text{CS} \mid R_I\} \geqq P^* \quad \text{whenever} \quad p_1 - \max_{j>1} p_j \geqq \Delta^* , \tag{1.1}$$

where the constants P^* (with $1/k < P^* < 1$) and Δ^* (with $\Delta^* > 0$) are preassigned. Approximations and a table for $r = r(P^*, \Delta^*)$ are given for selected values of k, P^* and Δ^*. Table 2 gives exact vs. approximate expected total number of observations $E\{N \mid R_I\}$ for $k = 2$ and some comparisons with a fixed sample size procedure.

The second procedure, R_I', uses vector-at-a-time (VT) sampling; it takes one observation from each of the k populations (simultaneously) until at least one of them has r successes. The winner (or one selected from the winners at random) is then chosen as best. We determine r by (1.1) with R_I replaced by R_I'. It is shown (Section 4) that the minimum $P\{\text{CS}\}$ and hence the value of r required to satisfy (1.1) is exactly the same for the PWC-rule (procedure R_I) and the VT-rule (procedure R_I'). (In [5] a similar result was found for fixed-sample stopping rules; also see the discussion of procedure $\hat{R}_I$ below.)

A procedure, R_I^*, dual to R_I is studied (Section 6); it is based on waiting for a fixed number of failures and it is shown asymptotically to be an improvement on R_I when $p_1 < \frac{1}{2}$.

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Another variation, $\hat{R}_I$, of procedure R_I reorders the k populations according to the number of successes (using randomization for ties) after every complete cycle, consisting of k failures (one from each population). It is proven that this does not alter the PCS of R_I; hence the same value of r will be required for $\hat{R}_I$. This reduces $E\{N\}$ and $E\{L\}$ defined in (1.2) below, but for any k, large r and fixed $p_1 > 0$ (so that the p_j for $j > 1$ cannot get close to 1) the saving will be small and has not been evaluated.

Asymptotically ($\Delta^* \to 0$, $\Delta^* r^{\frac{1}{2}} \to$ constant > 0) the PWC is shown to be better than the VT-rule uniformly (i.e., throughout the parameter space) with regard to both $E\{N\}$ and $E\{L\}$. The smallest value (r_0, say) of r such that these asymptotic approximations hold for all $r \geqq r_0$ is also estimated, but no bound on the accuracy of this estimate is given.

Background. the PWC sampling rule (called PW for $k = 2$) was considered by Robbins and others in connection with the 2-arm bandit problem (see references in [12] and also a footnote on page 284 in [2]). The effect of the PW sampling rule for a stopping rule, based on the absolute difference in the number of successes reaching a fixed constant r, has been studied for $k = 2$ in [12].

In addition to using $E\{N \mid R\}$ as a criterion for evaluating any procedure R, we also consider the expected loss (risk function)

$$E\{L \mid R\} = \textstyle\sum_{i=2}^{k} (p_1 - p_i')E\{N_i' \mid R\}\,, \tag{1.2}$$

where N_i' is the number of observations from the population with success parameter p_i' ($i = 1, 2, \cdots, k$).

In [10] a VT procedure with a fixed number of stages is considered for the same formulation (1.1). The selection criterion is the number of successes. Some comparisons with the present paper are given (Table 2). A sequential VT procedure $\mathscr{P}_B$ based on likelihoods is studied in [1]. A sequential scheme $\mathscr{P}_P$ with elimination is due to Paulson [7], [8]. His sampling scheme is based on *ad hoc* independent Poisson random variables that have no intuitive relation to the problem and are not counted by the author in evaluating $E\{N\}$. Although there are many results on the PCS and $E\{N\}$ for $\mathscr{P}_B$ in [1] (lower bounds for $E\{N\}$ on page 292, approximations to $E\{N\}$ in Section 14.2, exact results for $E\{N\}$ for $k = 2$ on page 324), we do not make comparisons below with either $\mathscr{P}_B$ or $\mathscr{P}_P$ because

(i) The main interest here is to study the effect of different types of sampling with other features (such as the stopping rule) remaining the same. Hence we emphasize the comparison of R_I and R_I'.

(ii) The emphasis in this paper is on two types of losses, the loss due to sampling poorer populations as well as the expected total number of observations. The former is not explicitly considered in the above references.

(iii) There is a lack of any theoretical evaluation of $E\{N\}$ as well as a paucity of empirical results for $E\{N\}$ in [7]. Similarly, for $k > 2$ there are no empirical (Monte Carlo) results for Case D in [1].

(iv) We do not claim any optimal properties for R_I or R_I', but the simplicity of the procedure enables us to obtain exact results for the $P\{\text{CS}\}$, $E\{N\}$, and $E\{L\}$ for all k.

(v) Sequential procedures (like $\mathscr{P}_B$ and $\mathscr{P}_P$) are difficult to compare with fixed-sample size or inverse sampling procedures because they are not always applicable to the same practical situation and/or because the usual complication of the former relative to the latter is difficult to evaluate numerically.

2. The procedures R_I: exact results. Under inverse sampling we stop as soon as one population has r successes and select it as best; the integer $r > 0$ is predetermined so that (1.1) is satisfied. We wish to find the probability of a correct selection $P\{\text{CS} \mid R_I\}$ under the procedure R_I, which uses the PWC sampling rule.

Let A_1 denote the best population, A_2 the one following A_1 in the initial randomization, etc. (continuing in cyclic order) and let $S(A_i) = S_i$ denote the current number of successes for A_i, so that $r - S_i = T_i$ is the number of successes A_i needs to qualify to be selected as best; let $\mathbf{T} = (T_1, T_2, \cdots, T_k)$. We define probabilities $U_i(\mathbf{m}) = U_i(m_1, m_2, \cdots, m_k)(1 \leqq i \leqq k)$ by

$$U_i(\mathbf{m}) = P\{\text{CS} \mid \mathbf{T} = \mathbf{m} \text{ and the next observation is on } A_i\} \tag{2.1}$$

and use p_i to denote the single-trial probability of success for population A_i ($i = 1, 2, \cdots, k$). For the PWC sampling rule the outcome of a trial on A_i is either a success (with probability p_i) in which case m_i is changed to $m_i - 1$, or a failure in which case the next observation is on A_{i+1}. Thus

$$U_i(\mathbf{m}) = p_i U_i(m_1, m_2, \cdots, m_i - 1, \cdots, m_k) + q_i U_{i+1}(\mathbf{m}) \tag{2.2}$$

where $U_{k+1} \equiv U_1$ and boundary conditions are given by

$$\begin{aligned} U_1(0, m_2, \cdots, m_k) &= 1 \quad \text{if } m_j > 0 \quad \text{for all } j \neq 1 \\ U_i(m_1, \cdots, m_{i-1}, 0, m_{i+1}, \cdots, m_k) &= 0 \quad \text{if } m_j > 0 \quad \text{for all } j \neq i. \end{aligned} \tag{2.3}$$

To find a solution of (2.2) satisfying (2.3) we use generating functions $V_i = V_i(\mathbf{x}) = V_i(x_1, x_2, \cdots, x_k)$ defined by

$$V_i = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} U_i(\mathbf{m}) x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k} \qquad (i = 1, 2, \cdots, k). \tag{2.4}$$

Then (2.2) leads to

$$\begin{aligned} (1 - p_1 x_1)V_1 - q_1 V_2 &= p_1 x_1 \prod_{j=2}^{k} \left(\frac{x_j}{1 - x_j}\right) \\ (1 - p_i x_i)V_i - q_i V_{i+1} &= 0 \qquad \text{for } i = 2, \cdots, k \end{aligned} \tag{2.5}$$

(where $V_{k+1} \equiv V_1$). Hence, letting $D = (1 - p_1 x_1)(1 - p_2 x_2) \cdots (1 - p_k x_k) - q_1 q_2 \cdots q_k$,

$$\begin{aligned} V_1 &= \frac{p_1 x_1}{D} \prod_{j=2}^{k} \left(\frac{x_j(1 - p_j x_j)}{1 - x_j}\right) \\ V_i &= \frac{p_1 x_1}{D} \left[\prod_{j=2}^{k} \left(\frac{x_j q_j}{1 - x_j}\right)\right] \prod_{\alpha=2}^{i-1} \left(\frac{1 - p_\alpha x_\alpha}{q_\alpha}\right) \qquad (i = 2, 3, \cdots, k) \end{aligned} \tag{2.6}$$

(where products with no factors are taken equal to one). Since we use randomization with equal probabilities $1/k$ for each population at the outset,

$$P\{\mathrm{CS} \mid R_I\} = (1/k) \textstyle\sum_{i=1}^{k} U_i(\mathbf{r}) , \tag{2.7}$$

the coefficient of $x_1^r x_2^r \cdots x_k^r$ in $(1/k) \sum_{i=1}^{k} V_i$, where $\mathbf{r} = (r, r, \cdots, r)$ and r is chosen to satisfy (1.1).

To get an explicit expression for (2.7) we use the expansion

$$\begin{aligned} \frac{1}{D} &= \textstyle\sum_{i=0}^{\infty} \frac{(q_1 q_2 \cdots q_k)^i}{[\prod_{j=1}^{k} (1 - p_j x_j)]^{i+1}} \\ &= \textstyle\sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \cdots \sum_{\omega=0}^{\infty} (p_1 x_1)^\alpha (p_2 x_2)^\beta \cdots (p_k x_k)^\omega \\ &\quad \times \textstyle\sum_{i=0}^{\infty} \binom{i+\alpha}{\alpha}\binom{i+\beta}{\beta} \cdots \binom{i+\omega}{\omega}(q_1 q_2 \cdots q_k)^i . \end{aligned} \tag{2.8}$$

Using the well-known identity for the incomplete beta function (see e.g., (2.3) in [6])

$$q^r \textstyle\sum_{j=0}^{s-1} \frac{\Gamma(r+j)}{\Gamma(r) j!} p^j = I_q(r, s) = p^s \sum_{j=r}^{\infty} \frac{\Gamma(s+j)}{\Gamma(s) j!} q^j \tag{2.9}$$

(where the first equality holds for any real $r > 0$ and the second for any real $s > 0$), we find from (2.6) and (2.8) that the coefficient of $(x_1 x_2 \cdots x_k)^r$ in V_1 is

$$U_1(\mathbf{r}) = p_1^r \textstyle\sum_{i=0}^{\infty} \binom{i+r-1}{i} q_1^i \prod_{j=2}^{k} I_{q_j}(i, r) , \tag{2.10}$$

where $I_q(0, r) = 1 = 1 - I_p(r, 0)$ for $r > 0$. From V_α with $\alpha \geqq 2$ we obtain

$$U_\alpha(\mathbf{r}) = p_1^r \textstyle\sum_{i=0}^{\infty} \binom{i+r-1}{i} q_1^i [\prod_{j=2}^{\alpha-1} I_{q_j}(i, r)][\prod_{j=\alpha}^{k} I_{q_j}(i+1, r)] . \tag{2.11}$$

Hence by (2.7) we can use (2.10) and (2.11) to write

$$\begin{aligned} P\{\mathrm{CS} \mid R_I\} = (1/k) E_r\{&[\textstyle\prod_{j=2}^{k} I_{q_j}(X, r)] \\ &+ \textstyle\sum_{\alpha=2}^{k} [\prod_{j=2}^{\alpha-1} I_{q_j}(X, r)][\prod_{j=\alpha}^{k} I_{q_j}(X+1, r)]\} , \end{aligned} \tag{2.12}$$

where the random variable X has the negative binomial distribution with index $r > 0$, success parameter p_1 and mean rq_1/p_1 (cf. (2.9) above).

Similar calculations are used to find the expected number of observations $E\{N_i \mid R_I\}$ on A_i under procedure R_I; the sum of these is the expected total number of observations $E\{N \mid R_I\}$. For any fixed i, let $S_j^{(i)}(\mathbf{m}) = S_j(\mathbf{m})$ be defined by

$$S_j(\mathbf{m}) = E\{N_i \mid \mathbf{T} = \mathbf{m} \text{ and the next trial is on } A_j\} \qquad (j = 1, 2, \cdots, k) . \tag{2.13}$$

As in (2.2) we obtain

$$S_j(\mathbf{m}) = p_j S_j(m_1, m_2, \cdots, m_i - 1, \cdots, m_k) + q_j S_{j+1}(\mathbf{m}) + \delta_{ji} , \tag{2.14}$$

where $\delta_{ji} = 1$ for $j = i$ and zero otherwise, and $S_{k+1} \equiv S_1$. The boundary conditions are

$$S_j(m_1, \cdots, m_{j-1}, 0, m_{j+1}, \cdots, m_k) = 0 \quad \text{if } m_\alpha > 0 \quad \text{for } \alpha \neq j . \tag{2.15}$$

The desired result is obtained by finding for $i = 1, 2, \cdots, k$

$$E\{N_i \mid R_I\} = (1/k) \sum_{j=1}^{k} S_j^{(i)}(\mathbf{r}) ; \qquad E\{N \mid R_I\} = \sum_{i=1}^{k} E\{N_i \mid R_I\} . \tag{2.16}$$

Using the generating functions as in (2.5), we define $T_i = T_i(\mathbf{x})$ and obtain

$$\begin{aligned} T_i &= \frac{1}{D} \prod_{\alpha=1}^{k} \left(\frac{x_\alpha}{1 - x_\alpha}\right) \prod_{j \neq i} (1 - x_j p_j) \\ T_j &= T_i \sum_{\alpha=j}^{i-1} \left(\frac{q_\alpha}{1 - x_\alpha p_\alpha}\right) & \text{for } j < i \\ T_j &= T_i \sum_{\alpha=j}^{k} \left(\frac{q_\alpha}{1 - x_\alpha p_\alpha}\right) \prod_{\beta=1}^{i-1} \left(\frac{q_\beta}{1 - x_\beta p_\beta}\right) & \text{for } j > i , \end{aligned} \tag{2.17}$$

where D, defined as above, is expanded in (2.8). From (2.17) we obtain for $j \leqq i$ and $j > i$, respectively,

$$\begin{aligned} S_j^{(i)}(\mathbf{r}) &= (1/q_i) \sum_{\alpha=0}^{\infty} I_{q_i}(\alpha + 1, r) [\textstyle\prod_{\beta=j}^{i-1} I_{q_\beta}(\alpha + 1, r)] [\prod_{\gamma<j, \gamma>i} I_{q_\gamma}(\alpha, r)] \\ S_j^{(i)}(\mathbf{r}) &= (1/q_i) \sum_{\alpha=0}^{\infty} I_{q_i}(\alpha + 1, r) [\textstyle\prod_{\beta<i, \beta \geqq j} I_{q_\beta}(\alpha + 1, r)] [\prod_{\gamma=i+1}^{j-1} I_{q_\gamma}(\alpha, r)] . \end{aligned} \tag{2.18}$$

By (2.16) the average of these k quantities ($j = 1, 2, \cdots, k$) in (2.18) is $E\{N_i \mid R_I\}$ and the sum ($i = 1, 2, \cdots, k$) of these k averages is $E\{N \mid R_I\}$.

REMARK 2.1. We show that $E\{N \mid R_I\}$ is uniformly bounded if the p_i are all bounded away from zero. It suffices to consider $S_j^{(i)}(\mathbf{r})$ for any i, j in (2.18) and set all the incomplete beta functions (after the first one) equal to unity. We then obtain for all j

$$\begin{aligned} S_j^{(i)}(\mathbf{r}) &\leqq \frac{1}{q_i} \sum_{\alpha=0}^{\infty} I_{q_i}(\alpha + 1, r) = \frac{r}{q_i} \int_0^{q_i} (1 - t)^{r-1} \sum_{\alpha=0}^{\infty} \binom{\alpha+r}{\alpha} t^\alpha \, dt \\ &= \frac{r}{q_i} \int_0^{q_i} \frac{dt}{(1 - t)^2} = \frac{r}{p_i} . \end{aligned} \tag{2.19}$$

Hence $E\{N \mid R_I\}$ is bounded by $r \sum_{i=1}^{k} (1/p_i)$.

REMARK 2.2. The experimenter may be concerned (as the referee was) about whether large deviations occur among the U_α in (2.10) and (2.11). The maximum difference $U_1 - U_2$ for the case $q_i = q_1 + \Delta^*$ ($i = 2, 3, \cdots, k$) is

$$U_1 - U_2 = E_r\{I_{q_1+\Delta^*}^{k-1}(X, r) - I_{q_1+\Delta^*}^{k-1}(X + 1, r)\} ; \tag{2.20}$$

for $k = 2$ this is

$$\begin{aligned} U_1 - U_2 &= [p_1(p_1 - \Delta^*)]^r \sum_{j=0}^{\infty} \binom{j+r-1}{j}^2 [q_1(q_1 + \Delta^*)]^j \\ &= [p_1(p_1 - \Delta^*)]^r \, {}_2F_1[r, r, 1; q_1(q_1 + \Delta^*)] , \end{aligned} \tag{2.21}$$

where ${}_2F_1(a, b, c; z)$ is the usual hypergeometric function. $U_1 - U_2$ can be interpreted as the probability that A_1 reaches r successes in the same cycle as the other population (if experimentation were continued). Let $\approx$ denote proximity and not be confused with $\sim$. For $\Delta^* = .2$ and $k = 2$ with $r = 20$ ($P^* \approx .95$) and $r = 40$ ($P^* \approx .99$) from (2.21) we find $U_1 - U_2 = .025$ (maximum attained at $p_1 = .90$) and $U_1 - U_2 = .004$ (maximum attained at $p_1 = .84$), respectively. Hence P^* varies between $.95 \pm \frac{1}{2}(.025)$ in the first case and between $.99 \pm \frac{1}{2}(.004)$ in the second case. An approximation to $U_1 - U_2$ for general k is given in Remark 6.1.

3. Approximation and the determination of r for procedure R_I. Since the incomplete beta function is decreasing (increasing) in the first (second) argument, we can get bounds on the $P\{\text{CS} \mid R_I\}$ in (2.12) with the same asymptotic value for $r \to \infty$ by replacing X by $X + 1$ or vice versa, obtaining

$$E_r\{\textstyle\prod_{j=2}^k I_{q_j}(X + 1, r)\} < P\{\text{CS} \mid R_I\} < E_r\{\prod_{j=2}^k I_{q_j}(X, r)\} . \tag{3.1}$$

Let X_{p_j} denote the number of failures observed (or the number of completed cycles required) to obtain r successes when p_j is the probability of success on a single trial ($j = 1, 2, \cdots, k$). Then the X_{p_j} ($j = 1, 2, \cdots, k$) are independent negative binomial chance variables with success parameter p_j and common index r, and $X = X_{p_1}$ in (3.1).

We can also obtain (3.1) by noting that the left and right members of (3.1) are $P\{X_{p_1} < X_{p_j}\ (j = 2, 3, \cdots, k)\}$ and $P(X_{p_1} \leqq X_{p_j}\ (j = 2, 3, \cdots, k)\}$ respectively, and that the PCS must lie between these. Hence the first inequality in (3.2) below; the error here is bounded by $\sum_{j=2}^k P\{X_{p_1} = X_{p_j}\}$, which $\to 0$ as $\Delta^* \to 0$. Hence letting $\Delta_{1j} = p_1 - p_j$ and assuming that $p_1 > p_j$ ($j = 2, 3, \cdots, k$) we have for any r

$$\begin{aligned} P\{\text{CS} \mid R_I\} &> P\{X_{p_i} < X_{p_j}\ (j = 2, 3, \cdots, k)\} \\ &= P\left\{\left(\frac{X_{p_j} - rq_j/p_j}{(rq_j)^{\frac{1}{2}}/p_j}\right) > \left(\frac{X_{p_1} - rq_1/p_1}{(rq_1)^{\frac{1}{2}}/p_1}\right)\frac{p_j}{p_1}\left(\frac{q_1}{q_j}\right)^{\frac{1}{2}} \right. \\ &\qquad \left. - \frac{\Delta_{1j}}{p_1}\left(\frac{r}{q_j}\right)^{\frac{1}{2}}\ (j = 2, 3, \cdots, k)\right\} \\ &= \int_{-\infty}^{\infty} \prod_{j=2}^k \left[1 - G_{jr}\left(\frac{yp_j q_1^{\frac{1}{2}} - \Delta_{1j} r^{\frac{1}{2}}}{p_1 q_j^{\frac{1}{2}}}\right)\right] dG_{1r}(y) , \end{aligned} \tag{3.2}$$

where $G_{jr}(y)$ is the cdf of Y_{p_j} defined by the parentheses in (3.2) with success parameter p_j. It is well known that the Y_{p_j} tend to standard normal chance variable for each j ($j = 1, 2, \cdots, k$). In (3.2) we use a version of the Helly-Bray theorem and Pólya's theorem (cf. [9] page 97, 100) as in the following

LEMMA 1. *If $G_{jn}(x)$ and $G_j(x)$ are* cdf'*s* (*or complements of* cdf'*s*) *with $G_j(x)$ continuous and such that $G_{jn}(x) \to G_j(x)$ for all x ($j = 1, 2, \cdots, k$), then*

$$\int_{-\infty}^{\infty} [\textstyle\prod_{j=2}^k G_{jn}(x)]\, dG_{1n}(x) \to \int_{-\infty}^{\infty} [\prod_{j=2}^k G_j(x)]\, dG_1(x) . \tag{3.3}$$

PROOF. By Pólya's theorem $G_{jn}(x) \to G_j(x)$ uniformly in x for each j. Hence for the product of two

$$\begin{aligned} |G_{2n} G_{3n} - G_2 G_3| &\leqq |G_{3n}(G_{2n} - G_2)| + |G_2(G_{3n} - G_3)| \\ &\leqq |G_{2n} - G_2| + |G_{3n} - G_3| , \end{aligned} \tag{3.4}$$

we also have uniform convergence in x; similarly for any finite product of such functions. Let $H_n(x) = \prod_{j=2}^k G_{jn}(x)$, $H(x) = \prod_{j=2}^k G_j(x)$. Then the difference D of the two members of (3.3) is

$$D \leqq |\int [H_n(x) - H(x)]\, dG_{1n}(x)| + |\int H(x)\, dG_{1n}(x) - \int H(x)\, dG_1(x)| \,. \tag{3.5}$$

The last term $\to 0$ by the Helly-Bray theorem. By the uniform convergence $|H_n(x) - H(x)|$ can be made less than a fixed ε for all x, so that the first term also $\to 0$. □

In applying Lemma 1 to (3.2) we take $G_j(x) = \Phi(x)$ the standard normal cdf for each j, (n is replaced by r), and we use the sign "$\sim$" to mean asymptotic approximation with $\Delta^* \to 0$. This implies that to satisfy (1.1) we need $r \to \infty$; we assume that $\Delta^* r^{\frac{1}{2}}$ tends to a positive constant (cf. (3.8) below). Hence we obtain from (3.2) for $0 < p_j < p_1 < 1$ ($j = 2, 3, \cdots, k$)

$$\begin{aligned} P\{\mathrm{CS} \mid R_I\} &\sim \int_{-\infty}^{\infty} \prod_{j=2}^k \left[1 - \Phi\left(\frac{yp_j q_1^{\frac{1}{2}} - \Delta_{1j} r^{\frac{1}{2}}}{p_1 q_j^{\frac{1}{2}}}\right)\right] d\Phi(x) \\ &= \int_{-\infty}^{\infty} \prod_{j=2}^k \Phi\left(\frac{yp_j q_1^{\frac{1}{2}} + \Delta_{1j} r^{\frac{1}{2}}}{p_1 q_j^{\frac{1}{2}}}\right) d\Phi(y) \,. \end{aligned} \tag{3.6}$$

As a step in the minimization of (3.6) subject to the conditions $\Delta_{1j} \geqq \Delta^*$ ($j = 2, 3, \cdots, k$), note from (2.12) that the exact PCS is strictly decreasing in each p_j for $j \geqq 2$ and hence set $p_j = p_2$ and $\Delta_{1j} = \Delta^*$ for $j \geqq 2$. Then for $0 < p_2 < p_1 < 1$ (so that $0 < \rho < 1$ in (3.8)) we can write (3.6) in the form

$$\min P\{\mathrm{CS} \mid R_I\} \sim \int_{-\infty}^{\infty} \Phi^{k-1}\left(\frac{x\rho^{\frac{1}{2}} + H}{(1-\rho)^{\frac{1}{2}}}\right) d\Phi(x) \equiv A_{k-1}(\rho, H) \,, \tag{3.7}$$

where the last equality defines $A_{k-1}(\rho, H)$ and where

$$H = \frac{\Delta^* r^{\frac{1}{2}}}{(q_1 p_2^2 + q_2 p_1^2)^{\frac{1}{2}}} \quad \text{and} \quad \rho = \frac{q_1 p_2^2}{q_1 p_2^2 + q_2 p_1^2} \,. \tag{3.8}$$

(H is the same as for $k = 2$ in [11].) For $k = 2$ we obtain

$$A_1(\rho, H) = \int_{-\infty}^{\infty} \Phi\left(\frac{x\rho^{\frac{1}{2}} + H}{(1-\rho)^{\frac{1}{2}}}\right) d\Phi(x) = \Phi(H) \,, \tag{3.9}$$

independent of ρ.

For the second part of the minimization, we obtain an approximate result and a correction term (which is small). We obtain an approximate minimum of (3.7) by minimizing H in (3.8) and disregarding the fact that ρ is also varying; this shows (as for $k = 2$ in (2.9) of [11]) that the least favorable (LF) configuration is

$$\begin{aligned} p_1 &= \tfrac{2}{3} + \tfrac{1}{2}\Delta^* + \mathcal{O}\{(\Delta^*)^2\} \,; \\ p_2 = p_3 = \cdots = p_k &= \tfrac{2}{3} - \tfrac{1}{2}\Delta^* + \mathcal{O}\{(\Delta^*)^2\} \,. \end{aligned} \tag{3.10}$$

Putting (3.10) in the expression for ρ in (3.8) gives

$$\rho = \tfrac{1}{2} - \tfrac{1}{8}3\Delta^* + \mathcal{O}\{(\Delta^*)^2\} \,; \tag{3.11}$$

which indicates that for $\rho = \frac{1}{2}$ we get a first approximation.

To approximately satisfy (1.1) we solve for r by solving

$$A_{k-1}(\tfrac{1}{2}, H) = P^* \tag{3.12}$$

with existing tables (e.g., [3] or [4]). If H_0 or $\lambda = \lambda(P^*, k)$ is the value of H that satisfies (3.12) then for small Δ^* we have as in (2.11) or [11]

$$r \sim \frac{8}{27}\left(\frac{\lambda}{\Delta^*}\right)^2, \tag{3.13}$$

except that λ is now a function of both P^* and k. Table 1, based on (3.13), contains values of r for selected k, Δ^* and P^*.

For example, if we take $k = 3$, $P^* = .90$ and $\Delta^* = .10$ and use the tables in [2] or [3], we find for the solution of (3.12) that $H = 1.58$. Using $H = 1.58$ from (3.13) we obtain $r = 74$. A correction term in the original technical report on this paper indicates that $r = 76$ is a better solution for this example; the

TABLE 1

Values of r required by procedures[3] R_I, *and* R_I' *based on* (3.13)

k	Δ^*	$P^* = .90$	$P^* = .95$	$P^* = .99$
2	.1	48.67[2]	80.17	160.35
	.2	12.17	20.04	40.09[4]
	.3	5.41	8.91	17.82
	.4	3.04	5.01	10.02[4]
	(EP .4)[1]	(.886)	(.946)	(.990)
3	.1	75.50	110.57	195.27
	.2	19.31	28.04	49.14
	.3	8.77	12.65	21.97
	.4	5.04	7.20	12.43
	(EP .4)	(.901)	(.952)	(.991)
5	.1	104.50	142.07	230.58
	.2	26.97	36.38	58.31
	.3	12.47	16.67	26.21
	.4	7.16	9.48	14.89
10	.1	138.84	179.61	272.62
	.2	36.24	46.30	69.17
	.3	16.69	21.25	31.21
	.4	9.63	12.28	17.78

[1] Exact probabilities for $\Delta^* = .4$ based on (2.12) using randomization, e.g., for $k = 2$ and $P^* = .90$ we calculate the exact values ν_3 and ν_4 for $r = 3$ and 4 using (2.12), and then randomize by entering $.04\nu_4 + .96\nu_3$ in the table.

[2] For the procedures R_I and R_I' we use the integer $r + 1$ when the appropriate entry is $r + \varepsilon$ with $0 < \varepsilon < 1$. To obtain a PCS closer to the nominal P^* one can use the decimal entries and randomize between r and $r + 1$ as explained in the previous footnote.

[3] By the remarks about procedures $\hat{R}_I$ and $\hat{R}_I'$ in the introduction and in Section 6, the same value of r is also required by these procedures.

[4] Use $[r]$ (i.e., 40 and 10) in these cases since the exact PCS using (2.7) is .990 to 3 decimals.

correction term and its derivation were deleted for brevity. These corrections, however, were used in computing Table 1, e.g. the entry for the above problem is 75.50.

For the *expected number of trials* we use (2.18) and the fact that for $r \to \infty$

$$I_q(\alpha, r) = I_q(\alpha + 1, r) + \binom{j+r-1}{j} p^r q^j = I_q(\alpha + 1, r)[1 + o(1)] . \tag{3.14}$$

Thus we can approximate $S_j^{(i)}(\mathbf{r})$ in (2.18) for all j and also $E\{N_i \mid R_I\}$ by

$$E\{N_i \mid R_I\} \sim (1/q_i) \sum_{\alpha=0}^{\infty} \prod_{\beta=1}^{k} I_{q_\beta}(\alpha + 1, r) \sim S_j^{(i)}(\mathbf{r}) \quad (j = 1, 2, \cdots, k) \tag{3.15}$$

where the infinite sum Z does not depend on i or j. Hence for $r \to \infty$ (or $\Delta^* \to 0$) we have

$$E\{N \mid R_I\} \sim (\textstyle\sum_{i=1}^{k} 1/q_i) Z . \tag{3.16}$$

Using the second identity in (2.9) for $\beta = 1$ in (3.15) and interchanging summations, we can write for Z

$$Z = p_1^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} q_1^j \sum_{\alpha=0}^{j-1} \prod_{\beta=2}^{k} [1 - I_{p_\beta}(r, \alpha + 1)] . \tag{3.17}$$

To simplify (3.17) we first assume $p_1 > 0$ and prove the

LEMMA 2. *For any positive integers r, j and any $p \geqq 0$*

$$\sum_{\alpha=0}^{j-1} I_p(r, \alpha + 1) = (r + j) I_p(r, j) - (r/p) I_p(r + 1, j) . \tag{3.18}$$

The same result holds for any real $r \geqq 0$ and in the limit as $p \to 0$, if we define $I_p(0, \alpha + 1)$ to be 0 for $\alpha \geqq 0$.

PROOF. Using (2.9) and the integral form for $I_p(r, \alpha + 1)$ with $r > 0$,

$$\begin{aligned} \sum_{\alpha=0}^{j-1} I_p(r, \alpha + 1) &= \sum_{\alpha=0}^{j-1} \frac{\Gamma(r + \alpha + 1)}{\Gamma(r)\alpha!} \int_0^p t^{r-1}(1 - t)^\alpha \, dt \\ &= r \int_0^p I_t(r + 1, j) \frac{dt}{t^2} , \end{aligned} \tag{3.19}$$

where we interchanged summation and integration and use (2.9) again in the last step. Integrating-by-parts and noting that $I_t(r + 1, j)/t \to 0$ as $t \to 0$, we obtain the lemma (3.18).

Returning to (3.17) we multiply out the last product and use Lemma 2 for $k - 1$ of the terms to obtain for Z

$$\begin{aligned} Z = \frac{rq_1}{p_1} &+ \sum_{i=2}^{k} \left[\frac{r}{p_i} E_r\{I_{p_i}(r + 1, X)\} - \frac{r}{p_1} E_{r+1}\{I_{p_i}(r, X)\} \right] \\ &+ \cdots + (-1)^{k-1} E_r\{\textstyle\sum_{\alpha=0}^{X-1} \prod_{\beta=2}^{k} I_{p_\beta}(r, \alpha + 1)\} . \end{aligned} \tag{3.20}$$

For $p_1 > p_j$ ($j = 2, 3, \cdots, k$) and $r \to \infty$ all the expectations in (3.20) tend to zero exponentially fast. We drop all terms after the first line in (3.20); for example the last term is bounded above in absolute value by

$$\begin{aligned} E_r\{X \textstyle\prod_{\beta=2}^{k} I_{p_\beta}(r, X + 1)\} &\sim \frac{rq_1}{p_1} \prod_{\beta=2}^{k} I_{p_\beta}\left(r, \frac{rq_1}{p_1}\right) \sim \frac{C' r e^{-Cr}}{C'' r^{(k-1)/2}} \\ &= \mathcal{O}\left\{\exp\left(-\frac{r}{2} \sum_{j=2}^{k} \frac{\Delta_{1j}^2}{p_1^2 q_1}\right)\right\} . \end{aligned} \tag{3.21}$$

The normal approximation and the first term of the Feller-Laplace expansion for the tail of the normal distribution were both used in (3.21). Using the normal approximation for the first line of (3.20), we obtain for large r

$$Z \sim \frac{rq_1}{p_1} + \sum_{j=2}^{k} \frac{r\Delta_{1j}}{p_1 p_j} \Phi\left(-\Delta_{1j}\left(\frac{r}{D_j}\right)^{\frac{1}{2}}\right) \sim \frac{rq_1}{p_1}, \tag{3.22}$$

where $D_j = q_1 p_j^2 + q_j p_1^2$ and $r = r(\Delta^*, P^*, k)$ is determined by (3.13) so as to satisfy (1.1). For small Δ^*, r is large and we can take the first term alone in (3.22) to estimate Z. Hence for $\Delta^* \to 0$ (or $r \to \infty$) we have from (3.16) and (3.22)

$$E\{N \mid R_I\} \sim \frac{rq_1}{p_1}\left(\sum_{i=1}^{k} \frac{1}{q_i}\right) < \frac{kr}{p_1}. \tag{3.23}$$

This upper bound for $\Delta^* \to 0$ is also obtained in (4.12) below as the asymptotic limit for $E\{N \mid R_I'\}$ as $\Delta^* \to 0$, thus showing that $E\{N \mid R_I\} < E\{N \mid R_I'\}$ for Δ^* small (or r large).

If we define the expected loss or risk $E\{L \mid R\}$ under procedure R by

$$E\{L \mid R\} = \sum_{i=1}^{k} (p_1 - p_i) E\{N_i \mid R\}, \tag{3.24}$$

then from (3.15) and (3.22) we have for $\Delta^* \to 0$ (or $r \to \infty$)

$$E\{L \mid R_I\} \sim \frac{rq_1}{p_1} \sum_{i=1}^{k} \left(\frac{p_1 - p_i}{q_i}\right). \tag{3.25}$$

4. Procedure R_I' and comparisons with procedure R_I. Let R_I' denote the procedure that uses the same inverse-sampling termination rule as R_I together with the vector-at-a-time (VT) sampling rule. Ties are decided by randomization, i.e., we select one of the c contenders that reached r successes at the final stage using an independent experiment with probability $1/c$ for each.

To obtain the $P\{\text{CS} \mid R_I'\}$ we consider the event that on the mth stage (i.e., after m vectors of observations and not before) the best player A_1 has his rth success ($r \leqq m$) and each of the remaining A_i ($i \geqq 2$) has at most $r - 1$ successes. Summing on m, we obtain

$$\begin{aligned} P\{\text{CS} \mid R_I'\} - Q &= \sum_{m=r}^{\infty} \binom{m-1}{r-1} p_1^r q_1^{m-r} \prod_{i=2}^{k} \left[\sum_{j=0}^{r-1} \binom{m}{j} p_i^j q_i^{m-j}\right] \\ &= p_1^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} q_1^j \prod_{i=2}^{k} [I_{q_i}(j+1, r)] \\ &= E_r\{\prod_{i=2}^{k} I_{q_i}(X+1, r)\}, \end{aligned} \tag{4.1}$$

where Q is the contribution to the $P\{\text{CS} \mid R_I'\}$ arising from randomization over ties. Since each I_q-function in (4.1) is strictly increasing in q, we minimize the right side of (4.1) by setting $p_i = p_2$ ($i = 3, 4, \cdots, k$); this does not prove that we have a minimum for the $P\{\text{CS} \mid R_I'\}$, although it is a proof for the asymptotic ($r \to \infty$) case. To prove that $p_i = p_2$ ($i = 3, 4, \cdots, k$) also yields a minimum of the $P\{\text{CS} \mid R_I'\}$ for small r, we write Q in the form

$$\tfrac{1}{2}\{T_{1,2} + \cdots + T_{1,k}\} + \tfrac{1}{3}\{T_{1,2,3} + \cdots + T_{1,k-1,k}\} + \cdots + (1/k)\{T_{1,2,\cdots,k}\}, \tag{4.2}$$

where, e.g., $T_{1,2}$ is the probability that A_1 and A_2 (and only these two) tie for first place by getting their rth success on the same vector and before the others. Thus for the pair $(1, \alpha)$ with any $\alpha > 1$

$$T_{1,\alpha} = p_1^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} q_1^j [I_{q_\alpha}(j, r) - I_{q_\alpha}(j+1, r)] \prod_{i=2, i\neq\alpha}^{k} [I_{q_i}(j+1, r)], \tag{4.3}$$

for the triple $(1, \alpha, \beta)$ with $\alpha \neq \beta$ arbitrary (but not equal to 1)

$$T_{1,\alpha} = p_1^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} q_1^j \prod_{x=\alpha,\beta} [I_{q_x}(j, r) - I_{q_x}(j+1, r)] \times \prod_{i=2, i\neq\alpha, i\neq\beta}^{k} [I_{q_i}(j+1, r)], \tag{4.4}$$

etc. Multiplying the differences in square brackets and using (4.2) to combine terms, we find that a typical term has h factors of the form $I_{q_\alpha}(j+1, r)$ and $k-1-h$ factors of the form $I_{q_\beta}(j, r)$, where α runs over a fixed set S_h of h values among $(2, 3, \cdots, k)$ and β runs over the complementary set CS_h; let $\mathscr{S}_h$ denote the set of size $\binom{k-1}{h}$ consisting of all such sets S_h of size h. The coefficient W_h of this typical term, starting from the right end of (4.2), is

$$\begin{aligned} W_h &= (-1)^h \left\{\frac{1}{k} - \binom{h}{1}\frac{1}{k-1} + \binom{h}{2}\frac{1}{k-2} - \cdots + (-1)^h \binom{h}{h}\frac{1}{k-h}\right\} \\ &= \int_0^1 \left(\frac{1}{x} - 1\right)^h x^{k-1}\, dx = \frac{1}{k\binom{k-1}{h}} = \frac{\Gamma(k-h)\Gamma(h+1)}{\Gamma(k+1)} > 0. \end{aligned} \tag{4.5}$$

Hence we can write the exact value of the $P\{\text{CS} \mid R_I'\}$ in the form

$$P\{\text{CS} \mid R_I'\} = p_1^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} q_1^j \sum_{h=0}^{k-1} W_h \sum_{S_h \in \mathscr{S}_h} \left[\prod_{\alpha \in S_h} I_{q_\alpha}(j+1, r)\right] \times \left[\prod_{\beta \in CS_h} I_{q_\beta}(j, r)\right]. \tag{4.6}$$

Since $W_h > 0$, all terms in (4.6) are positive and it follows as above that we minimize $P\{\text{CS} \mid R_I'\}$ by setting $p_i = p_2$ $(i = 3, 4, \cdots, k)$. This simplifies (4.6) considerably and a lower bound to the $P\{\text{CS} \mid R_I'\}$ for $p_i \geqq p_2$ becomes

$$\begin{aligned} &\min_{p_i \geqq p_2, (i=3,4,\cdots,k)} P\{\text{CS} \mid R_I'\} \\ &\quad = p_1^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} q_1^j \frac{1}{k} \sum_{h=0}^{k-1} I_{q_2}^h(j+1, r) I_{q_2}^{k-1-h}(j, r) \\ &\quad = p_1^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} q_1^j \frac{[I_{q_2}^k(j, r) - I_{q_2}^k(j+1, r)]}{k[I_{q_2}(j, r) - I_{q_2}(j+1, r)]}. \end{aligned} \tag{4.7}$$

The same minimization can also be applied in (2.12) for procedure R_I and we clearly note that the result is exactly the same as in (4.7) above. Hence, after the first step of minimization, the $P\{\text{CS}\}$ expressions for R_I and R_I' are exactly the same. It follows that *the least favorable configuration is the same for R_I and R_I' and hence they require exactly the same value of r to satisfy* (1.1). This same result was also found for a related procedure R^* explained in Section 6.

To obtain the expected total number of observations $E\{N \mid R_I'\}$ under procedure R_I', we use the fact that we have an expression F_1 like (4.6) with the extra factor $km = k(j + r)$ if we select A_1 and $k - 1$ similar expressions F_i corresponding to the selection of A_i $(i = 2, 3, \cdots, k)$. Thus

$$(4.8) \qquad \begin{aligned} F_1 &= kp_1^r \sum_{j=0}^{\infty} (j + r)\binom{j+r-1}{j} q_1^j \sum_{h=0}^{k-1} W_h \\ &\quad \times \sum_{S_h \in \mathscr{S}_h} [\prod_{\alpha \in S_h} I_{q_\alpha}(j + 1, r)][\prod_{\beta \in CS_h} I_{q_\beta}(j, r)] \\ &= \frac{kr}{p_1} E_{r+1}\{\sum_{h=0}^{k-1} W_h \sum_{S_h \in \mathscr{S}_h} [\prod_{\alpha \in S_h} I_{q_\alpha}(j + 1, r)][\prod_{\beta \in CS_h} I_{q_\beta}(j, r)]\} \end{aligned}$$

where W_h is given by (4.5), the F_i $(i = 2, 3, \cdots, k)$ are obtained by interchanging p_1 with p_i (and q_1 with q_i), and

$$(4.9) \qquad E\{N \mid R_I'\} = \sum_{i=1}^{k} F_i \, .$$

To get an asymptotic approximation for (4.9) when $p_1 > p_i$ for $i \geqq 2$, we first show that every F_i $(i \geqq 2)$ tends to zero as $r \to \infty$. It suffices to show that for $q_2 > q_1$ and $r \to \infty$

$$(4.10) \qquad p_2^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} q_2^j I_{q_1}(j, r) \sim I_{q_1}\left(\frac{rq_2}{p_2}, r\right) = o\left(\frac{1}{r}\right).$$

We interpret (4.10) as the probability that $Y_{p_1} \geqq Y_{p_2}$ where $Y_{p_i} = X_{p_i}/r$ and X_{p_i} is the negative binominal with parameter p_i and common index r; the expectation of Y_{p_i} is q_i/p_i and the variance is $q_i/rp_i^2 \to 0$ $(i = 1, 2)$ as $r \to \infty$. Thus for an asymptotic $(r \to \infty)$ analysis we can replace Y_{p_2} by q_2/p_2 (or X_{p_2} by rq_2/p_2) and this gives the middle expression in (4.10). Using a normal approximation to the beta as in (3.21), we obtain for $\Delta = p_1 - p_2 = q_2 - q_1 > 0$ and $r \to \infty$

$$(4.11) \qquad I_{q_1}\left(\frac{rq_2}{p_2}, r\right) \sim \Phi\left(\frac{-\Delta r^{\frac{1}{2}}}{p_2 q_2^{\frac{1}{2}}}\right) \sim \frac{C_1}{r} e^{-C_2 r} = o\left(\frac{1}{r}\right).$$

For the nonzero term F_1 in (4.9) we do a similar analysis and every I_q-function approaches 1 in expectation. Hence by (4.8) we obtain

$$(4.12) \qquad E\{N \mid R_I'\} \sim \frac{kr}{p_1}.$$

To obtain the total expected number of observations from the non-best populations we replace k in (4.12) by $k - 1$. Using the expected loss defined in (3.24) we obtain for $\Delta > 0$

$$(4.13) \qquad E\{L \mid R_I'\} = \frac{1}{k} [\sum_{i=1}^{k} F_i] \sum_{j=1}^{k} (p_1 - p_j) \sim \frac{r}{p_1} \sum_{j=1}^{k} (p_1 - p_j) \, ,$$

where the last expression holds for large r.

Since $q_1 < q_i$ $(i = 2, 3, \cdots, k)$, we find by comparing (3.23) and (4.12) that for large r the procedure R_I requires a uniformly smaller expected total number of trials when $\Delta > 0$. In addition, for large r procedure R_I has a uniformly smaller expected loss when $\Delta > 0$.

To approximate the value of r above which these results hold we now return to (3.14). A finer analysis of the application of (3.14) to (2.18) shows that a constant (with respect to r) is obtained from the omitted term in (3.14) whenever

$\gamma = 1$ in (2.18). For any i, we find that $\gamma = 1$ in exactly $i - 1$ of the equations in (2.18), namely for $j = 2, 3, \cdots, i$ in the first line of (2.18). Moreover, for each i, the contribution to $E\{N \mid R_I\}$ is $1/kq_i$. For $\gamma > 1$ we can use an argument similar to that in (4.11) to show that the omitted sums approach zero as $r \to \infty$. Hence we can replace (3.23) by the finer result

$$E\{N \mid R_I\} = \frac{rq_1}{p_1}\left(\sum_{i=1}^{k} \frac{1}{q_i}\right) + \frac{1}{k} \sum_{i=1}^{k} \left(\frac{i-1}{q_i}\right) + o(1) , \tag{4.14}$$

and a similar result holds for $E\{L \mid R_I\}$ if we replace q_i by $q_i/(p_1 - p_i)$ for $i \geqq 2$. For procedure R_I there are no corresponding nonzero terms omitted in (4.12) and (4.13). Hence we approximate the smallest value (r_0, say) of r, such that the stated result for $E\{N\}$ holds for all $r \geqq r_0$, by the solution in r of

$$\frac{rq_1}{p_1}\left(\sum_{i=1}^{k} \frac{1}{q_i}\right) + \frac{1}{k} \sum_{i=1}^{k} \frac{i-1}{q_i} \doteq \frac{kr}{p_1} . \tag{4.15}$$

For $p_2 = p_3 = \cdots = p_k$, this has the solution $p_1/2\Delta$, i.e.,

$$r_0 \sim p_1/2\Delta , \tag{4.16}$$

which is the same as that obtained in [11] for $k = 2$.

Similarly we approximate the smallest value (r_0, say) of r, such that the stated result on $E\{L\}$ holds for all $r \geqq r_0$, by the solution in r of

$$\frac{rq_1}{p_1} \sum_{i=1}^{k} \left(\frac{p_1 - p_i}{q_i}\right) + \frac{1}{k} \sum_{i=1}^{k} \frac{(i-1)(p_1 - p_i)}{q_i} = \frac{r}{p_1} \sum_{i=1}^{k} (p_1 - p_i) . \tag{4.17}$$

For $p_2 = p_3 = \cdots = p_k$ this is the same equation as (4.15) and hence (4.16) again gives the required solution.

5. Comparisons in the case of k equal success-parameters. Starting with R_I', we use the fact that the expectation of the minimum of k independent negative-binomial NB(p, r) chance variables, each with common success probability p and index r, is asymptotically ($r \to \infty$) equivalent to the $[100/(k + 1)]$st percentile of the underlying NB(p, r) distribution. Thus the common number of observations M from each of the populations until any one of them reaches r successes has asymptotic ($r \to \infty$) expectation equal to the solution in s of

$$I_p(r, s) = p^r \sum_{j=0}^{s-r} \binom{j+r-1}{j} q^j = (k + 1)^{-1} . \tag{5.1}$$

Since s and r will both be large, we use the normal approximation to the NB(p, r) and replace (5.1) by

$$P\left\{\frac{M - r/p}{(rq)^{\frac{1}{2}}/p} < \frac{s + \frac{1}{2} - r/p}{(rq)^{\frac{1}{2}}/p}\right\} \sim \Phi\left(\frac{(s + \frac{1}{2})p - r}{(rq)^{\frac{1}{2}}}\right) = \frac{1}{k + 1} , \tag{5.2}$$

where M is the common number of observations per population (failures plus successes). Multiplying the solution of (5.2) by k, we obtain the result

$$E\{N \mid R_I'\} \sim ks = \frac{kr}{p} - \frac{k\lambda}{p}(rq)^{\frac{1}{2}} - \frac{k}{2} \tag{5.3}$$

where $\lambda = \lambda(k)$ is the $[100k/(k + 1)]$st percentile of the standard normal distribution, independent of P^*.

For the procedure R_I we superimpose the PWC procedure on the same data that was obtained a vector-at-a-time and note that the same population that reaches r successes first under procedure R_I' (or one of them if there was more than one) will also reach r successes first under procedure R_I; this is because at any time the number of failures from different populations can differ by at most one. Hence the asymptotic value of s for the winning population is again the solution of (5.2) and the number of failures is $s - r$. By PWC sampling, all the populations have $s - r$ failures and hence each of the $k - 1$ non-winners has $p(s - r)/q$ successes. Thus we obtain for the total

TABLE 2
Comparison of approximate and exact results for $k = 2$ and inverse sampling vs. *fixed sample size procedures*
($\Delta^* = .2$, $P^* = .95$)

Type of Stopping Rule	$\bar{p} = \frac{1}{2}(p_1 + p_2)$	Maximum of $E(N)$ Values ($p_1 = p_2$)			
		Procedure R_I		Procedure R_I'	
		Approx.[a]	Exact[b]	Approx.[c]	Exact[d]
Inverse Sampling $P(r = 20) = .958$ $P(r = 21) = .042$	0.1	361.18	348.48	363.26	353.22
	0.2	179.89	172.97	182.17	177.98
	0.3	119.34	114.32	121.86	119.62
	0.4	88.92	84.83	91.74	90.50
	0.5	70.35	66.95	73.72	73.08
	0.6	57.94	54.79	61.74	61.54
	0.7	48.56	45.77	53.21	53.36
	0.8	40.64	38.43	46.95	47.32
	0.9	31.59	31.44	42.18	42.82
	1.0	—	20.04	39.08	40.08
Fixed Sample Size		67.64[e]	67[f]	67.64[e]	68[g]

($\Delta^* = .2$, $P^* = .99$)

Type of Stopping Rule	$\bar{p}$	Approx. (R_I)	Exact (R_I)	Approx. (R_I')	Exact (R_I')
Inverse Sampling $P(r = 40) = 1$	0.1	744.39	725.54	747.32	732.50
	0.2	371.47	360.86	374.64	368.18
	0.3	247.00	239.04	250.48	246.84
	0.4	184.60	177.86	188.45	186.22
	0.5	146.94	140.86	151.29	149.94
	0.6	121.53	115.80	126.59	125.84
	0.7	102.38	97.34	109.02	108.74
	0.8	87.87	82.52	95.96	96.04
	0.9	72.86	68.56	85.97	86.42
	1.0	—	40.00	79.00	80.00
Fixed Sample Size		135.3	134	135.3	134

[a] Based on (5.4). [b] Based on the sum of $E\{N_i/R_I\}$ in (2.16). [c] Based on (5.3).
[d] Based on (4.9). [e,g] Based on Table I of [10]. [f] Based on Table 1 of [5].

$$(5.4)\qquad E\{N\mid R_I\} \sim \frac{(s-r)(k-1)}{q} + s = \frac{kr - \lambda(k-p)(r/q)^{\frac{1}{2}}}{p} - \left(\frac{k-p}{2q}\right)$$

where $\lambda = \lambda(k)$ is as in (5.3). Comparing the two leading terms in (5.3) and (5.4), we find for large r that $E\{N\mid R_I\} \leqq E\{N\mid R_I'\}$ for all p since $k \geqq 1 \geqq p/(1-pq)$. For $k=1$ we note that $\lambda = 0$ and (5.3) and (5.4) are equal, both having a slight negative bias. This negative bias appears to be the case for most values of k. On the other hand, if we omit the last (correction) term in (5.3) and (5.4), then the resulting approximation has a slight positive bias, i.e., it is conservative, for most values of k. For the accuracy of (5.4) we compare exact and approximate results in Table 2 and discuss some examples below.

For $k = 3$, $\Delta^* = .2$ and $P^* = .95$ we need an r-value of 29 for procedures R_I and R_I'. Using $r = 29$ the exact value of $E\{N\mid R_I\}$ for $p_1 = p_2 = p_3 = .9$ is 59.8 and the approximate value from (5.4) is 59.4. For $k = 2$, $\Delta^* = .2$ and $P^* = .95$ we need an r value of 21 (with randomization we take $r = 20$ with probability .958 and 21 with probability .042). For $r = 20$ (and 21) the maximum error in using (5.4) to approximate $E\{N\mid R_I\}$ over all $p_1 = p_2$ is roughly 6 percent.

We have not proved for each r that the maximum of $E\{N\}$ for fixed p_1 occurs when all the p_i $(i \geqq 2)$ are equal to p_1, but we note from (3.23) and (4.12) that this holds asymptotically $(r \to \infty)$ for both R_I and R_I'.

For the expected loss criterion with a common p and any r, we find that $E\{L\} = 0$ for both procedures. From (3.25) and (4.13) we note that the maximum for fixed p_1 may occur when the p_i $(i \geqq 2)$ are equal, but not equal to p_1.

In summary, the procedure R_I with PWC sampling is asymptotically $(r \to \infty)$ superior to R_I' with VT sampling throughout the parameter space with respect to both $E\{N\}$ and $E\{L\}$.

6. A dual procedure R_I^*. Another procedure R_I^* that is comparable with R_I and in some sense dual to it is defined by waiting until every population has exactly r failures under PWC sampling. The population with the larger (or largest) number of successes is declared to be the best. In case of ties we randomize between all contenders for first place. Since each population has exactly r failures at termination we can treat the populations separately and do not need the recursive-equation approach. The results are quite similar to those obtained above and it was therefore decided to include them.

Let Y_{p_i} $(i = 1, 2, \cdots, k)$ denote the random total number of observations required to obtain r failures from the population with success parameter p_i, where p_1 is the largest of the p_i and the rest are defined by the same cycle (starting with the best player A) as is used by the PWC sampling rule. Then for a population with arbitrary p

$$(6.1)\qquad P\{Y_p = y\} = q^r\binom{y-1}{r-1}p^{y-r} \qquad y = r, r+1, \cdots,$$

$$(6.2)\qquad P\{Y_p < y\} = q^r \textstyle\sum_{j=0}^{y-r-1} \binom{j+r-1}{j} p^j = I_q(r, y-r),$$

the mean $E\{Y_p\} = r/q$ and the variance $\sigma^2(Y_p) = rp/q^2$. Hence the probability of a correct selection (CS) is given by

$$P\{\text{CS} \mid R_I^*\} - Q = q_1^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} p_1^j \prod_{\alpha=2}^{k} I_{q_\alpha}(r, j) , \tag{6.3}$$

where Q is the contribution that arises from randomization when there are ties for first place. If we let T_{12} denote the probability that the two populations with parameters p_1 and p_2 (and only these two) tie for first place, etc., then

$$Q = \tfrac{1}{2}\{T_{1,2} + T_{1,3} + \cdots + T_{1,k}\} + \tfrac{1}{3}\{T_{1,2,3} + \cdots + T_{1,k-1,k}\} + \cdots + (1/k)T_{1,2,\cdots,k} , \tag{6.4}$$

where, for example,

$$T_{1,2} = q_1^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} p_1^j [I_{q_2}(r, j+1) - I_{q_2}(r, j)] \prod_{\alpha=3}^{k} I_{q_\alpha}(r, j) \tag{6.5}$$

and $T_{1,2,3}$ contains two such differences in square brackets, etc. We wish to show that all negative signs, as in (6.5), disappear when we multiply out all the square brackets that arise. Consider any term that contains a fixed subset of h functions $I_{q_\alpha}(r, j+1)$ with argument $j+1$. For any $h(0 \leqq h \leqq k)$ and any subset of size h, the final coefficient which we denote by W_h will be

$$\sum_{i=0}^{k-h-1} \frac{(-1)^{k-h-i-1}}{k-i} \binom{k-h-1}{i} \doteq \int_0^1 x^{k-1} \left(\frac{1}{x} - 1\right)^{k-h-1} dx = \frac{1}{k\binom{k-1}{h}} . \tag{6.6}$$

Hence, if we let S_h denote any fixed subset of size h, CS_h its complement, and $\mathscr{S}_h$ denote the $\binom{k-1}{h}$ possible subsets of size h, then

$$P\{\text{CS} \mid R_I^*\} = q_1^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} p_1^j \sum_{h=0}^{k-1} W_h \times \sum_{S_h \in \mathscr{S}_h} [\prod_{\alpha \in S_h} I_{q_\alpha}(r, j+1)][\prod_{\beta \in CS_h} I_{q_\beta}(r, j)] . \tag{6.7}$$

It follows that in the minimization subject to $q_i \geqq q_1 + \Delta^*$ we can set $q_i = q_1 + \Delta^* = \hat{q}$ (say) $(i = 2, 3, \cdots, k)$ and obtain

$$\min P\{\text{CS} \mid R_I^*\} = q_1^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} p_1^j \frac{[I_{\hat{q}}^k(r, j+1) - I_{\hat{q}}^k(r, j)]}{k[I_{\hat{q}}(r, j+1) - I_{\hat{q}}(r, j)]} . \tag{6.8}$$

The expected total sample size is easily seen to be

$$E\{N \mid R_I^*\} = r \sum_{i=1}^{k} q_i^{-1} . \tag{6.9}$$

To evaluate r so as to satisfy (1.1) with R_I replaced by R_I^*, we use the normal approximation as in (3.6) and obtain

$$\min P\{\text{CS} \mid R_I^*\} \sim \int_{-\infty}^{\infty} \Phi^{k-1}\left(\frac{x\rho^{\frac{1}{2}} + H}{(1-\rho)^{\frac{1}{2}}}\right) d\Phi(x) \equiv A_{k-1}(\rho, H) \tag{6.10}$$

where (similar to (3.8), but with p and q interchanged)

$$H = \frac{\Delta^* r^{\frac{1}{2}}}{(p_1\hat{q}^2 + p\hat{q}_1^2)^{\frac{1}{2}}} \quad \text{and} \quad \rho = \frac{p_1\hat{q}^2}{p_1\hat{q}^2 + p\hat{q}_1^2} . \tag{6.11}$$

An approximate minimization therefore leads (as at (3.11)) to

$$q_1 = \tfrac{2}{3} - \tfrac{1}{2}\Delta^* + \mathscr{O}\{(\Delta^*)^2\} ; \quad q_2 = q_3 = \cdots = q_k = \tfrac{2}{3} + \tfrac{1}{2}\Delta^* + \mathscr{O}\{(\Delta^*)^2\}(=\hat{q}) . \tag{6.12}$$

Putting this in the second expression in (6.11) gives

$$\rho = \tfrac{1}{2} + \tfrac{3}{2}\Delta^* + \mathcal{O}\{(\Delta^*)^2\}\,; \tag{6.13}$$

$\rho = \frac{1}{2}$ now will provide a lower bound for small Δ^*. [A correction term as in the original technical report may be desirable but this is omitted for brevity.] The first approximation for r is the solution of

$$A_{k-1}(\tfrac{1}{2}, H) = P^* \tag{6.14}$$

and if $\lambda = \lambda(P^*)$ is the table value of H that satisfies (6.14), then

$$r \mathrel{\dot\sim} \frac{8}{27}\left(\frac{\lambda}{\Delta^*}\right) \tag{6.15}$$

is the first approximation for r.

Hence $E\{N \mid R_I{}^*\}$ is given by (6.9) with r replaced by the right side of (6.15). Comparing with (3.23) we find that procedure R_I is preferred when

$$q_1/p_1 < 1 \qquad \text{or} \qquad p_1 > \tfrac{1}{2} \tag{6.16}$$

and procedure $R_I{}^*$ is preferred when $p_1 < \frac{1}{2}$.

Two other procedures $\hat{R}_I$ and $\hat{R}_I{}'$ (suggested by the referee) are to reorder the k populations after each complete cycle, consisting of k failures, according to the total number of successes obtained up to that point. Ties are settled by randomization. $\hat{R}_I$ uses the play-the-winner scheme and we refer to it as the PWO (play-the-winner, ordered variation) sampling rule; $\hat{R}_I{}'$ uses the VT-sampling rule. Since we do not stop in the middle of a vector, it is clear that this ordering does not affect the PCS for procedure $\hat{R}_I{}'$ with VT-sampling. It is shown (in Remark 6.2 below) that the PCS for procedure $\hat{R}_I$ is also the same as for procedure R_I. It follows that the LF configuration is the same and the same value of r is required for procedures R_I, $R_I{}'$, $\hat{R}_I$ and $\hat{R}_I{}'$ to satisfy (1.1).

For procedure $\hat{R}_I{}'$ the maximum saving in $E\{N\}$ incurred over procedure $R_I{}'$ is at most $(k-1)/2$, since, when a correct selection is made, the best population has probability $\frac{1}{2}$ of being before any other (a saving of one for each of the $k-1$ populations) and, when a wrong selection is made, the expected saving is no greater than $(k-1)/2$.

For procedure $\hat{R}_I$ the saving in $E\{N\}$ incurred in comparison with R_I is more than $(k-1)/2$ (for P^* close to 1) and is at least $(P^*/2)\sum_{i=2}^{k} 1/q_i$ since each population has probability $\frac{1}{2}$ of getting a turn before the best population and $1/q_i$ is the expected number of trials up to and including the first failure ($i = 2, 3, \cdots, k$). When a wrong selection is made the saving is disregarded since $P^* \to 1$. Hence for $P^* \to 1$ an upper bound on the saving is $\frac{1}{2}\sum_{i=2}^{k}(1/q_i)$. For fixed k and large r, this represents a fixed constant that does not enter into asymptotic ($r \to \infty$) considerations and, if the $q_i \geqq q_1$ are not near zero, we can treat this saving as negligible. A numerical investigation of this saving has not been carried out.

REMARK 6.1. The problem of estimating the maximum of $U_1 - U_2$ (Remark 2.2) can also be handled with the help of Lemma 2 when P^* is close to 1 and

Δ^* is not too large (i.e., small enough so we can disregard powers of Δ^*). We start with the asymptotic normal approximation to the probability that the best population π_1 ties with at least 1 other population (note that $(U_1 - U_2)/2$ is the probability of both a correct selection and a tie involving the best population). Letting $\eta = p_2/(2(rq_2)^{\frac{1}{2}})$ and replacing X_{p_1} in (3.2) first by $X_{p_1} - \frac{1}{2}$ for U_1 and then by $X_{p_1} + \frac{1}{2}$ for U_2, we obtain, as in Section 3 for $p_2 = p_3 = \cdots = p_k$,

$$(6.17) \qquad U_1 - U_2 \sim \int_{-\infty}^{\infty} \left[\Phi^{k-1}\left(\frac{x\rho^{\frac{1}{2}} + H}{(1-\rho)^{\frac{1}{2}}} + \eta\right) - \Phi^{k-1}\left(\frac{x\rho^{\frac{1}{2}} + H}{(1-\rho)^{\frac{1}{2}}} - \eta\right)\right] d\Phi(x) ,$$

where ρ and H are given by (3.8). Since r is large and η is small for P^* close to 1,

$$(6.18) \qquad U_1 - U_2 \sim \frac{(k-1)p_2}{(rq_2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \Phi^{k-2}\left(\frac{x\rho^{\frac{1}{2}} + H}{(1-\rho)^{\frac{1}{2}}}\right) \varphi\left(\frac{x\rho^{\frac{1}{2}} + H}{(1-\rho)^{\frac{1}{2}}}\right) \varphi(x)\, dx$$

$$= (k-1)p_2\left(\frac{1-\rho}{rq_2}\right)^{\frac{1}{2}} \varphi(H) \int_{-\infty}^{\infty} \Phi^{k-2}(y\rho^{\frac{1}{2}} + H(1-\rho)^{\frac{1}{2}})\, d\Phi(y) ,$$

by Lemma 2 of original technical report. The last integral $A_{k-2}(\rho/(1+\rho), H[(1-\rho)/(1+\rho)]^{\frac{1}{2}})$ in (6.18) is a slowly varying function of ρ and for $k = 2$ equals 1 and for $k = 3$ equals $\Phi(H)$, independent of ρ. If we now set $p_1 = \frac{2}{3} + \frac{1}{2}\Delta^* + \varepsilon$, $p_2 = \frac{2}{3} - \frac{1}{2}\Delta^* + \varepsilon$ and maximize $p_2\varphi(H)((1-\rho)/q_2)^{\frac{1}{2}}$ as a function of ε, we find $\varepsilon = 4/(9\lambda^2 - 15)$ and we obtain as a first approximation

$$(6.19) \qquad \max(U_1 - U_2) \sim 6^{\frac{1}{2}}\varphi(\lambda)\left(\frac{\lambda^2 - 1}{3\lambda^2 - 5}\right) \frac{(k-1)}{r^{\frac{1}{2}}} A_{k-2}\left(\frac{\rho}{1+\rho}, H\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}}\right)$$

where ρ and H are given by (3.8) with the above values of p_1, p_2, and λ is the root in H_0 of (3.12) as in Section 3. However more conservative results are obtained by trial and error in (6.18) using ε above as a starting value. Note that if we divide Δ^* by 2 and keep the same P^* then by (3.13) r is multiplied by 4 and by (6.19) the max $(U_1 - U_2)$ is cut in half.

ILLUSTRATION. For $k = 2$, $P^* = .99$, $\Delta^* = .2$ we find that $r = 40$ and $\lambda = H_0 = 2.326$. Then $\varepsilon = .118$ and $p_1 = \frac{2}{3} + \frac{1}{2}\Delta^* + \varepsilon = .89$ and max $(U_1 - U_2) = .004$ by (6.19). Trial and error in (6.18) shows a maximum of .005 occurring at $p_1 = .83$. More exact methods based on Remark 2.2 show a maximum of .004 occurring at $p_1 = .84$. Hence the U-values vary between $P^* \pm \frac{1}{2}(.004)$, i.e., between .988 and .992. Note that the value $(1 - \Delta^*)^r$ at $p_1 = 1$ of $U_1 - U_2$ is sometimes a good estimate of max $(U_1 - U_2)$ but for large r, as in this example, it is very poor.

REMARK 6.2. Following is a proof that the PWO and the PWC sampling rules have the same PCS when the inverse sampling termination rule (wait for r successes from any one population) is used. Define the 1st full cycle to run up to and including the kth failure; the 2nd full cycle then starts and runs up to and including the next k failures, etc. Assume the last cycle is completed to form a full cycle. Then the procedures R_I and $\hat{R}_I$ can differ in their final selection only when m populations ($m \geqq 2$) reach r successes in the same full cycle. These m populations are then tied in both successes and failures. Conditional on this equality, we get a CS under procedure R_I if A_1 is first among these m, i.e., with probability $1/m$. Similarly, under procedure $\hat{R}_I$ we condition on the fact that m populations are tied in both successes and failures and disregard the other populations. In the initial randomization A_1 has probability $1/m$ of being ahead. Subsequent randomizations do not change this. For example, if $m_1 \leqq m$ of these populations, including A_1, are subsequently included in a randomization then the resulting overall probability that A_1 is ahead is $(m_1/m)(1/m_1) = 1/m$ since A_1 can change places with any one of these m_1 populations and still have probability $1/m_1$ of being ahead in the second randomization. It follows that the PCS values for the PWC and the PWO sampling rules are the same.

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