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LOUD SHOT NOISE

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We consider problems involving large or loud values of the shot noise process $X(t) := \sum_{i:\, \tau_i \le t} h(t - \tau_i)$, $t \ge 0$, where $h: [0,\infty) \to [0,\infty)$ is nonincreasing and $(\tau_i,\ i \ge 0)$ is the sequence of renewal times of a renewal process. Results are obtained by extending the renewal sequence to all $i \in \mathbb{Z}$ and considering the stationary sequence (ξ_n) given by $\xi_n = \sum_{i \le n} h(\tau_n - \tau_i)$. We show that ξ_n has a thin tail in the sense that under broad circumstances $\Pr\{\xi_n > x + \delta | \xi_n > x\} \to 0$ as $x \to \infty$, where $\delta > 0$. We also show that $\Pr\{\max(\xi_1, \dots, \xi_n) \le u_n\} - (\Pr\{\xi_0 \le u_n\})^n \to 0$ for real sequences (u_n) for which $\limsup n \Pr\{\xi_0 > u_n\} < \infty$.

1. Introduction. Let $\{X(t),\ t \ge 0\}$ be a shot noise process of the form

$$X(t) = \sum_{i:\, \tau_i \le t} h(t - \tau_i), \tag{1.1}$$

where the response function $h: [0,\infty) \to [0,\infty)$ is nonincreasing, $h(0+) > 0$ and $\{\tau_i,\ i \ge 0\}$ are the renewal times of an ordinary renewal process with $\tau_0 \equiv 0$, $\tau_n = \sum_1^n \eta_i$ for $n \ge 1$. The idea is that a "shot" occurs at each renewal time and the resulting noise level s units of time after a shot is $h(s)$. Then $X(t)$ is the total noise level at time t, obtained by adding the effects of all shots prior to time t.

Many papers have been published on shot noise, in both mathematics and physical science journals. A large selection of these can be traced via the references given by Hsing and Teugels (1989) and Vervaat (1979). The exact mathematical models vary considerably. In particular, some authors assume $\{\tau_i\}$ is a Poisson process and some assume h is random and chosen independently for each τ_i. Many authors give particular consideration to the case where the shot noise decreases exponentially at a deterministic rate, since this case is both natural and relatively tractable.

We investigate properties involving large or "loud" values of the shot noise process X. Since X clearly jumps upward at each τ_i and decreases between the τ_i's, it is natural to consider the embedded sequence $Y_n = X(\tau_n)$ and a corresponding stationary sequence $\{\xi_n,\ n \ge 0\}$ defined by

$$\xi_n = \sum_{m \le n} h(\tau_n - \tau_m), \tag{1.2}$$

where for $m < 0$, $\tau_m = -\sum_{m+1}^{0} \eta_i$, $\{\eta_i,\ -\infty < i < \infty\}$ being a sequence of positive i.i.d. r.v.'s.

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The first topic we consider is the tail behaviour of the (stationary) distribution of ξ_n; in particular, under what circumstances and for what values of δ does

(1.3) $$\lim_{x \to \infty} \Pr\{\xi_0 > x + \delta | \xi_0 > x\} = 0$$

hold? Rather surprisingly (1.3), which quantifies a sense in which the tail of the distribution of ξ_0 decreases at a faster than exponential rate, holds under very weak assumptions. Indeed, making only the natural assumptions that $\Pr\{\eta_i \le t\} > 0$ for all $t > 0$ and $\Pr\{\eta_i \le 0\} = 0$, we show in Section 2 that if $\Delta := \inf\{t\colon h(t) = 0\} < \infty$, then (1.3) holds for every $\delta > h(\Delta -)$, and if $\Delta = \infty$, then either (1.3) holds for all $\delta > 0$ or else $E\xi_0 = +\infty$. A sufficient condition for $E\xi_0 < \infty$ and hence for (1.3) when $\Delta = \infty$ is that $\int_0^\infty h(t)\,dt < \infty$; this is also necessary if $\mu := E\eta_1 < \infty$.

The second topic we consider is the asymptotic behaviour of $M_n := \max_{0 \le i \le n} \xi_i$. We show that if $\Delta < \infty$ or if $h(x) \equiv e^{-x}$, then

(1.4) $$\Pr\{M_n \le u_n\} - \{G(u_n)\}^n \to 0 \quad \text{as } n \to \infty,$$

where $G(t) := \Pr\{\xi_0 \le t\}$ and $\{u_n\}$ is any sequence of real numbers for which $\limsup n \Pr\{\xi_0 > u_n\} < \infty$. Formula (1.4) is an assertion that M_n mimics the behaviour of the maximum of a sequence of i.i.d. r.v.'s with the same distribution function as ξ_0 and is often expressed by stating that $\{\xi_n\}$ has extremal index 1.

A result similar to (1.4) has been obtained by Hsing and Teugels (1989) with the additional assumptions that $\Delta < \infty$, that $\mu < \infty$, that (1.3) holds for $\delta = h(0)$ and that h satisfies an additional technical condition, which implies in particular that h is strictly decreasing. They did not prove (1.3). Our method of proof is completely different from theirs. Although we do not give any details, standard techniques of extreme value theory can be used to extend (1.4) to various extremal results for X, for example, the convergence of the exceedance point process of $\{Y(n), n \ge 0\}$.

The question as to whether or not (1.4) holds for h with unbounded support seems to be a difficult one: We show that it does hold in the important special case $h(x) \equiv e^{-x}$, without any extra assumptions on the renewal process $\{\tau_i\}$, but our proof leans heavily on the fact that, in this particular case, $\{\xi_n, n \ge 0\}$ has the Markov property. We indicate how our methods could be extended to the case that h is "close to exponential," but for general h, the question remains open.

2. Tail behaviour of ξ_0. Since studying the distribution of ξ_0 involves only the renewal process $\{\tau_{-i}, i \ge 0\}$, it is convenient to reverse the direction of time. At the same time we make a slight generalization and, throughout Section 2, study the distribution of

(2.1) $$\xi := \sum_0^\infty h(t_m),$$

where $\{t_m, m \geq 0\}$ is a modified renewal process on $[0, \infty)$, i.e., $t_0 \equiv 0$, $t_m = \Sigma_1^m \theta_i$ for $m \geq 1$, where $\theta_1, \theta_2, \ldots$ are independent positive random variables, $\theta_2, \theta_3, \ldots$ have common distribution function F_2 and θ_1 has distribution function F_1. We assume throughout that for $i = 1, 2$,

$$F_i(0) = 0, \qquad F_i(t) > 0 \quad \forall\, t > 0. \tag{2.2}$$

It turns out that (1.3) can be established in the case that h has unbounded support by an approximation argument based on the corresponding result for h with bounded support. Until further notice we will therefore assume that

$$h \text{ is nonincreasing}, \qquad h(0) = 1, \qquad h(1+) = 0 \quad \text{and} \quad h(t) > 0 \quad \text{for } 0 \leq t < 1. \tag{2.3}$$

We proceed via a sequence of lemmas, in which we use the following notation. For $x > 0$ let $N(x) = \min\{n: \Sigma_0^n h(t_m) > x\}$, with $\min \varnothing = \infty$, so that $\{\xi > x\} = \{N(x) < \infty\} = \bigcup_0^\infty \{N(x) = n\}$, and write $S(x) = t_{N(x)}$ when $N(x) < \infty$, $S(x) = \infty$ when $N(x) = \infty$. Note that, for $y \leq 1$,

$$S(x) \leq y \Leftrightarrow \sum_0^{M(y)} h(t_m) > x, \tag{2.4}$$

where $M(y) := \max\{n: t_n \leq y\}$. Our first lemma exploits the monotonicity of h to show that, conditioned on $S(x) \leq y$, θ_2 is stochastically smaller than each θ_i, for $i > 2$. In the proof and throughout the paper, indicator functions are denoted by I.

LEMMA 1. *For $t > 0$, $x > 0$, $0 < y \leq 1$ and $i > 2$,*

$$\Pr\{\theta_i > t | S(x) \leq y\} \geq \Pr\{\theta_2 > t | S(x) \leq y\}. \tag{2.5}$$

PROOF. We show first that for each fixed $i > 2$ and $0 < s_2 < s_i$,

$$\Pr\{S(x) \leq y | \theta_2 = s_2, \theta_i = s_i\} \geq \Pr\{S(x) \leq y | \theta_i = s_2, \theta_2 = s_i\}. \tag{2.6}$$

To see this, take arbitrary positive $s_1, s_3, s_4, \ldots, s_{i-1}$ and write $\sigma_0 = \sigma_0' = 0$, $\sigma_m = \Sigma_1^m s_j$ and $\sigma_m' = \Sigma_1^m s_j'$ for $1 \leq m \leq i$, where $s_1' = s_1$, $s_2' = s_i$, $s_j' = s_j$ for $2 < j < i$ and $s_i' = s_2$. Note that $\sigma_m \leq \sigma_m'$ for $m \leq i$. Suppose $\sigma_i \leq y$ so that if $t_i = \sigma_i$, then $M(y) \geq i$. Then

$$\begin{aligned} &\Pr\{S(x) \leq y | \theta_j = s_j, 1 \leq j \leq i\} \\ &\quad = \Pr\left\{ \sum_{i+1}^{M(y)} h(t_m) > x - \sum_0^i h(\sigma_m) | \theta_j = s_j, 1 \leq j \leq i \right\} \\ &\quad \geq \Pr\left\{ \sum_{i+1}^{M(y)} h(t_m) > x - \sum_0^i h(\sigma_m') | \theta_j = s_j', 1 \leq j \leq i \right\} \\ &\quad = \Pr\{S(x) \leq y | \theta_j = s_j', 1 \leq j \leq i\}, \end{aligned} \tag{2.7}$$

where we have used the fact that, given $t_i = \sigma_i$, where $\sigma_i \leq y$, $M(y)$ and

$t_{i+1}, t_{i+2}, \dots$ are independent of θ_j, $j \le i$. Also, when $\sigma_i > y$, we have

$$\begin{aligned}\Pr\{S(x) \le y | \theta_j = s_j, 1 \le j \le i\} &= I\Big\{\sum_{m:\, \sigma_m \le y} h(\sigma_m) > x\Big\} \\ &\ge I\Big\{\sum_{m:\, \sigma'_m \le y} h(\sigma'_m) > x\Big\} \qquad (2.8) \\ &= \Pr\{S(x) \le y | \theta_j = s'_j, 1 \le j \le i\}.\end{aligned}$$

We now obtain (2.6) by integrating out the dependence on $s_1, s_3, s_4, \dots, s_{i-1}$ in (2.7) and (2.8). From (2.6) we have

$$\begin{aligned}&\Pr\{S(x) \le y, \theta_i > t\} - \Pr\{S(x) \le y, \theta_2 > t\} \\ &\quad = \Pr\{S(x) \le y, \theta_2 \le t, \theta_i > t\} - \Pr\{S(x) \le y, \theta_2 > t, \theta_i \le t\} \\ &\quad = \int_{v>t}\int_{u \le t} [\Pr\{S(x) \le y | \theta_2 = u, \theta_i = v\} \\ &\qquad\qquad - \Pr\{S(x) \le y | \theta_2 = v, \theta_i = u\}]\, dF_2(u)\, dF_2(v) \\ &\quad \ge 0,\end{aligned}$$

and (2.5) is immediate. □

We can now conclude that θ_2 is likely to be small when $S(x) \le y$:

LEMMA 2. *For each* $0 < y \le 1$,

$$\Pr\{\theta_2 \le 2x^{-1} | S(x) \le y\} \ge \tfrac{1}{4} \quad \textit{for } x \ge 4. \qquad (2.9)$$

PROOF. From $h(t) \le 1$ it follows that $N(x) \ge [x]$, where $[x]$ is the integer part of x. Thus from $S(x) \le y \le 1$ it follows that at most $[\frac{1}{2}x]$ of $\theta_1, \theta_2, \dots, \theta_{[x]}$ exceed $2x^{-1}$ and hence at least $[\frac{1}{2}x]$ of them are at most $2x^{-1}$. Therefore, by Lemma 1,

$$\begin{aligned}[\tfrac{1}{2}x] - 1 &\le E\{\#\{i: 2 \le i \le [x] \text{ and } \theta_i \le 2x^{-1}\} | S(x) \le y\} \\ &= \sum_{2}^{[x]} \Pr\{\theta_i \le 2x^{-1} | S(x) \le y\} \le [x-1] \Pr\{\theta_2 \le 2x^{-1} | S(x) \le y\},\end{aligned}$$

which establishes (2.9) since $[\frac{1}{2}x] - 1 \ge \frac{1}{4}[x-1]$ for $x \ge 4$. □

The next lemma embodies the intuitively appealing idea that for large x, if $S(x) \le y$, then with high probability $S(x)$ is close to y. This turns out to be a key ingredient in the proof of our main results.

LEMMA 3. *For each fixed* $0 < \gamma < 1$,

$$\Pr\{S(x) \le y - \gamma | S(x) \le y\} \to 0 \quad \textit{as } x \to \infty \textit{ uniformly over } y \in (0,1]. \qquad (2.10)$$

PROOF. We remark first that, for $x > 2$,

$$\Pr\{S(x) \le y - \gamma | \theta_2 = z\} \le \Pr\{S(x-1) \le y - \gamma - z\} \qquad (2.11)$$

and hence, by Lemma 2, for $x \geq 4$,

$$\begin{aligned}(2.12)\qquad \tfrac{1}{4}\Pr\{S(x) \leq y - \gamma\} &\leq \Pr\{S(x) \leq y - \gamma, \theta_2 \leq 2x^{-1}\} \\ &\leq \Pr\{S(x-1) \leq y - \gamma\}\Pr\{\theta_2 \leq 2x^{-1}\}.\end{aligned}$$

Now note that if k is chosen (independently of y) with $k^{-1} < h(1 - \gamma/2)$, then

$$\Pr\{t_k < \gamma/2\} > 0$$

by (2.2) and

$$\begin{aligned}&\Pr\{S(x-1) \leq y - \gamma\}\Pr\{t_k < \gamma/2\} \\ &\quad = \Pr\{S(x-1) \leq y - \gamma, t_{N(x-1)+k} - t_{N(x-1)} < \gamma/2\} \\ &\quad \leq \Pr\{S(x) \leq y - \gamma/2\} \leq \Pr\{S(x) \leq y\}.\end{aligned}$$

Together with (2.12) this establishes (2.10), since $\Pr\{\theta_2 \leq 2x^{-1}\} \to 0$ as $x \to \infty$. □

The next result extends the idea of Lemma 3 to show that if $S(x) \leq y$ for large x, then the distances between successive renewal points in $[0, y]$ tend to be small.

LEMMA 4. *Let* $B(x) = \max\{\theta_i\colon 1 \leq i \leq N(x)\}$ *when* $N(x) < \infty$, $B(x) = 0$ *when* $N(x) = \infty$. *Then for each fixed* $t > 0$,

$$(2.13)\qquad \Pr\{B(x) > t | S(x) \leq y\} \to 0 \quad \text{as } x \to \infty, \text{ uniformly over } y \in (0, 1].$$

PROOF. Since $B(x) \leq S(x)$, we may assume $y > t$. Interchanging the order of summation and noting that $\Pr\{N(x) \geq i, S(x) \leq y | \theta_i = z\}$ is decreasing in z, we get

$$\begin{aligned}\Pr\{B(x) > t, S(x) \leq y\} &= \sum_{n=1}^{\infty} \Pr\{N(x) = n, S(x) \leq y, B(x) > t\} \\ &\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} \Pr\{N(x) = n, S(x) \leq y, \theta_i > t\} \\ (2.14)\qquad &= \sum_{i=1}^{\infty} \Pr\{N(x) \geq i, S(x) \leq y, \theta_i > t\} \\ &\leq \sum_{i=1}^{\infty} \{1 - F_i(t)\}\Pr\{N(x) \geq i, S(x) \leq y | \theta_i = t\},\end{aligned}$$

where $F_i(t) = F_2(t)$ for $i \geq 2$. Now if $N(x) \geq i$ and if θ_i is decreased, $S(x)$ decreases by at least the same amount and the inequality $N(x) \geq i$ remains valid, so $w_i(z) := \Pr\{S(x) \leq y - z, i \leq N(x) | \theta_i = t - z\}$ is increasing in $z < t$. Thus for $0 < u < v < t$,

$$\begin{aligned}&\{F_i(v) - F_i(u)\}w_i(0) \\ (2.15)\qquad &\quad \leq \int_u^v w_i(t - z)\, dF_i(z) \\ &\quad \leq \Pr\{N(x) \geq i, S(x) \leq y + v - t, u < \theta_i \leq v\}.\end{aligned}$$

Now fix $0 < u < v < t$ with $F_i(v) > F_i(u)$ for $i = 1, 2$, write

$$c = \max_i \{(1 - F_i(t))(F_i(v) - F_i(u))^{-1}\}$$

and note that

$$S(x) \le y + v - t \Rightarrow N(x) < \infty \Rightarrow \sum_1^{N(x)} \theta_i \le 1 \Rightarrow \#\{i \le N(x) : \theta_i \ge u\} \le u^{-1}.$$

Then (2.14) and (2.15) give

$$\begin{aligned}
\Pr\{B(x) > t, S(x) \le y\} &\le \sum_1^\infty \{1 - F_i(t)\} w_i(0) \\
&\le c \sum_1^\infty \Pr\{N(x) \ge i, S(x) \le y + v - t, u \le \theta_i \le v\} \\
&\le c \Pr\{S(x) \le y + v - t\} \\
&\quad \times \sum_1^\infty \Pr\{N(x) \ge i, \theta_i \ge u | S(x) \le y + v - t\} \\
&= c \Pr\{S(x) \le y + v - t\} \\
&\quad \times E\{\#\{i \le N(x) : \theta_i \ge u\} | S(x) \le y + v - t\} \\
&\le c u^{-1} \Pr\{S(x) \le y + v - t\},
\end{aligned}$$

so that (2.13) follows from Lemma 3. □

Returning to the situation of Section 1, let us write R for the renewal function of $\{\tau_n, n \ge 0\}$, viz. $R(t) = E\{\max\{n : \tau_n \le t\}\}$, so that $E\{\xi_0\} = \int_0^\Delta h(t)\, dR(t)$ [recall that $\Delta := \inf\{t : h(t) = 0\}$]. Our result on the marginal distribution of ξ_0 is:

THEOREM 1. *Suppose that* $0 < h(0) < \infty$, $h(0+) > 0$, *h is nonincreasing and the distribution of the* η_i *satisfies* (2.2). *If* $\Delta < \infty$, *then*

$$\lim_{x \to \infty} \Pr\{\xi_0 > x + \delta | \xi_0 > x\} = 0 \tag{2.16}$$

for every $\delta > h(\Delta-)$ *and, if h is constant on some interval* $(\Delta - \varepsilon, \Delta)$, *for* $\delta = h(\Delta-)$. *If* $\Delta = \infty$ *and*

$$E\{\xi_0\} < \infty, \tag{2.17}$$

then (2.16) *holds for every* $\delta > 0$. *If* $\Delta = \infty$ *and* $E\{\xi_0\} = \infty$, (2.16) *holds for no* $\delta > 0$.

PROOF. When $\Delta < \infty$ it is easy to see [by considering $\hat{h}(x) = h(\Delta^{-1}x)\{h(0)\}^{-1}$, $\hat{\tau}_m = \Delta\tau_m$] that there is no loss of generality in taking $h(0) = 1$ and $\Delta = 1$. Then, setting $-\tau_{-m} = t_m$, we have the situation of the early part of Section 2, with $F_1(t) = F_2(t) = F(t) = \Pr\{\eta_i \le t\}$. For δ as indicated, we have $D := \sup\{y\colon h(y) > \delta\} < 1$. Thus

$$\Pr\{\xi_0 > x + \delta, t_{N(x)+1} > 1 | \xi_0 > x\} \le \Pr\{h(S(x)) > \delta | \xi_0 > x\}$$
$$\le \Pr\{S(x) \le D | \xi_0 > x\} \to 0 \quad \text{as } x \to \infty,$$

by Lemma 3. [Recall that $\xi_0 > x$ iff $S(x) \le 1$.] But also, when $x > 1$,

$$\Pr\{\xi_0 > x + \delta, t_{N(x)+1} \le 1\} \le \Pr\{t_{N(x)+1} \le 1\}$$
$$\le \int_0^1 F(1-y)\Pr\{S(x) \in dy\}$$
$$= \int_0^1 \Pr\{S(x) \le 1 - y\}\, dF(y),$$

so it follows from Lemma 3 and dominated convergence that

$$\Pr\{\xi_0 > x + \delta, t_{N(x)+1} \le 1 | \xi_0 > x\} \to 0 \quad \text{as } x \to \infty.$$

When $\Delta = \infty$, it is obvious that if (2.16) holds for some $\delta > 0$, then (2.17) holds. To see the reverse implication assume (2.17), write $Z(t) = \sum_{m \ge 0} h(t + t_m)$ and note that $E(Z(t)) = \int_0^\infty h(t+s)\, dR(s) \to 0$ as $t \to \infty$ and hence

(2.18) $\qquad \Pr\{Z(t) \ge \delta\} \to 0 \quad \text{as } t \to \infty$ for each fixed $\delta > 0$.

Conditioning on the value of $S(x)$ gives for any $K > 0$,

$$\Pr\{\xi_0 > x + \delta | \xi_0 > x\} \le \int_0^\infty \Pr\Big\{ \sum_{m \ge 0} h\big(t_{N(x)+m}\big) > \delta, S(x) \in dt | \xi_0 > x\Big\}$$
$$\le \int_0^\infty \Pr\{Z(t) \ge \delta\}\Pr\{S(x) \in dt | \xi_0 > x\}$$
$$\le \Pr\{S(x) \le K | \xi_0 > x\} + \Pr\{Z(K) \ge \delta\}.$$

In view of (2.18), (2.16) will follow if we can show that for each fixed $K > 0$,

(2.19) $\qquad \Pr\{S(x) \le K | \xi_0 > x\} \to 0 \quad \text{as } x \to \infty.$

But another scaling argument shows that if (2.19) holds for one $K > 0$, then it holds for any $K > 0$. We therefore establish (2.19) with $K = \frac{1}{2}$. To this end we observe that $S(x) \le \frac{1}{2} \Leftrightarrow \tilde{S}(x) \le \frac{1}{2}$, where $\tilde{S}$ refers to the same renewal process but with the truncated response function $\tilde{h}(x) := I\{0 \le x < 1\}h(x)$. Since (2.3) holds for $\tilde{h}$, Lemma 3 gives $\Pr\{\tilde{S}(x) \le \frac{1}{2}\} = o\{\Pr\{\tilde{\xi}_0 > x\}\} = o\{\Pr\{\xi_0 > x\}\}$, which establishes (2.19) with $K = \frac{1}{2}$, and hence yields (2.16). □

REMARK 1. We do not know when (2.16) holds for $\delta \le h(\Delta -)$. In the example, $h(x) = I\{0 \le x \le 1\}$, $h(\Delta -) = 1$ and (2.16) holds iff $\delta \ge 1$.

REMARK 2. When $\Delta = \infty$ and $\mu := E(\eta_i) < \infty$, an integration by parts and the fact that $R(t) \sim \mu^{-1}t$ as $t \to \infty$ show that

$$E\{\xi_0\} = \int_0^\infty h(t)\, dR(t) = \infty \Leftrightarrow \int_0^\infty h(t)\, dt = \infty \Leftrightarrow \sum_0^\infty h(n\varepsilon) = \infty$$

for every $\varepsilon > 0$.

But by the strong law $\exists\ 0 < \varepsilon(\omega) < \infty$ with $t_n(\omega) \le n\varepsilon(\omega)$ for all n and hence

$$\xi_0(\omega) = \sum_0^\infty h(t_n(\omega)) \ge \sum_0^\infty h(n\varepsilon(\omega)) = \infty \quad \text{a.s.}$$

whenever $E(\xi_0) = \infty$.

REMARK 3. When $\Delta = \infty$ and $\mu = \infty$, a truncation argument shows that $\int_0^\infty h(t)\,dt < \infty$ is a sufficient condition for (2.17) and hence (2.16) to hold. A result in Erickson (1973, Lemma 1) shows that (2.17) is equivalent to

$$\int_0^\infty t\{\mu(t)\}^{-1} d(1 - h(t)) < \infty,$$

where $\mu(t) = \int_0^t \{1 - F(x)\}\,dx$. In particular, if $1 - F(t) \sim ct^{-\alpha}$ as $t \to \infty$ and $0 < \alpha < 1$, then (2.17) is equivalent to $\int_0^\infty t^{\alpha-1} h(t)\,dt < \infty$.

REMARK 4. We have not been able to establish whether or not $E\{\xi_0\} = \infty \Leftrightarrow \Pr\{\xi_0 = \infty\} = 1$ when $\mu = \infty$.

REMARK 5. Consider the situation where μ is finite, $\{\tau_i', -\infty < i < \infty\}$ is a *stationary* renewal process with interpoint distribution F and

$$X'(t) = \sum_{i:\, \tau_i' \le t} h(t - \tau_i'). \tag{2.20}$$

Then $\xi_n' := X'(\tau_n')$ is a stationary sequence with the same marginal distribution as ξ_0 and $\{X'(t), -\infty < t < \infty\}$ is a stationary process which has a different marginal distribution. The point of allowing $\{t_n, n \ge 0\}$ to be a modified renewal process in the early part of this section is revealed in the following theorem.

THEOREM 2. *If $\mu < \infty$, the conclusions of Theorem 1 hold with ξ_0 replaced by $X'(0)$.*

PROOF. Write $\xi' = X'(0) + h(0)$. Then it is clear from (2.20) that ξ' has a representation of the form (2.1) with $F_2 = F$ and $F_1 = \overline{F}$, where $\overline{F}$ is the stationary distribution of the backward recurrence time, viz. $\overline{F}(t) = \mu^{-1}\int_0^t (1 - F(x))\,dx$. Since we are assuming that (2.2) holds for F, it also holds for F_1 and F_2, so we can repeat the proof of Theorem 1 with ξ_0 replaced by ξ' and some very minor changes. Of course (2.16) holds for $X'(0)$ iff it holds for ξ', and similarly for the other conditions. □

REMARK 6. In the original situation of Section 1, where $\{\tau_i, i \ge 0\}$ is an ordinary renewal process and (1.1) holds, viz.

$$X(t) = \sum_{i:\, \tau_i \le t} h(t - \tau_i),$$

it might be thought that if $\mu < \infty$, then for fixed t, $X(t)$ would have a representation of the form (2.1) with $F_2 = F$ and $F_1 = F^{(t)}$, where $F^{(t)}$ stands for the (nonstationary) distribution of the backward recurrence time and with h replaced by $h^{(t)}(s) := h(s)I\{0 \le s \le t\}$. However, this overlooks the fact that $\{\tau_i,\ i \ge 0\}$ has a renewal point at zero. For the sequence $\{t_n,\ n \ge 0\}$ this translates into a point at t, so in fact

$$\Pr\{X(t) + h(0) > x\} = \Pr\left\{ \sum_0^\infty h^{(t)}(t_n) > x | t_m = t \text{ for some } 1 \le m < \infty \right\}.$$

Nevertheless, our methods can be extended to establish an appropriate conditional version of Lemma 3 and hence to show that Theorem 1 also holds with ξ_0 replaced by $X(t)$ for fixed $t > 0$.

3. Extreme value theory. Suppose now that $\{\tau_i,\ -\infty < i < \infty\}$ is a two-sided renewal process with $\tau_0 \equiv 0$, that

$$\xi_n = \sum_{i \le n} h(\tau_n - \tau_i),$$
$$M_n = \max_{0 \le m \le n} \{\xi_m\},$$

and write G for the distribution function of ξ_0. Our main result on the asymptotic behaviour of M_n is given by the following theorem.

THEOREM 3. *Suppose that h satisfies (2.3), F satisfies (2.2), k_n is any sequence of integers which increase to ∞ and u_n is any sequence of real numbers satisfying*

$$\limsup_{n \to \infty} k_n\{1 - G(u_n)\} < \infty. \tag{3.1}$$

Then

$$\Pr\{M_{k_n} \le u_n\} - \{G(u_n)\}^{k_n} \to 0 \quad \textit{as } n \to \infty. \tag{3.2}$$

PROOF. As pointed out in O'Brien (1987, Corollary 2.3), with no loss of generality we can and do take $k_n = n$. Also Lemma 3.1 of Hsing and Teugels (1989), whose proof uses only the fact that h has bounded support, shows that $\{\xi_n,\ n \ge 0\}$ is strongly mixing so that, using Theorem 2.1 of O'Brien (1987), (3.2) will follow if we can show that, for any sequence of positive integers $\{p_n,\ n \ge 0\}$ with $p_n \to \infty$ and $p_n = o(n)$,

$$\Pr\{M(0, p_n) > u_n | \xi_0 > u_n\} \to 0 \quad \text{as } n \to \infty, \tag{3.3}$$

where $M(i, j) = \max\{\xi_{i+1}, \xi_{i+2}, \dots, \xi_j\}$. Given an arbitrary $\varepsilon > 0$, fix $r \ge 1$ such that $\Pr\{A\} < \varepsilon$, where $A = \{\tau_r \le 1\}$. Then

$$\begin{aligned} \Pr\{M(0, p_n) > u_n | \xi_0 > u_n\} &\le \sum_1^r \Pr\{\xi_i > u_n | \xi_0 > u_n\} \\ &\quad + \Pr\{M(r, p_n) > u_n | \xi_0 > u_n\} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\Pr\{M(r,p_n) > u_n|\xi_0 > u_n\} &\\ &\le \Pr\{A|\xi_0 > u_n\} + \Pr\{A^c, M(r,p_n) > u_n|\xi_0 > u_n\}\\ &= \Pr\{A\} + \Pr\{A^c, M(r,p_n) > u_n\}\\ &\le \Pr\{A\} + \Pr\{M(r,p_n) > u_n\}\\ &\le \varepsilon + p_n \Pr\{\xi_1 > u_n\}\\ &\le 2\varepsilon\end{aligned}$$

for n sufficiently large, since $p_n = o(n)$ and (3.1) holds with $k_n = n$. In view of the obvious estimate $\Sigma_1^r \Pr\{\xi_i > u_n|\xi_0 > u_n\} \le r\Pr\{\xi_1 + r > u_n|\xi_0 > u_n\}$, it is now clear that (3.3) will follow from (3.4) if we can show that, for fixed r,

$$\lim_{x\to\infty} \Pr\{\xi_1 > x - r|\xi_0 > x\} = 0. \tag{3.5}$$

Again denote $-\tau_{-n}$ by t_n, $n \ge 0$. For fixed $t \in (0,1)$ write $L = \#\{n\colon 1 - t < t_n \le 1 - \frac{1}{2}t\}$ and $\lambda = h(1 - \frac{1}{2}t)$, so that $\lambda > 0$. Then, given $\tau_1 = t$,

$$\xi_1 = 1 + \sum h(t + t_n) \le 1 + \sum_{n\colon t_n \le 1-t} h(t_n) \le 1 + \xi_0 - \lambda L$$

and hence

$$\begin{aligned}\Pr\{\xi_1 > x - r|\xi_0 > x, \tau_1 = t\} &\le \Pr\{\xi_0 > x + \lambda L - r - 1|\xi_0 > x\}\\ &\le \Pr\{\xi_0 > x + 1|\xi_0 > x\} + \Pr\{L < l|\xi_0 > x\},\end{aligned}$$

where $l = (r+2)\lambda^{-1}$. The first term tends to 0 as $x \to \infty$, by Theorem 1. Then

$$L < l \Rightarrow \sup\{\theta_i;\, i \text{ such that } t_{i-1} \le 1 - \tfrac{1}{2}t\} \ge \tfrac{1}{2}t\lambda(r+2)^{-1} = b, \quad \text{say},$$

so

$$\Pr\{L < l, S(x) > 1 - \tfrac{1}{2}t|\xi_0 > x\} \le \Pr\{B(x) \ge b|\xi_0 > x\}$$

and this tends to zero by Lemma 4. On the other hand,

$$\Pr\{L < l, S(x) \le 1 - \tfrac{1}{2}t|\xi_0 > x\} \le \Pr\{S(x) \le 1 - \tfrac{1}{2}t|\xi_0 > x\}$$

and this tends to zero by Lemma 3. Conditioning on τ_1 and using dominated convergence then establishes (3.5) and hence the theorem. □

REMARKS. Using the methods of Hsing and Teugels (1989) [see, in particular, (2.5)], it is not difficult to show that Theorem 3 holds with M_n replaced by $\max_{0\le r\le n} Y_r$, where $Y_r = X(\tau_r) = \Sigma_{0\le m\le r} h(\tau_r - \tau_m)$.

Note that (3.3) means that $\{\xi_n\}$ has "extremal index 1," in the usual terminology of extreme value theory [cf. O'Brien (1987)]. This is usually interpreted as meaning that large values in the sequence $\{\xi_n\}$ do not cluster together, with seems to contradict the fact that $\xi_n > x$ implies $\xi_{n-1} > x - 1$. However, "ξ_n is large" in this context means that $G(\xi_n)$ is close to 1, so that, in view of Theorem 1, ξ_n being large does not imply ξ_{n-1} is large.

There are two major difficulties in extending Theorem 3 to the case that h has unbounded support. (Of course, if h is supported by $[0, \Delta]$ with $0 < \Delta < \infty$, the usual scaling argument shows that Theorem 3 holds.) First, it is not difficult to see that, in general, $\{\xi_n,\ n \geq 0\}$ will not be strongly mixing in this case. It would therefore be necessary to show that $\{\xi_n,\ n \geq 0\}$ has some weaker mixing property, such as the A.I.M. property of O'Brien (1987). Second, the argument used to establish (3.3) is clearly going to fail in this situation. We content ourselves with settling the question for the important special case $h(x) = e^{-x},\ 0 \leq x < \infty$ (but see remark at the end of this section). Here $\{\xi_n,\ n \geq 0\}$ has the Markov property and we use this extensively in proving the next result. It should be noted that even in this Markov case, it is not clear that $\{\xi_n\}$ is strongly mixing since it may not be Harris recurrent.

THEOREM 4. *If* $h(x) \equiv e^{-x}$ *and* F *satisfies* (2.2), *then* (3.2) *holds.*

PROOF. We proceed via several lemmas, all of which refer to the case $h(x) = e^{-x}$.

LEMMA 5. *G, the distribution function of ξ_0, is continuous.*

PROOF. From $\xi_1 = \sum_{n \leq 1} e^{-(\tau_1 - \tau_n)} = 1 + e^{-\tau_1}\xi_0$ and $\xi_1 =_d \xi_0$, we have the distributional identity

$$\xi_0 =_d 1 + A\xi_0, \tag{3.6}$$

where $A = e^{-\tau_1}$. From this it is clear that the distribution of ξ_0 is nondegenerate, since otherwise A would have a degenerate distribution, contradicting (2.2). Now (3.6) is a special case of a much-studied identity and the result follows from Theorem 3.2 of Vervaat (1979). [See also Grincevičius (1974).] □

LEMMA 6. *Let $G_n(y|x) = \Pr\{\xi_n \leq y | \xi_0 = x\}$. Then for each fixed x and y,*

$$G_n(y|x) \to G(y) \quad \text{as } n \to \infty. \tag{3.7}$$

PROOF. This is immediate from the representation

$$\xi_n = e^{-\tau_n}\xi_0 + Z_n, \tag{3.8}$$

where

$$Z_n = \sum_{1 \leq m \leq n} e^{-(\tau_n - \tau_m)},$$

since $\tau_n \to +\infty$ a.s. and $Z_n \to_d \xi_0$ as $n \to \infty$. It is also a special case of Theorem 1.5 of Vervaat (1979). □

We now show that $\{\xi_n\}$ has the A.I.M. property of O'Brien (1987).

LEMMA 7. *Let* $\{q_n\}$ *be any sequence of positive integers with* $\lim_{n\to\infty} q_n = +\infty$, $\{c_n\}$ *be any sequence of positive numbers and*

$$D_n = \sup_{\substack{i>0\\ j>0}} \Big| \Pr\{M(0,i) \le c_n,\, M(i+q_n, i+q_n+j) \le c_n\} - \Pr\{M(0,i) \le c_n\}\Pr\{M(0,j) \le c_n\}\Big|.$$

Then $D_n \to 0$ *as* $n \to \infty$.

PROOF. Writing $G_n(x,y) = \Pr\{\xi_0 \le x,\ \xi_n \le y\}$ and using the stationarity and Markov property of $\{\xi_n\}$ gives

$$\begin{aligned}
&\Pr\{M(0,i) \le c_n,\, M(i+q_n, i+q_n+j) \le c_n\}\\
&= \int_0^\infty \int_0^\infty \Pr\big\{M(0,i) \le c_n,\, M(i+q_n, i+q_n+j) \le c_n | \xi_i = x,\, \xi_{i+q_n} = y\big\}\\
&\qquad \times G_{q_n}(dx, dy)\\
&= \int_0^\infty \int_0^\infty \Pr\big\{M(0,i) \le c_n | \xi_i = x,\, \xi_{i+q_n} = y\big\}\\
&\qquad \times \Pr\big\{M(i+q_n, i+q_n+j) \le c_n | M(0,i) \le c_n,\, \xi_i = x,\, \xi_{i+q_n} = y\big\}\\
&\qquad \times G_{q_n}(dx, dy)\\
&= \int_0^\infty \int_0^\infty \Pr\{M(0,i) \le c_n | \xi_i = x\} g(y,j,n) G_{q_n}(dy|x) G(dx),
\end{aligned}$$

where $g(y,j,n) = \Pr\{M(0,j) \le c_n | \xi_0 = y\}$. Since

$$\begin{aligned}
&\Pr\{M(0,i) \le c_n\}\Pr\{M(0,j) \le c_n\}\\
&\qquad = \int_0^\infty \int_0^\infty \Pr\{M(0,i) \le c_n | \xi_i = x\} g(y,j,n) G(dy) G(dx),
\end{aligned}$$

it follows that

$$D_n \le \sup_{j>0} \int_0^\infty \left| \int_0^\infty g(y,j,n)\{G_{q_n}(dy|x) - G(dy)\} \right| G(dx).$$

For fixed n and j, g is nonincreasing in y and $\lim_{y\to\infty} g(y,j,n) = 0$, so an integration by parts gives, for fixed $0 < x < \infty$,

$$\int_0^\infty g(y,j,n)\{G_{q_n}(dy|x) - G(dy)\} = -\int_0^\infty \{G_{q_n}(y|x) - G(y)\} g(dy,j,n).$$

Thus $D_n \le \int_0^\infty \Delta_n(x) G(dx)$, where

$$\Delta_n(x) = \sup_y \big| G_{q_n}(y|x) - G(y) \big|.$$

But in view of Lemmas 5 and 6, $G_n(y|x) \to G(y)$ uniformly in y for each fixed x. Thus $\Delta_n(x) \to 0$ and, since $\Delta_n(x) \le 1$, the result follows by dominated convergence. □

LEMMA 8. *Relation* (3.3) *holds, viz.* $\Pr\{M(0, p_n) > u_n | \xi_0 > u_n\} \to 0$ *as* $n \to \infty$.

PROOF. Since $u_n \to \infty$ we may choose integer-valued $r_n \uparrow \infty$ such that

$$\frac{r_n}{u_n} \to 0 \quad \text{and} \quad \frac{\log u_n}{r_n} \to 0 \quad \text{as } n \to \infty. \tag{3.9}$$

Now $\xi_i \le \xi_1 + i$ so, from Theorem 1 (with $\Delta = \infty$ and $E\xi_0 < \infty$),

$$\begin{aligned} \Pr\{M(0, r_n) > u_n | \xi_0 > u_n\} &\le \Pr\{\xi_1 > u_n - r_n | \xi_0 > u_n\} \\ &\sim \Pr\{\xi_1 > u_n - r_n | u_n + 1 \ge \xi_0 > u_n\} \\ &= \Pr\{1 + e^{-\tau_1}\xi_0 > u_n - r_n | u_n + 1 \ge \xi_0 > u_n\} \\ &\le \Pr\left\{e^{-\tau_1} > 1 - \frac{r_n + 2}{u_n + 1}\right\} \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{3.10}$$

by (3.9) and (2.2). Given $\varepsilon > 0$, choose K such that $\Pr\{\xi_0 > K\} < \varepsilon$ and, recalling the representation (3.8), observe that Z_n is independent of ξ_0 so, writing $r(n)$ for r_n,

$$\begin{aligned} \Pr\{\xi_{r(n)} > 2K | \xi_0 > u_n\} &\sim \Pr\{\xi_{r(n)} > 2K | u_n + 1 \ge \xi_0 > u_n\} \\ &\le \Pr\{\xi_0 e^{-\tau_{r(n)}} > K | u_n + 1 \ge \xi_0 > u_n\} \\ &\quad + \Pr\{Z_{r(n)} > K\} \\ &\le \Pr\{\tau_{r(n)} < \log\{K^{-1}(u_n + 1)\}\} + \Pr\{\xi_0 > K\} \\ &\le 2\varepsilon \quad \text{for all sufficiently large } n, \end{aligned} \tag{3.11}$$

since $m^{-1}\tau_m \to \mu > 0$ a.s. as $m \to \infty$. Also, writing $\tilde{G}_n(y) = \Pr\{\xi_{r(n)} \le y | \xi_0 > u_n\}$, the Markov property and the fact that $\Pr\{M(r_n, p_n) > u_n | \xi_{r(n)} = x\}$ is increasing in x give

$$\begin{aligned} &\Pr\{M(r_n, p_n) > u_n, \xi_{r(n)} \le 2K | \xi_0 > u_n\} \\ &= \int_0^{2K} \Pr\{M(r_n, p_n) > u_n | \xi_{r(n)} = x\} \tilde{G}_n(dx) \\ &\le \Pr\{M(r_n, p_n) > u_n | \xi_{r(n)} = 2K\} \\ &\le \{1 - G(2K)\}^{-1} \int_{2K}^{\infty} \Pr\{M(r_n, p_n) > u_n | \xi_{r(n)} = x\}\, dG(x) \\ &\le \{1 - G(2K)\}^{-1} \int_0^{\infty} \Pr\{M(r_n, p_n) > u_n | \xi_{r(n)} = x\}\, dG(x) \\ &\le \{1 - G(2K)\}^{-1} \Pr\{M(r_n, p_n) > u_n\} \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.12}$$

Finally the estimate

$$\Pr\{M(r_n, p_n) > u_n | \xi_0 > u_n\} \le \Pr\{\xi_{r(n)} > 2K | \xi_0 > u_n\} + \Pr\{M(r_n, p_n) > u_n, \xi_{r(n)} \le 2K | \xi_0 > u_n\}$$

together with (3.11), (3.12) and (3.10), establishes (3.3). □

We continue with the proof of Theorem 4. Lemma 7, with $u_n = c_n$, shows that $\{\xi_n, n \ge 0\}$ has the A.I.M. (u_n) property; together with Lemma 8, this allows us to use Theorem 2.1 of O'Brien (1987), and the result is immediate. □

REMARK 7. With rather more effort, Lemma 7 can be shown to hold whenever $\{\xi_n, n \ge 0\}$ has the asymptotic version of the Markov property:

$$\begin{aligned} &\Pr\{\xi_{n+1} \in dy_1, \dots, \xi_{n+m} \in dy_m | \xi_0 = x, A\} \\ &\quad = \{1 + o(1)\}\Pr\{\xi_{n+1} \in dy_1, \dots, \xi_{n+m} \in dy_m | \xi_0 = x\} \qquad \text{as } n \to \infty, \end{aligned} \tag{3.13}$$

uniformly in $m \ge 1$, $y_1, y_2, \dots, y_m$, x and $A \in \sigma\{\xi_i, i < 0\}$. Also Lemma 8 can be extended to those h which also have the properties

$$\begin{aligned} &h(t) > 0 \quad \text{for all } t > 0, \\ &\inf_{s \ge 0} \frac{h(t+s)}{h(s)} > 0 \quad \text{for all } t > 0, \\ &\lim_{t \to \infty} \left\{ t \sup_{s \ge 0} \frac{h(t+s)}{h(s)} \right\} = 0. \end{aligned} \tag{3.14}$$

The only example of nonexponential h satisfying (3.13) we have is the rather trivial one where $h(x) = ce^{-x}$ for all $x \ge x_0$. If, for example, h is also linear on $[0, x_0]$, then (3.14) also holds. Assuming that the distribution of ξ_0 is continuous, this would constitute an example of a nonexponential h for which the conclusion of Theorem 3 is also valid.

REMARK 8. Again the question arises as to whether the conclusion of Theorem 4 remains valid when the stationary sequence $\{\xi_n, n \ge 0\}$ is replaced by the nonstationary sequence $\{Y_n, n \ge 0\}$ based on an *ordinary* renewal process $\{\tau_n, n \ge 0\}$. [Recall $Y_n = X(\tau_n) = \sum_{0 \le m \le n} h(\tau_n - \tau_m)$, $h(x) = e^{-x}$.] Because the support of h is unbounded, this is a more delicate question than when $\Delta < \infty$. Nevertheless, by observing that the distribution of $\{Y_n, n > 0\}$ coincides with the conditional distribution of $\{\xi_n, n > 0\}$, given $\xi_0 = 1$, the question can, with some effort, be answered in the affirmative.

4. Concluding remarks.

REMARK 9. It is known [see de Haan (1970), Section 2.9] that (1.3) holds for *all* $\delta > 0$ iff G satisfies the maximum weak law of large numbers, and this in turn is equivalent to the statement that $1 - G(\log x)$ is rapidly varying with index $-\infty$ as $x \to \infty$ [see Bingham, Goldie and Teugels (1986), page 83]. We point out that Hsing and Teugels (1989) give the erroneous impression that (1.3) with $\delta = 1$ implies these conditions.

REMARK 10. It is easy to see that the only extreme value distribution function that G could be attracted to is $\Lambda(x) := \exp\{-e^{-x}\}$, but it seems to be very difficult to say much about when G is in its domain of attraction. It is known that for G to be in any domain of attraction, we must have

$$\lim_{x \to \infty} \left\{ \frac{1 - G(x)}{1 - G(x-)} \right\} = 1, \tag{4.1}$$

and it is clear that if $h(\Delta -) > 0$, (4.1) will not hold. The only situation where any progress has been made is in the special case that $\{\tau_n, n \geq 0\}$ is a Poisson process, when $\{X(t), t \geq 0\}$ is often called a *filtered Poisson* process and there is [see Parzen (1962)] an explicit formula for $E(e^{-s\xi_0})$ in terms of h. In this situation, recent results of Embrechts, Jensen, Maejima and Teugels (1985) [see also Jensen (1988)] were used in Hsing and Teugels (1989) to show that, when $\Delta < \infty$ and h has a certain asymptotic behaviour near $x = \Delta$ and satisfies certain other technical conditions, G belongs to the domain of attraction of Λ. We also remark that extensive work on the asymptotic behaviour of G in the case of the filtered Poisson process with $h(x) \equiv e^{-x}$ [see de Bruijn (1951) and Vervaat (1972)] has established results of the form

$$-\log\{1 - G(x)\} \sim x \log x \quad \text{as } x \to \infty.$$

These results do not imply (1.3) nor do they settle the domain of attraction question.

REMARK 11. In the case that $\{t_n, n \geq 0\}$ is an ordinary renewal process, it is easy to see that Lemma 1 holds with θ_2 replaced by θ_1, viz. conditioned on $S(x) \leq y$, θ_1 is stochastically dominated by θ_i for each fixed $i > 1$. We point out that a similar proof also establishes that the stochastic ordering $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_i$ also holds. In the case that h is the indicator function of a finite interval, this reduces to a result in Kremers (1988).

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