

A LIMIT RESULT RESPECTING GRAPH STRUCTURE FOR A FULLY CONNECTED LOSS NETWORK WITH ALTERNATIVE ROUTING¹

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Recently there has been a considerable amount of work on the transient behaviour of loss networks in two different limiting regimes. The first of these, which we do not consider here, is when link capacities and offered traffics become large but the number of links remains finite. The second is the diverse routing limit when the number of links increases with the offered load to each link held constant. Thus far, however, all results of this latter type have been for simplified models with exchangeable links. In adopting such a simplified model one loses the inherent graph structure of the original loss network, which appears to be a serious drawback.

In this paper we consider a loss network with graph structure and show that, subject to natural constraints on the initial configuration, the model behaves asymptotically exactly like one with exchangeable links. Our result is proved by combining the techniques of Gibbens, Hunt and Kelly with those of Hajek for a problem in random graph theory.

1. Introduction. The model we consider is a fully connected loss network operating under alternative routing. An example would be a telephone network in which every node (or exchange) is connected to every other node and in which an arriving call may, if the direct link between its source and destination is full, be routed via two or more alternative links. To be more precise, our network consists of $n + 2$ nodes, every pair of nodes being connected by a link consisting of C circuits, giving a total of $N = (n + 1)(n + 2)/2$ links. For all $\alpha \neq \beta$, calls between node α and node β arrive as a Poisson process of rate ν , all arrival streams being independent. If there is free capacity on the direct link between α and β , then the call is routed along this path. If not, we try to route the call along two links via a randomly chosen third node $\gamma \neq \alpha, \beta$. If there is free capacity on both these links, then the call is routed. Otherwise the call is lost. A call that has been successfully routed holds one circuit from each link on its path for the holding period of the call. This holding period is independent of earlier arrival times and holding periods, and is exponentially distributed with unit mean. In [3], Gibbens, Hunt and Kelly replace the model by a simplified version as follows.

There are N links, each link comprising C circuits. Calls requesting link l as their first choice arrive as a Poisson process of rate ν . If a call is blocked on

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its first choice link, it tries two other links chosen at random from the $N - 1$ remaining links, with each pair of links having equal probability of being chosen. If neither of the links in the chosen pair is full, the call is set up along these two links. Otherwise, the call is lost. When a circuit is used by a call, the circuit is held for an exponential time, mean 1. All circuit holding times are independent of one another and of earlier arrival times. In particular, a call that requires two links holds each link independently for an exponential length of time, and so these circuits will become free at different times. Thus the simplified model differs in two ways from the original network: circuit holding times in multilink calls are independent, and the graph structure relationship between links has been lost.

For this simplified model it is relatively straightforward to prove a limit result and they prove the following. Let $n_j^N(t)$ be the number of links with j circuits in use at time t , $j = 0, 1, \dots, C$. Let

$$x_j^N(t) = \frac{n_j^N(t)}{N}, \quad \mathbf{x}^N(t) = (x_j^N(t))_j.$$

THEOREM 1. *If $\mathbf{x}^N(0) \Rightarrow \mathbf{x}(0)$, then $\mathbf{x}^N(\cdot) \Rightarrow \mathbf{x}(\cdot)$, where $\mathbf{x}(\cdot)$ is the unique solution to the equations*

$$(1) \quad x_0(t) = x_0(0) + \int_0^t \{x_1(u) - (\nu + \lambda(u))x_0(u)\} du,$$

$$(2) \quad x_j(t) = x_j(0) + \int_0^t \{(\nu + \lambda(u))x_{j-1}(u) - (\nu + \lambda(u) + j)x_j(u) + (j + 1)x_{j+1}(u)\} du, \quad j \neq 0, C,$$

$$(3) \quad x_C(t) = x_C(0) + \int_0^t \{(\nu + \lambda(u))x_{C-1}(u) - Cx_C(u)\} du$$

and

$$(4) \quad \lambda(t) = 2\nu x_C(t)(1 - x_C(t)).$$

REMARK ON PROOF. Consider an arriving call when the N th network is in state $\mathbf{x}^N(t)$. This call is blocked directly with probability $x_C^N(t)$ and is then offered to two other links chosen at random. The probability that the call is now routed via these two links is $(1 - x_C^N(t))^2 + o(1)$, since the call must choose two links that are not full. If, however, the model retained its graph structure, it is not immediately clear that this result still holds. For example, suppose that all the links that are full have a node in common and that every link from this node is full. Then a blocked call cannot be rerouted even though the vast majority of links in the network are not full. It will be shown that such configurations have negligibly small probability (Section 2) and hence Theorem 1 remains valid for a loss network with graph structure (Section 3).

REMARK. Although the simplified model of Gibbens, Hunt and Kelly [3] assumes that the two circuits used by a rerouted call are released independently, Theorem 1 holds even without this; see Hunt [5], for example.

2. Analysis of graph structure. The techniques we use here are generalisations of those developed by Hajek [4] to solve Kelly’s triangle problem. The result Hajek proves is for a static system but the methods he uses are readily extended to a transient situation. To examine the graph structure as our network evolves, we consider the jump chain, so let k index the jumps of the process. We make the following definitions.

$R(k) :=$ the total number of directly routed calls at stage k ,

$W(k) :=$ the total number of alternatively routed calls at stage k ,

$D_k := R(k) + W(k) + \nu N$,

$y_i(k) :=$ the number of links with i circuits in use at stage k .

Now fix some $t_0 > 0$ and consider the first $n^2 t_0$ steps in the jump chain. At stage k the following occurs.

1. With probability $p_r(k) := (R(k))/D_k$, a randomly chosen directly routed call clears down.
2. With probability $p_w(k) := (W(k))/D_k$, a randomly chosen alternatively routed call clears down.
3. With probability $p_a(k) := (\nu N)/D_k$, a link is chosen at random. If the link is not full, a call is routed directly along this link. If, however, the link is full, then two links that form an alternative route to this link are chosen at random and a call is routed along these two links if possible. Otherwise, nothing happens.

For $i \neq j$, let $A_e^{i,j}(k)$ be the number of triangles containing edge e and i busy circuits on one side and j on the other, $0 \leq i, j \leq C$; for $i = j$, let $A_e^{i,j}(k)$ be twice this. We adopt this latter definition rather than the more obvious one for ease of notation; we are interested in the number of links in state i that form triangles with edge e and when relating this to the triangles that contain edge e we must double count those that contain two edges in state i . Let $\Lambda_e(k)$ be the number of calls alternatively routed from link e and set

$$\overline{A_e^{i,j}}(k) = 2n \frac{y_i(k)}{N} \frac{y_j(k)}{N}, \quad \overline{\Lambda}_e(k) = \frac{W(k)}{N},$$

$$\phi(k) = \max_e \left\{ \max_{i,j} \{ |A_e^{i,j}(k) - \overline{A_e^{i,j}}(k)| \} \vee |\Lambda_e(k) - \overline{\Lambda}_e(k)| \right\}.$$

We wish to establish tight bounds on $\phi(k)$ for $0 \leq k \leq n^2 t_0$ and we do this via Theorem 3. First, we need a lemma. Let Δ be the one-step difference operator,

$\Delta X(k) = X(k + 1) - X(k)$ for any variable X depending on k and let F_k be the σ -algebra generated by configurations at stages less than or equal to k .

LEMMA 2. *There exist constants L and $\delta > 0$ such that, for n sufficiently large, we have*

$$|\mathbb{E}(\Delta X_e(k) - \Delta \overline{X}_e(k)|F_k)| \mathbb{1}_{\{\phi(k) \leq n\delta\}} \leq \frac{L}{n^2}(\phi(k) + 1), \quad X_e = A_e^{i,j} \text{ or } \Lambda_e,$$

for every edge e and all $k \leq n^2 t_0$.

REMARK. The condition that $\phi(k) < n\delta$ is not necessary for the lemma but is used in the proof of Theorem 3.

PROOF. We consider two cases separately.

(i) $X_e = \Lambda_e$.

$$(5) \quad \mathbb{E}(\Delta \overline{\Lambda}_e(k)|F_k) = \frac{1}{N} \left\{ -\frac{W(k)}{D_k} + p_a(k) \frac{1}{N} \sum_{e' \text{ full}} \left[\frac{1}{n} \left(n - \sum_{l \leq C} A_e^{l,C}(k) \right) \right] \right\},$$

$$(6) \quad \mathbb{E}(\Delta \Lambda_e(k)|F_k) = -\frac{\Lambda_e(k)}{D_k} + p_a(k) \frac{\mathbb{1}_{\{e \text{ full}\}}}{N} \frac{1}{n} \left(n - \sum_{l \leq C} A_e^{l,C}(k) \right).$$

The last terms in equations (5) and (6) are each bounded in modulus by $1/N$ and thus

$$|\mathbb{E}(\Delta \Lambda_e(k) - \Delta \overline{\Lambda}_e(k)|F_k)| \leq \frac{1}{D_k} |\Lambda_e(k) - \overline{\Lambda}_e(k)| + \frac{2}{N}$$

so the result follows easily.

(ii) $X_e = A_e^{i,j}$.

First, define

$$A_e^{l,-1}(k) = A_e^{l,C+1}(k) = y_{-1}(k) = y_{C+1}(k) = 0 \quad \text{for all } k, e \text{ and } l,$$

to deal with the extreme cases when i or $j \in \{0, C\}$. Defining

$$B_e^{i,j}(k) = \frac{2n}{N^2} (y_i(k) \Delta y_j(k) + y_j(k) \Delta y_i(k)),$$

we have

$$(7) \quad \begin{aligned} & |\mathbb{E}(\Delta A_e^{i,j}(k) - \Delta \overline{A}_e^{i,j}(k)|F_k)| \\ & \leq |\mathbb{E}(\Delta A_e^{i,j}(k) - B_e^{i,j}(k)|F_k)| + |\mathbb{E}(B_e^{i,j}(k) - \Delta \overline{A}_e^{i,j}(k)|F_k)|. \end{aligned}$$

The final term here is bounded by $8n/N^2$, since $\Delta \overline{A}_e^{i,j}(k) = B_e^{i,j} + 2n \Delta y_i(k) \Delta y_j(k) / N^2$ and $\Delta y_i(k)$ is bounded by 2. Now consider the first term

on the right of (7). Direct calculation (with care when $|i - j| \leq 1$) yields

$$\begin{aligned}
 & \mathbb{E}(\Delta A_e^{i,j}(k) | F_k) \\
 &= \frac{1}{D_k} \left\{ \left[\left(\sum_{e_{i+1}(j)} (i+1) - \sum_{\alpha_{i+1,j}} \lambda_e(\alpha_{i+1,j}) \right) - \left(\sum_{e_i(j)} i - \sum_{\alpha_{i,j}} \lambda_e(\alpha_{i,j}) \right) \right] \right. \\
 & \quad \left. + \left[\left(\sum_{e_{j+1}(i)} (j+1) - \sum_{\alpha_{i,j+1}} \lambda_e(\alpha_{i,j+1}) \right) - \left(\sum_{e_j(i)} j - \sum_{\alpha_{i,j}} \lambda_e(\alpha_{i,j}) \right) \right] \right\} \\
 & \hspace{15em} \text{(direct calls clearing down)} \\
 & + \frac{1}{D_k} \left\{ \sum_{\alpha_{i+1,j+1}} \lambda_e(\alpha_{i+1,j+1}) - \sum_{\alpha_{i,j}} \lambda_e(\alpha_{i,j}) \right\} (1 + \mathbb{1}_{\{i=j\}}) \\
 & \hspace{15em} \text{(rerouted calls clearing down)} \\
 & + p_a(k) \left\{ \frac{A_e^{i,j-1}(k)}{N} + \frac{A_e^{i-1,j}(k)}{N} - \frac{(\mathbb{1}_{\{i \neq C\}} + \mathbb{1}_{\{j \neq C\}}) A_e^{i,j}(k)}{N} \right\} \\
 & \hspace{15em} \text{(directly routed arrivals)} \\
 & + \frac{p_a(k)}{N} \left\{ \sum_{e_{i-1}(j)} \frac{1}{n} \left(\sum_{l < C} A_{e_{i-1}(j)}^{l,C}(k) - \mathbb{1}_{\{e \text{ full}\}} \right) \right. \\
 & \quad \left. - \sum_{e_i(j)} \frac{1}{n} \left(\sum_{l < C} A_{e_i(j)}^{l,C}(k) - \mathbb{1}_{\{e \text{ full}\}} \right) \right\} \\
 & + \frac{p_a(k)}{N} \left\{ \sum_{e_{j-1}(i)} \frac{1}{n} \left(\sum_{l < C} A_{e_{j-1}(i)}^{l,C}(k) - \mathbb{1}_{\{e \text{ full}\}} \right) \right. \\
 & \quad \left. - \sum_{e_j(i)} \frac{1}{n} \left(\sum_{l < C} A_{e_j(i)}^{l,C}(k) - \mathbb{1}_{\{e \text{ full}\}} \right) \right\} \\
 & \hspace{15em} \text{(calls rerouted, but not from } e \text{)} \\
 & + \frac{p_a(k)}{N} \left\{ \mathbb{1}_{\{e \text{ full}\}} \left(\frac{A_e^{i-1,j-1}(k)}{n} - \frac{A_e^{i,j}(k)}{n} \mathbb{1}_{\{i \neq C, j \neq C\}} \right) \right\} \\
 & \hspace{15em} \text{(calls rerouted from } e \text{)}.
 \end{aligned}$$

Here $\alpha_{i,j}$ indexes all nodes connected to edge e by one link in state i and one in state j and $\lambda_e(\alpha_{i,j})$ gives the number of alternatively routed calls for edge e that pass through node $\alpha_{i,j}$. Also, $e_i(j)$ indexes those links with i circuits in use that, along with a link in state j , form an alternative route to edge e .

After some simplification we find

$$\begin{aligned} & \mathbb{E}(\Delta A_e^{i,j}(k) | F_k) \\ &= \frac{1}{D_k} \{ [(i+1)A_e^{i+1,j}(k) - iA_e^{i,j}(k)] + [(j+1)A_e^{i,j+1}(k) - jA_e^{i,j}(k)] \} \\ &+ \frac{p_a(k)}{N} \{ [A_e^{i-1,j}(k) - \mathbb{1}_{\{i \neq C\}} A_e^{i,j}(k)] \\ &\quad + [A_e^{i,j-1}(k) - \mathbb{1}_{\{j \neq C\}} A_e^{i,j}(k)] \} \\ &+ \frac{p_a(k)}{N} \left\{ \left[\sum_{e_{i-1}(j)} \frac{1}{n} \sum_{l < C} A_{e_{i-1}(j)}^{l,C}(k) - \sum_{e_i(j)} \frac{1}{n} \sum_{l < C} A_{e_i(j)}^{l,C}(k) \right] \right. \\ &\quad \left. + \left[\sum_{e_{j-1}(i)} \frac{1}{n} \sum_{l < C} A_{e_{j-1}(i)}^{l,C}(k) - \sum_{e_j(i)} \frac{1}{n} \sum_{l < C} A_{e_j(i)}^{l,C}(k) \right] \right\} \\ &+ \frac{\overline{\Lambda}_e^{i,j}(k)}{D_k} + \frac{p_a(k)}{N} \mathbb{1}_{\{e \text{ full}\}} \frac{\overline{\overline{A}}_e^{i,j}(k)}{n}. \end{aligned}$$

Here $\overline{\Lambda}$ collects all the terms involving λ_e and $\mathbb{1}_{\{e \text{ full}\}} \overline{\overline{A}}$ collects all the terms involving $\mathbb{1}_{\{e \text{ full}\}}$.

Now consider $y_i(k)$. We find that

$$\begin{aligned} \mathbb{E}(\Delta y_i(k) | F_k) &= \frac{1}{D_k} [(i+1)y_{i+1}(k) - iy_i(k)] \\ &+ \frac{p_a(k)}{N} (y_{i-1}(k) - \mathbb{1}_{\{i \neq C\}} y_i(k)) \\ &+ \frac{p_a(k)}{N} \left[\sum_{e' \text{ full}} \sum_{l < C} \left(\frac{A_{e'}^{i-1,l}(k)}{n} \middle| F_k \right) - \sum_{e' \text{ full}} \sum_{l < C} \left(\frac{A_{e'}^{i,l}(k)}{n} \middle| F_k \right) \right] \end{aligned}$$

and thus

$$\begin{aligned} & \frac{2n}{N^2} y_j(k) \mathbb{E}(\Delta y_i(k) | F_k) \\ &= \frac{1}{D_k} \{ (i+1)\overline{A}_e^{i+1,j}(k) - i\overline{A}_e^{i,j}(k) \} + \frac{p_a(k)}{N} \{ \overline{A}_e^{i-1,j}(k) - \overline{A}_e^{i,j}(k) \} \\ &+ \frac{2n}{N^2} y_j(k) \frac{p_a(k)}{N} \left[\sum_{e' \text{ full}} \sum_{l < C} \left(\frac{A_{e'}^{i-1,l}(k)}{n} \middle| F_k \right) - \sum_{e' \text{ full}} \sum_{l < C} \left(\frac{A_{e'}^{i,l}(k)}{n} \middle| F_k \right) \right]. \end{aligned}$$

Now, after some thought, we can see that

$$\left| \sum_{e_i(j)} \sum_{l < C} A_{e_i(j)}^{l,C}(k) - \frac{2n}{N^2} y_j(k) \sum_{e' \text{ full}} \sum_{l < C} A_{e'}^{l,C}(k) \right| \leq L'n\phi(k)$$

for some L' , and so, noting that $\bar{\Lambda}(k) \leq \bar{\Lambda}(k) + \phi(k) \leq C + \phi(k)$, the result follows easily. \square

We are now in a position to prove the result that we need in the next section.

THEOREM 3. *Suppose that the initial configuration for our sequence of networks is given and is such that $(\phi^N(0))/n \rightarrow 0$ as $N \rightarrow \infty$. Then*

$$\lim_{N \rightarrow \infty} \mathbb{P}(\phi^N(k) \leq \delta n \text{ for all } 0 \leq k \leq n^2 t_0) = 1$$

for all $\delta > 0$.

SKETCH OF PROOF. The proof is very similar to the proof of [4], Proposition 2.1, and so we only highlight the differences. Fix $\delta > 0$ and $\frac{1}{2} < \gamma < 1$ and let $f_n = 2(\phi^N(0) \vee n^\gamma)$. Define $r_k, 0 \leq k \leq n^2 t_0$, by

$$r_k = f_n + \sum_{j=0}^{k-1} \frac{L}{n^2} (r_j + 1)$$

and notice that this gives

$$0 \leq r_k = \left(1 + \frac{L}{n^2}\right)^k (f_n + 1) - 1 \leq r_{n^2 t_0} \leq e^{t_0 L} (f_n + 1) \leq n\delta$$

for all n sufficiently large. Set $D = \min\{k \geq 0: \phi^N(k) \geq r_k \text{ or } k > n^2 t_0\}$. Then [4] it follows that

$$\lim_{N \rightarrow \infty} \mathbb{P}(\phi^N(k) \geq r_k \text{ for some } k \leq D) = 0.$$

Since $D < n^2 t_0$ if and only if $\phi^N(k) \geq r_k$ for some $k \leq D$ and since this event has probability tending to zero, we have $\mathbb{P}(D > n^2 t_0) \rightarrow 1$, which implies the result. \square

3. Functional law of large numbers. Theorem 3 gives a tight bound on the spread of alternative routes in the network and we can now apply this to prove a limit theorem respecting graph structure.

THEOREM 4. *Suppose $\mathbf{x}^N(0) \Rightarrow \mathbf{x}(0)$ and $(\phi^N(0))/n \rightarrow 0$ as $N \rightarrow \infty$. Then $\mathbf{x}^N(\cdot) \Rightarrow \mathbf{x}(\cdot)$ where $\mathbf{x}(\cdot)$ is the unique solution to equations (1)–(4).*

PROOF. First, define

$$\delta_{i,j}^N(t) = \sum_e \mathbb{1}_{\{e \text{ full}\}} \frac{A_e^{i,j}(t)}{n} - 2x_C^N(t)x_i^N(t)x_j^N(t),$$

where $A_e^{i,j}(t)$ is the continuous-time version of $A_e^{i,j}(k)$ and let $\delta_{i,j}^N(k)$ be the discrete-time version of $\delta_{i,j}^N(t)$. Let $\mathcal{E} = \{\text{more than } 4n^2T(\nu + C) \text{ jumps by time } T\}$. Then, given any $\varepsilon, T > 0$ and any sequence f_n ,

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq T} |\delta_{i,j}^N(t)| > \varepsilon\right) &\leq \mathbb{P}(\mathcal{E}) + \mathbb{P}\left(\left\{\sup_{0 \leq t \leq T} |\delta_{i,j}^N(t)| > \varepsilon\right\} \cap \mathcal{E}^C\right) \\ &\leq \mathbb{P}(\mathcal{E}) + \mathbb{P}\left(\sup_{k \leq 4n^2T(\nu + C)} |\delta_{i,j}^N(k)| > \varepsilon\right) \\ (8) \qquad &\leq \mathbb{P}(\mathcal{E}) \end{aligned}$$

$$(9) \qquad \qquad \qquad + \mathbb{P}\left(\sup_{k \leq 4n^2T(\nu + C)} |\delta_{i,j}^N(k)| > \varepsilon \mid \phi^N(0) < f_n\right)$$

$$(10) \qquad \qquad \qquad + \mathbb{P}(\phi^N(0) \geq f_n).$$

The jump rate for the N th network is bounded by $2n^2(\nu + C)$ and so (8) converges to zero. But our initial conditions and Theorem 3 ensure that we can choose a sequence f_n such that (9) and (10) also converge to zero and so $\delta_{i,j}^N(t) \rightarrow 0$ in probability for all $t > 0$.

Now let $\mathbf{z}^N(\cdot)$ be a vector Markov process that completely defines the state of the N th network and let $\mathcal{F}_t = \sigma(\mathbf{z}^N(s) : s \leq t)$. Define

$$\mathbf{M}^N(t) = \mathbf{x}^N(t) - \mathbf{x}^N(0) - \int_0^t \mathcal{G}^N \mathbf{x}^N(s) ds,$$

an $\{\mathcal{F}_t\}$ -martingale, where \mathcal{G}^N , the generator for $\mathbf{z}^N(\cdot)$, is given by

$$\begin{aligned} (\mathcal{G}^N \mathbf{x}(t))_i &= (\nu x_{i-1}^N(t) - \nu \mathbb{1}_{\{i \neq C\}} x_i^N(t)) + ((i + 1)x_{i+1}^N(t) - ix_i^N(t)) \\ &\quad + 2\nu(1 - x_C^N(t))(x_{i-1}^N(t) - x_i^N(t)) \\ &\quad + \nu \sum_{j=0}^{C-1} (\delta_{i-1,j}^N(t) - \delta_{i,j}^N(t)), \end{aligned}$$

and $x_{-1}^N = x_{C+1}^N = 0$. The martingale central limit theorem [2] proves that $\mathbf{M}^N \Rightarrow \mathbf{0}$ since $[\mathbf{M}^N] \rightarrow \mathbf{0}$, and a modification of (Whitt [6], page 1843) shows that $\{\mathbf{x}^N\}$ is relatively compact. Applying the continuous mapping theorem [2] to \mathbf{M}^N and using the fact that $\delta_{i,j}^N(t) \rightarrow 0$ in probability, we have that (1)–(4) is obeyed by any subsequence of the $\{\mathbf{x}^N\}$. But the result now follows since (1)–(4) has a unique solution. (See Arnold [1], pages 50, 57.) \square

The above result is for a network employing alternative routing with each call having only one retry. These techniques extend to more general networks, for example, networks employing trunk reservation, multiple retries, least

busy alternative and repacking. In all of these cases, we can derive a simple and easily explained functional law of large numbers. It would be interesting to extend the methods here to prove a functional central limit theorem for these processes; this has been done for approximations based on exchangeable link models [5] and presumably the graph structure is again irrelevant.

One situation in which we might expect the graph structure to be more important is in the context of large deviations. If our network is bistable, with two stable regimes (high and low blocking), the theory of large deviations can be applied to derive estimates of the time to tunnel from one regime to the other (see [3]). One would perhaps expect that the model with graph structure would tunnel more quickly than the exchangeable links model because the extra structure would allow local phenomena. It would be interesting to have some results in this area.

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