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ON THE ERGODICITY OF TAR(1) PROCESSES[1]

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This paper establishes a necessary and sufficient condition for geometrical ergodicity for the general first-order threshold autoregressive processes. This is achieved by investigating the nonlinear dynamic behavior generated by the delay parameter of a threshold model. The ergodic region turns out to be unbounded which is different from that of a linear process.

1. Introduction. Several nonlinear time series models have been proposed in recent years to capture various nonlinear phenomena commonly observed in practice. For example, Ozaki (1981) and Haggan and Ozaki (1981) used exponential autoregressive models to reproduce amplitude-dependent frequency. Tong (1983) considered the self-exciting threshold autoregressive (SETAR) models that are capable of describing time-irreversibility, jump resonances, and limit cycles. It is now evident that nonlinear times series models will play an important role in modern time series analysis. Unlike the linear models, however, it is rather hard to obtain a necessary and sufficient condition for ergodicity or geometrical ergodicity for a given class of nonlinear time series models [see Nummelin (1984) or Tong (1990), pages 126 and 456 for definitions of these terms]. Consider, for example, the SETAR models. There are no general necessary and sufficient conditions available for the ergodicity of a higher order model, even though interesting results exist for some special cases. In particular, consider the first-order threshold autoregressive, TAR(1), model

$$x_t = \begin{cases} \phi_1 x_{t-1} + \varepsilon_t, & \text{if } x_{t-d} \le 0, \\ \phi_2 x_{t-1} + \varepsilon_t, & \text{if } x_{t-d} > 0, \end{cases} \qquad t = 1, 2, \dots, \tag{1}$$

where $x_{1-d}, \dots, x_0$ are real numbers denoting the initial values of the process, d is a fixed positive integer commonly referred to as the *delay* parameter of x_t and the ε_t's are independent and identically distributed (iid) random variables with absolutely continuous marginal distribution and positive probability density function over the real line $\Re^1$ and $E|\varepsilon_t| < \infty$. Petruccelli and Woolford (1984) showed that for $d = 1$, the necessary and sufficient condition for the ergodicity of x_t is

$$\phi_1 < 1, \qquad \phi_2 < 1 \qquad \text{and} \qquad \phi_1\phi_2 < 1. \tag{2}$$

This result has several nice features including that (a) it has a neat geometric interpretation and (b) in contrast with the result of linear models,

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the ergodic region is unbounded. A question to ask then is to what degree can the result in (2) be generalized to the other SETAR models.

The main objective of this paper, therefore, is to extend the result of Petruccelli and Woolford to the general TAR(1) model, that is, to the case of a general delay parameter d. This is an important extension as (a) it might shed some light on the condition for the general SETAR models and (b) it provides a rigorous proof for the geometrical ergodicity of a nonlinear process that clearly illustrates the difference between linear and nonlinear models. Another objective is to understand the nonlinear dynamic behavior of a threshold process x_t with respect to the delay parameter d. It turns out that for the general TAR(1) model in (1), the nonlinear dynamic pattern of x_t depends critically on d. Indeed, the necessary and sufficient condition for geometrical ergodicity of x_t depends on d. Details of the condition are given in the following main result of the paper.

THEOREM 1. *For the first-order threshold autoregressive process x_t in* (1), *the necessary and sufficient condition for the geometrical ergodicity of x_t is*

$$\phi_1 < 1, \quad \phi_2 < 1, \quad \phi_1\phi_2 < 1, \quad \phi_1^{s(d)}\phi_2^{t(d)} < 1 \quad \text{and} \quad \phi_1^{t(d)}\phi_2^{s(d)} < 1,$$

where $s(d)$ and $t(d)$ are nonnegative integers depending on d, and $s(d)$ and $t(d)$ are odd and even numbers, respectively.

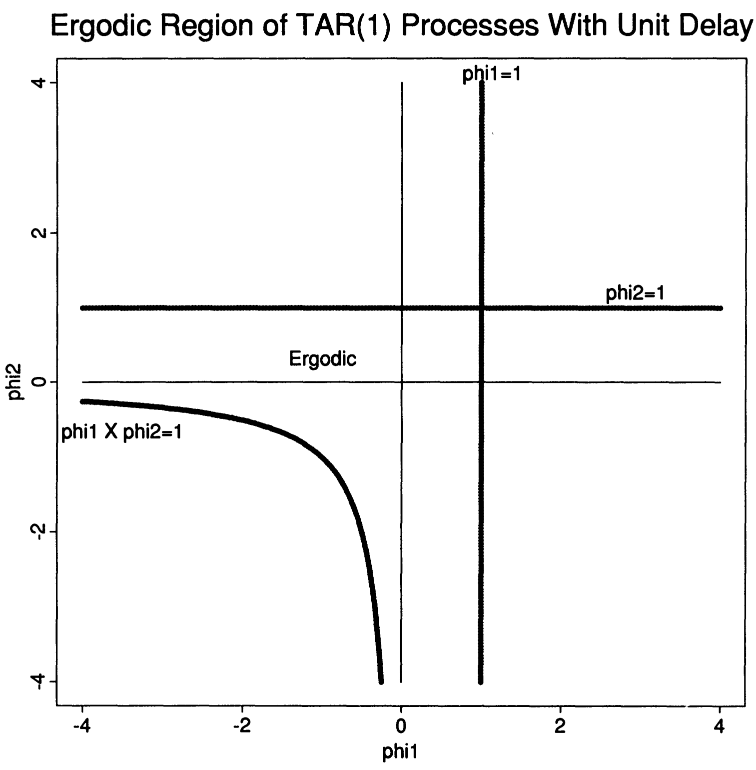

FIG. 1(a).

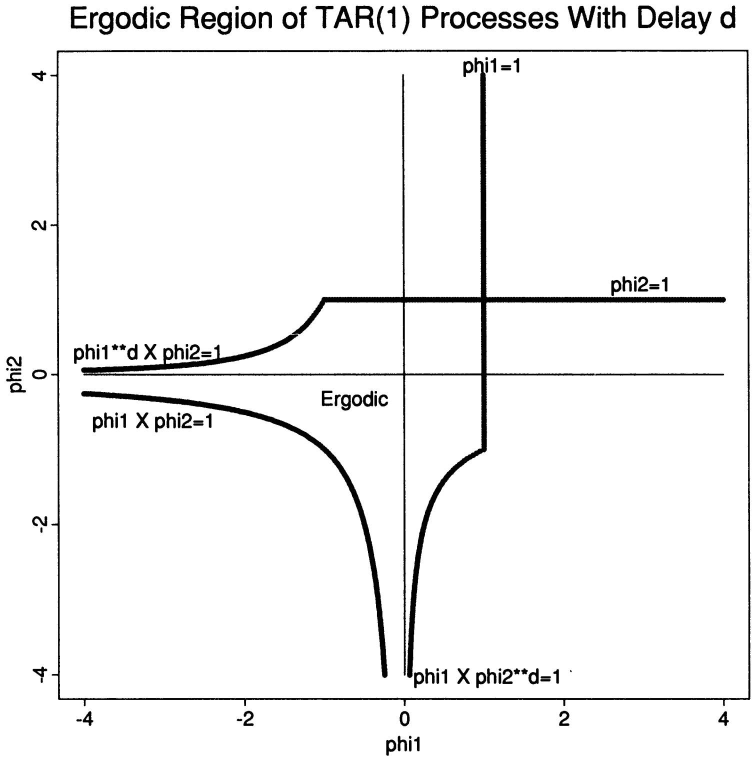

FIG. 1(b).

Details of the functions $s(d)$ and $t(d)$ will be given in Section 4. Here we give some of their values. These values were obtained by a computer program that uses the approach of this paper.

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$s(d)$	1	1	3	7	1	31	63	1	33	3	1	3	15	1	16383	37	1	15
$t(d)$	0	2	4	8	2	32	64	2	40	4	2	4	16	2	16384	48	2	16

Obviously, Theorem 1 reduces to that of Petruccelli and Woolford when $d = 1$. The ergodic region of (2) is shown in Figure 1(a) whereas that of Theorem 1 for $d = 2$ is in Figure 1(b). Comparing these two plots, we see clearly the effect of the delay parameter d.

The following result of Tweedie (1975) [see also Nummelin (1984)] will be used throughout the paper.

LEMMA 1.1. *Assume that* $\{X_t\}$ *is an aperiodic* ϕ*-irreducible Markov chain, and let* g *be a nonnegative measurable function. Then* $\{X_t\}$ *is geometrically ergodic if there exist a small set* K *with complement* K^c, $\varepsilon > 0$, $M < \infty$, $r > 1$, *such that*

$$rE\{g(X_{t+1})|X_t = x\} \le g(x) - \varepsilon, \qquad x \in K^c,$$
$$E\{g(X_{t+1})|X_t = x\} \le M, \qquad x \in K.$$

The proof of Theorem 1 also makes use of the following results: (a) By Chan and Tong (1985), the process $\{X_t\}$ is an aperiodic and μ_d-irreducible Markov chain where $X_t = (x_t, x_{t-1}, \dots, x_{t-d})'$ with x_t being the TAR(1) process in (1) and μ_d the Lebesgue measure on $\Re^d$; and (b) by Tjøstheim (1990), we have:

LEMMA 1.2. *If $\{X_t\}$ is an aperiodic Markov chain and h is a fixed positive integer, then*

$$\{X_{th}\} \text{ is } \begin{cases} \text{recurrent}, \\ \text{geometrically ergodic}, \\ \text{transient}, \end{cases} \Rightarrow \{X_t\} \text{ is } \begin{cases} \text{recurrent}, \\ \text{geometrically ergodic}, \\ \text{transient}. \end{cases}$$

Following the approach of Petruccelli and Woolford (1984), we also divide the condition and proof of Theorem 1 into four regions of (ϕ_1, ϕ_2). Section 2 proves the result for $\phi_1 \ge 0$, $\phi_2 \ge 0$; Section 3 deals with $\phi_1 \le 0$, $\phi_2 \le 0$; and Section 4 considers the case of ($\phi_1 > 0$, $\phi_2 < 0$) or ($\phi_1 < 0$, $\phi_2 > 0$). Combining all the conditions and results of these three sections, we obtain the general result of Theorem 1.

2. Conditions and proof for $\phi_1 \ge 0$ and $\phi_2 \ge 0$. In this case, the necessary and sufficient condition for the process x_t in (1) to be geometrically ergodic is $\phi_1 < 1$, $\phi_2 < 1$. The sufficient condition is based on Lemma 1.1 under the observation that the individual linear models for the two regimes $\{x_{t-d} \le 0\}$ and $\{x_{t-d} > 0\}$ are geometrically ergodic. Now we prove that the condition is also necessary. This is achieved by showing that when $\phi_1 \ge 1$ or $\phi_2 \ge 1$, the process is explosive. Without loss of generality, we consider the case $\phi_2 \ge 1$. First, if $\phi_2 > 1$, then there exists a real number η such that $1 < \eta < \phi_2$. Given $x_t = x > 0, x_{t-d+1} > 0$, we have, by Chebyshev's inequality, that

$$\begin{aligned} P\left(x_{t+1} \le \frac{1+\eta}{2} x_t \middle| x_t, x_{t-d+1}\right) &= P\left(\phi_2 x_t + \varepsilon_{t+1} \le \frac{1+\eta}{2} x_t \middle| x_t, x_{t-d+1}\right) \\ &= P\left(-\varepsilon_{t+1} \ge \left(\phi_2 - \frac{1+\eta}{2}\right) x_t \middle| x_t, x_{t-d+1}\right) \\ &\le P\left(|\varepsilon_{t+1}| \ge \left(\phi_2 - \frac{1+\eta}{2}\right) x \middle| x_t, x_{t-d+1}\right) \\ &\le \frac{E|\varepsilon_{t+1}|}{(\phi_2 - ((1+\eta)/2))x} \\ &\le \frac{2E|\varepsilon_{t+1}|}{(1-\eta)x}. \end{aligned}$$

Choosing an $M > 0$ such that $c = 2E|\varepsilon_{t+1}|/(1-\eta)M < 1$, then whenever $x_t > M$, $x_{t-d+1} > 0$, we have

$$P\left(x_{t+1} > \frac{1+\eta}{2}x_t \middle| x_t > M, x_{t-d+1} > 0\right) \geq 1 - c.$$

Noting that $(1+\eta)/2x_t > x_t > M$ and by repeating the above argument, we obtain that given $x_t > M$, $x_{t-d+2} > 0$, $x_{t-d+1} > 0$,

$$\begin{aligned} &P\left(x_{t+2} > \frac{1+\eta}{2}x_{t+1}, x_{t+1} > \frac{1+\eta}{2}x_t \middle| x_t > M, x_{t-d+2} > 0, x_{t-d+1} > 0\right) \\ &\quad \geq \left(1 - \frac{2}{1+\eta}c\right)(1-c). \end{aligned} \tag{3}$$

Let $\beta = 2/(1+\eta) < 1$ and note that

$$\begin{aligned} \sum_{t=1}^{\infty} \log(1 - c\beta^{t-1}) &= \sum_{t=1}^{\infty}\sum_{k=1}^{\infty}\left(-\frac{c^k\beta^{k(t-1)}}{k}\right) \\ &= \sum_{k=1}^{\infty}\left(-\frac{c^k}{k(1-\beta^k)}\right) \\ &\geq \sum_{k=1}^{\infty}\left(-\frac{c^k}{k(1-\beta)}\right) \\ &= \frac{\log(1-c)}{1-\beta}. \end{aligned}$$

The above result says that for $x_d > M$ and $x_i > 0$ where $i = 1, \ldots, d-1$, we have

$$\begin{aligned} &P\left(x_{t+1} > \frac{1+\eta}{2}x_t, t = d, \ldots, s+d \middle| x_d > M, x_i > 0, i = 1, \ldots, d\right) \\ &\quad \geq \prod_{t=1}^{s}(1 - c\beta^{t-1}) \geq (1-c)^{1/(1-\beta)}. \end{aligned}$$

Consequently, for any $x_k \in \Re^1$, $k = -d+1, \ldots, 0$,

$$\begin{aligned} &P(x_t \to \infty | x_k, k = -d+1, \ldots, 0) \\ &\quad \geq (1-c)^{1/(1-\beta)} \\ &\qquad \times P(x_d > M, x_i > 0, i = 1, \ldots, d-1 | x_k, k = -d+1, \ldots, 0) > 0. \end{aligned}$$

Hence, $\{x_t\}$ is not geometrically ergodic.

Next, consider the case $\phi_2 = 1$. Given $x_i > 0$, $i = 1, \ldots, d$, $x_t = x_{t-1} + \varepsilon_t$ is a random walk until the first time $x_{t-d} < 0$. From well-known results, for example, see Karlin and Taylor [(1975), pages 261–263], $E(T|x_i > 0, i = 1, \dots, d) = \infty$, where $T = \inf\{t > 0 | x_{t-d} \le 0\}$. Since $P(x_i > 0, i = 1, \dots, d | x_k, k = -d+1, \dots, 0) > 0$ for any $x_k \in \Re^1$, $k = 1-d, \dots, 0$, Theorem 7 of Tweedie (1974) implies that $\{x_t\}$ is not geometrically ergodic.

3. Conditions and proof for $\phi_1 \le 0$, $\phi_2 \le 0$. In this case the necessary and sufficient condition for the process x_t in (1) to be geometrically ergodic is $\phi_1\phi_2 < 1$.

SUFFICIENT CONDITION. Rewriting the TAR(1) model in (1) as

$$x_{t+1} = \phi_1 x_t I(x_{t-d+1} \le 0) + \phi_2 x_t I(x_{t-d+1} > 0) + \varepsilon_{t+1},$$

where $I(\cdot)$ is an indicator function which equals unity if its argument holds and is zero otherwise, we have

$$\begin{aligned}
x_{t+2} &= \phi_1 x_{t+1} I(x_{t-d+2} \le 0) + \phi_2 x_{t+1} I(x_{t-d+2} > 0) + \varepsilon_{t+2} \\
&= \phi_1^2 x_t I(x_{t-d+2} \le 0) I(x_{t-d+1} \le 0) \\
&\quad + \phi_1\phi_2 x_t I(x_{t-d+2} \le 0) I(x_{t-d+1} > 0) \\
&\quad + \phi_1\phi_2 x_t I(x_{t-d+2} > 0) I(x_{t-d+1} \le 0) \\
&\quad + \phi_2^2 x_t I(x_{t-d+2} > 0) I(x_{t-d+1} > 0) \\
&\quad + \phi_1 \varepsilon_{t+1} I(x_{t-d+2} \le 0) + \phi_2 \varepsilon_{t+1} I(x_{t-d+2} > 0) + \varepsilon_{t+2} \\
&= \phi_1\phi_2 x_t + (\phi_1^2 - \phi_1\phi_2) x_t I(x_{t-d+2} \le 0) I(x_{t-d+1} \le 0) \\
&\quad + (\phi_2^2 - \phi_1\phi_2) x_t I(x_{t-d+2} > 0) I(x_{t-d+1} > 0) \\
&\quad + \phi_1 \varepsilon_{t+1} I(x_{t-d+2} \le 0) + \phi_2 \varepsilon_{t+1} I(x_{t-d+2} > 0) + \varepsilon_{t+2} \\
&\equiv T_1 + T_2 + T_3 + T_4,
\end{aligned}$$

where T_4 is the sum of the last three terms. By using the same technique, we can further expand T_2 until it becomes a function of $x_{t-d+2} I(x_{t-d+2} \le 0) I(x_{t-d+1} \le 0)$. More specifically, we have

$$T_2 = f(\phi_1, \phi_2, x_{t-d-k} | k = 0, \dots, d+3) x_{t-d+2} I(x_{t-d+2} \le 0) I(x_{t-d+1} \le 0),$$

where $f(\phi_1, \phi_2, x_{t-d-k} | k = 0, \dots, d+3)$ is a function of ϕ_1 and ϕ_2 and some indicator functions of x_{t-d-k} for $k = 0, \dots, d+3$. For each t, let μ_{t-d+2} denote the distribution of the random pair $(x_{t-d+1}, \varepsilon_{t-d+2})$. Also, let A_i denote the set in $\Re^2$ satisfying

$$A_i = \{(x, \varepsilon) \colon x < 0, \varepsilon \le -\phi_i x\}, \qquad i = 1, 2.$$

Then for $\phi_1 < 0$ and $\phi_2 < 0$,

$$\begin{aligned}
&E|x_{t-d+2}I(x_{t-d+2} \le 0)I(x_{t-d+1} \le 0)| \\
&\quad \le E\big[|\phi_1 x_{t-d+1} + \varepsilon_{t-d+2}|I(\phi_1 x_{t-d+1} + \varepsilon_{t-d+2} \le 0) \\
&\qquad\qquad \times I(x_{t-d+1} \le 0)I(x_{t-2d+2} \le 0)\big] \\
&\qquad + E\big[|\phi_2 x_{t-d+1} + \varepsilon_{t-d+2}|I(\phi_2 x_{t-d+1} + \varepsilon_{t-d+2} \le 0) \\
&\qquad\qquad \times I(x_{t-d+1} \le 0)I(x_{t-2d+2} > 0)\big] \\
&\quad \le \int_{A_1} |\phi_1 x + \varepsilon|\mu_{t-d+2}(dx, d\varepsilon) + \int_{A_2} |\phi_2 x + \varepsilon|\mu_{t-d+2}(dx, d\varepsilon) \\
&\quad \le \int_{A_1} |\varepsilon|\mu_{t-d+2}(dx, d\varepsilon) + \int_{A_2} |\varepsilon|\mu_{t-d+2}(dx, d\varepsilon) \\
&\quad \le E|\varepsilon_{t-d+2}| + E|\varepsilon_{t-d+2}| \\
&\quad \le G,
\end{aligned}$$

where G is a fixed real number, which only depends on $E|\varepsilon_t|$, and we have used the results $-\phi_1 x_{t-d+1} < 0$ and $-\phi_2 x_{t-d+1} < 0$ when $x_{t-d+1} < 0$. Consequently, $E|T_2|$ is bounded. Similarly, we can prove $E|T_3|$ is also bounded. Furthermore, from the assumption on ε_t in (1), it is easy to see that $E|T_4| < \infty$. Therefore,

$$E\{|x_{t+2}||x_t = x\} \le \phi_1\phi_2|x| + \text{constant}.$$

From the above result, if $0 \le \phi_1\phi_2 < 1$, then there exist $M > 0$ and $0 < \eta < 1$ such that for $|x| > M$,

$$E\{|x_{t+2}||x_t = x\} \le \eta|x|.$$

Hence, given $x_{t-d+1} > M, \dots, x_t > M$,

$$E(\|X_{t+2}\| \, |X_t) < \eta\|X_t\|,$$

where $X_t = (x_t, x_{t-1})'$. Letting $g(X_t) = \|X_t\|$ and using Lemma 1.1, X_{2t} is geometrically ergodic and by Lemma 1.2, $\{X_t\}$ is geometrically ergodic.

Necessary Condition. (a) If $\phi_1 < 0$, $\phi_2 < 0$, $\phi_1\phi_2 > 1$, then there exists an η such that $1 < \eta < \phi_1\phi_2$. In what follows, we divide the proof into two cases depending on whether or not d is an even number.

When d is an even number, by using a similar method to that of Section 2, we can show that there exist $M > 0$ and $0 < c < 1$ such that given $x_t > M$, $x_{t-d+2} > 0$, $x_{t-d+1} < 0$,

$$P\left(x_{t+2} > \frac{1+\eta}{2}x_t \,\middle|\, x_t, x_{t-d+2}, x_{t-d+1}\right) > 1 - c$$

and, given $x_t < -M$, $x_{t-d+2} < 0$, $x_{t-d+1} > 0$,

$$P\left(x_{t+2} < \frac{1+\eta}{2}x_t \,\middle|\, x_t, x_{t-d+2}, x_{t-d+1}\right) > 1 - c.$$

Therefore, given $x_d > M > 0$, $(-1)^i x_i > 0$ for $i = 1, \dots, d-1$, we have

$$
\begin{aligned}
P\Bigg(& x_{2t+1+d} < \frac{1+\eta}{2} x_{2(t-1)+1+d} < 0, \\
& x_{2t+d} > \frac{1+\eta}{2} x_{2(t-1)+d} > 0, t = 1, \dots, s \Bigg| x_d, \dots, x_1 \Bigg) \\
& \geq \prod_{t=1}^{s} \left(1 - c\beta^{t-1}\right)^2 \\
& \geq (1-c)^{2/(1-\beta)},
\end{aligned}
$$

where $\beta = 2/(1+\eta) < 1$. Then for $x_k \in \Re^1$, $k = -d+1, \dots, 0$,

$$
\begin{aligned}
& P(|x_t| \to \infty | x_0, \dots, x_{-d+1}) \\
& \quad \geq (1-c)^{2/(1-\beta)} P\left(x_d > M, (-1)^i x_i > 0 | x_0, \dots, x_{-d+1}\right) > 0.
\end{aligned}
$$

Hence, $\{x_t\}$ is not geometrically ergodic. If d is odd, by choosing $x_t > M$, $x_{t-d+2} < 0$ and $x_{t-d+1} > 0$, we can prove the same result.

(b) For $\phi_1 < 0$, $\phi_2 < 0$, $\phi_1\phi_2 = 1$, we may assume, by the symmetry of ϕ_1 and ϕ_2, that $\phi_1 \leq -1$. First, consider the case that d is even. Given $x_{t-2} < 0$, $x_{t-4} < 0, \dots, x_{t-d} < 0$, we have

$$
\begin{aligned}
x_t &= \phi_1 x_{t-1} + \varepsilon_t \\
&= \phi_1^2 x_{t-2} I(x_{t-d-1} < 0) + \phi_1\phi_2 x_{t-2} I(x_{t-d-1} \geq 0) + \varepsilon_t + \phi_1 \varepsilon_{t-1} \\
&\leq \phi_1\phi_2 x_{t-2} I(x_{t-d-1} < 0) + \phi_1\phi_2 x_{t-2} I(x_{t-d-1} \geq 0) + \varepsilon_t + \phi_1 \varepsilon_{t-1} \\
&= x_{t-2} + \varepsilon_t + \phi_1 \varepsilon_{t-1} \quad \text{a.s.}
\end{aligned} \tag{4}
$$

Based on Equation (4), we define $y_i = x_{2i}$ for $i = 1, \dots, d/2$;

$$
y_t = y_{t-1} + \eta_t \quad \text{for } t > \frac{d}{2},
$$

where $\eta_t = \varepsilon_{2t} + \phi_1 \varepsilon_{2t-1}$ and

$$
T_2 = \inf\left\{ t | t > \frac{d}{2}, y_t \geq 0 \right\}.
$$

Since $\{\eta_t\}$ is a sequence of iid random variables, y_t is a random walk for $t > d/2$ and the stopping time T_2 satisfies $E(T_2) = \infty$. Let

$$
T_1 = \inf\left\{ t | t > \frac{d}{2}, x_{2t} \geq 0 \right\}.
$$

Then, given $x_2 < 0$, $x_4 < 0, \dots, x_d < 0$, it follows from the inequality in (4) that $\{T_2 > n\} \subset \{T_1 > n\}$ and, hence, $E(T_1) \geq E(T_2) = \infty$. Since for any initial values x_i, $i = 1-d, \dots, 0$,

$$
P\left(x_2 < 0, x_4 < 0, \dots, x_d < 0 | x_{1-d}, \dots, x_0\right) > 0,
$$

by Tweedie (1974), the subsequence $\{x_{2t}\}$ is not ergodic and, by Lemma 1.2, neither is the process $\{x_t\}$. Next, consider the case that d is odd. If $d = 1$, then $\{x_t\}$ is not ergodic; see Petruccelli and Woolford (1984). For $d > 1$ and given $x_{t-2} < 0, x_{t-4} < 0, \dots, x_{t-d-1} < 0$, we have

$$\begin{aligned} x_t &= \phi_1 x_{t-1} I(x_{t-d} < 0) + \phi_2 x_{t-1} I(x_{t-d} \geq 0) + \varepsilon_t \\ &= \phi_1(\phi_1 x_{t-2} + \varepsilon_{t-1}) I(x_{t-d} < 0) + \phi_2(\phi_1 x_{t-2} + \varepsilon_{t-1}) I(x_{t-d} \geq 0) + \varepsilon_t \\ &\leq x_{t-2} I(x_{t-d} < 0) + x_{t-2} I(x_{t-d} \geq 0) + \varepsilon_t \\ &\quad + [\phi_1 I(x_{t-d} < 0) + \phi_2 I(x_{t-d} \geq 0)]\varepsilon_{t-1} \\ &= x_{t-2} + \varepsilon_t + [\phi_1 I(x_{t-d} < 0) + \phi_2 I(x_{t-d} \geq 0)]\varepsilon_{t-1} \quad \text{a.s.} \end{aligned} \tag{5}$$

Define $y_i = x_{2i}$, $i = 1, \dots, (d+1)/2$ and

$$y_t = y_{t-1} + \eta_t \quad \text{for} \quad t > \frac{d+1}{2},$$

where $\eta_t = \varepsilon_{2t} + \varepsilon_{2t-1} z_t$ with $z_t = \phi_1 I(x_{2t-d} < 0) + \phi_2 I(x_{2t-d} \geq 0)$. For $d > 1$, since ε_{2t} and ε_{2t-1} are independent of y_{t-i} for $i \geq 1$ and z_{t-i} for $i \geq 0$, it is easily seen that

$$\begin{aligned} E[y_t | y_{t-1}, \dots, y_1] &= E[y_{t-1} + \eta_t | y_{t-1}, \dots, y_1] \\ &= y_{t-1} + E[\varepsilon_{2t} + \varepsilon_{2t-1} z_t | y_{t-1}, \dots, y_1] \\ &= y_{t-1} + E[\varepsilon_{2t}] + E[\varepsilon_{2t-1}] E[z_t | y_{t-1}, \dots, y_1] \\ &= y_{t-1}, \end{aligned}$$

which says that $\{y_t\}$ is a martingale. Let

$$T_2 = \inf\left\{ t | t > \frac{d+1}{2}, y_t \geq 0 \right\}, \qquad T_1 = \inf\left\{ t | t > \frac{d+1}{2}, x_{2t} \geq 0 \right\}.$$

For $t < T_2$,

$$\begin{aligned} &E[|y_{t+1} - y_t| | y_1, \dots, y_t] \\ &\quad = E[|\eta_t| | y_1, \dots, y_t] \\ &\quad \leq E|\varepsilon_{2t+2}| + E|\varepsilon_{2t+1}| E[|\phi_1 I(x_{2t-d} < 0) + \phi_2 I(x_{2t-d} \geq 0)| | y_1, \dots, y_t] \\ &\quad \leq E|\varepsilon_{2t+2}| + (|\phi_1| + |\phi_2|) E|\varepsilon_{2t+1}| \\ &\quad \leq K. \end{aligned}$$

By Corollary 6.3.1 of Karlin and Taylor (1975), if $E(T_2) < \infty$, we have $E(y_{T_2}) = E(y_1)$. This, however, is impossible because, by definition, $y_{T_2} \geq 0$ and, by assumption, $y_1 = x_2 < 0$. Consequently, $E(T_2) = \infty$. Given $x_{2i} < 0$, $i = 1, \dots, (d+1)/2$, by the inequality in (5), we have $x_{2t} \leq y_t < 0$ for $i = 1, \dots, T_2 - 1$. Hence, $\{T_2 > n\} \subset \{T_1 > n\}$ and $E(T_1) \geq E(T_2) = \infty$. Again, by the same argument as that of d being even, $\{x_t\}$ is not ergodic.

4. The case of $\phi_1 > 0, \phi_2 < 0$ or $\phi_1 < 0, \phi_2 > 0$. In this case, we need some preliminaries. Let Ω be the set of all d-dimensional binary strings, that is,

$$\Omega = \{\mathbf{a} = (a_1, \dots, a_d) | a_i = 0 \text{ or } 1 \text{ for } i = 1, \dots, d\}.$$

Define a mapping $f: \Omega \to \Omega$ by

$$(6) \quad f(a_1, \dots, a_d) = \left(a_1 + a_d, \dots, \sum_{i=1}^{k} a_i + a_d, \dots, \sum_{i=1}^{d} a_i + a_d \right) \Bigg|_{(\mathrm{mod}\, 2)}.$$

LEMMA 4.1. *The mapping f in* (6) *is one-to-one and onto.*

PROOF. This lemma follows directly from the fact that

$$f^{-1}(a_1, \dots, a_d) = (a_1 + a_{d-1} + a_d, a_1 + a_2, \dots, a_{d-1} + a_d)|_{(\mathrm{mod}\, 2)}. \quad \square$$

LEMMA 4.2. $\forall\, \mathbf{a} = (a_1, \dots, a_d) \in \Omega$, $\exists\, k$, *such that* $f^k(\mathbf{a}) = \mathbf{a}$.

PROOF. Since Ω is a finite set with 2^d elements and $\{\mathbf{a}, f(\mathbf{a}), f^2(\mathbf{a}), \dots, f^{2^d}(\mathbf{a})\}$ is a subset of Ω, we can find $k_1 > k_2$ such that $f^{k_1}(\mathbf{a}) = f^{k_2}(\mathbf{a})$. Since f^{-1} exists, we have $f^{k_1 - k_2}(\mathbf{a}) = f^k(\mathbf{a}) = \mathbf{a}$, where $k = k_1 - k_2$. $\square$

For each element $\mathbf{a}$ in Ω, define

$$(7) \qquad l(\mathbf{a}) = \min\{k > 0 | f^k(\mathbf{a}) = \mathbf{a}\}.$$

By Lemma 4.2, $l(\mathbf{a})$ exists for every $\mathbf{a} \in \Omega$. Next, define a cycle in Ω starting with $\mathbf{a}$ by

$$(8) \qquad C(\mathbf{a}) = \{\mathbf{a}, f(\mathbf{a}), \dots, f^{l(\mathbf{a})-1}(\mathbf{a})\}.$$

Obviously, $l(\mathbf{a})$ is the number of elements in $C(\mathbf{a})$. Finally, for each element $\mathbf{a}$ of Ω, define

$$(9) \qquad \begin{aligned} \beta(\mathbf{a}) &= \sum_{i=1}^{d} a_i, & \alpha(\mathbf{a}) &= d - \beta(\mathbf{a}), \\ \mathscr{A}(\mathbf{a}) &= \sum_{\mathbf{b} \in C(\mathbf{a})} \alpha(\mathbf{b}), & \mathscr{B}(\mathbf{a}) &= \sum_{\mathbf{b} \in C(\mathbf{a})} \beta(\mathbf{b}). \end{aligned}$$

Note that $\alpha(\mathbf{a})$ and $\beta(\mathbf{a})$ are, respectively, the number of 0's and 1's in the string $\mathbf{a}$.

LEMMA 4.3. *For the mapping f in* (6) *and elements of Ω, the following results hold*:

(i) *If there exists a positive integer k such that $\mathbf{b} = f^k(\mathbf{a})$, then $\mathbf{b} \in C(\mathbf{a})$.*
(ii) *A cycle is an equivalence class in Ω, that is,* (a) *if $\mathbf{a}$ and $\mathbf{b}$ are in the same cycle, so are $\mathbf{b}$ and $\mathbf{a}$,* (b) *if $\mathbf{a}$ and $\mathbf{b}$ are in the same cycle, $\mathbf{b}$ and $\mathbf{c}$ are in the same cycle, so are $\mathbf{a}$ and $\mathbf{c}$.*
(iii) $\mathscr{A}(\mathbf{a}) = \mathscr{A}(\mathbf{b})$ *and* $\mathscr{B}(\mathbf{a}) = \mathscr{B}(\mathbf{b})$ *if* $\mathbf{b} \in C(\mathbf{a})$.
(iv) $\mathscr{B}(\mathbf{a})$ *is an even number for any* $\mathbf{a} \in \Omega$.

PROOF. Part (iii) is trivial. For part (i), write $k = m \times l(\mathbf{a}) + n$, where m is a nonnegative integer and $0 \le n < l(\mathbf{a})$. Then $\mathbf{b} = f^k(\mathbf{a}) = f^n(\mathbf{a}) \in C(\mathbf{a})$. For part (ii), if $\mathbf{b} \in C(\mathbf{a})$, then there exists a k satisfying $0 \le k < l(\mathbf{a})$ such that $f^k(\mathbf{a}) = \mathbf{b}$. Next, for simplicity, write $l(\mathbf{a})$ as l. Then, $\mathbf{a} = f^l(\mathbf{a}) = f^{l-k}[f^k(\mathbf{a})] = f^{l-k}(\mathbf{b})$. By part (i), $\mathbf{a} \in C(\mathbf{b})$. Furthermore, it is easy to show that if $\mathbf{c} \in C(\mathbf{b})$ and $\mathbf{b} \in C(\mathbf{a})$, then $\mathbf{c} \in C(\mathbf{a})$. Consequently, a cycle is an equivalence class. To prove part (iv), let $\mathbf{a}^{(i)} = f^i(\mathbf{a})$, then

$$a_d^{(1)} = \sum_{j=1}^{d} a_j + a_d \pmod 2 = \beta(\mathbf{a}) + a_d \pmod 2$$

and

$$a_d^{(2)} = \sum_{j=1}^{d} a_j^{(1)} + a_d^{(1)} \pmod 2 = \beta(\mathbf{a}^{(1)}) + \beta(\mathbf{a}) + a_d \pmod 2.$$

In general, we obtain

$$a_d^{(m)} = \sum_{k=1}^{m-1} \beta(\mathbf{a}^{(k)}) + \beta(\mathbf{a}) + a_d \pmod 2.$$

Again, write $l(\mathbf{a}) = l$. Since $f^l(\mathbf{a}) = \mathbf{a}$, we have $\mathbf{a}^{(l)} = \mathbf{a}$ and, in particular,

$$\begin{aligned} a_d &= a_d^{(l)} \\ &= \sum_{k=1}^{l-1} \beta(\mathbf{a}^{(k)}) + \beta(\mathbf{a}) + a_d \pmod 2 \\ &= \mathscr{B}(\mathbf{a}) + a_d \pmod 2. \end{aligned}$$

Therefore, $\mathscr{B}(\mathbf{a}) = 0 \pmod 2$, which says that $\mathscr{B}(\mathbf{a})$ is an even number. □

For the TAR(1) process x_t in (1) and an element $\mathbf{a} = (a_1, \dots, a_d) \in \Omega$, define

$$(10) \quad J(a_i, t) = I\big((-1)^{a_i+1} x_t > 0\big) \quad \text{and} \quad L(\mathbf{a}, t) = \prod_{i=1}^{d} J(a_i, t - d + i),$$

where the $I(\cdot)$ is the usual indicator function defined before. Also, for given $x_{t-d+1}, \dots, x_t$, define a mapping

$$(11) \qquad \gamma: (x_{t-d+1}, \dots, x_t) \to \mathbf{a} = (a_1 \cdots a_d) \quad \text{with} \quad a_i = I(x_{t-d+i} > 0).$$

LEMMA 4.4. *For the* TAR(1) *process in* (1), *given* $x_{t-d+1}, \dots, x_t$ *and* $\mathbf{a} = \gamma(x_{t-d+1}, \dots, x_t)$, *where* $\gamma(\cdot)$ *is defined in* (11), *we have*

$$x_{t+nd} = \phi_1^{\alpha(\mathbf{a}) + \Sigma_{i=1}^{n-1} \alpha(f^i(\mathbf{a}))} \phi_2^{\beta(\mathbf{a}) + \Sigma_{i=1}^{n-1} \beta(f^i(\mathbf{a}))} x_t + R_1(n, d, x_t, \varepsilon_t)$$

and

$$x_{t+nd} = \phi_1^{\alpha(\mathbf{a}) + \Sigma_{i=1}^{n-1} \alpha(f^i(\mathbf{a}))} \phi_2^{\beta(\mathbf{a}) + \Sigma_{i=1}^{n-1} \beta(f^i(\mathbf{a}))} x_t \prod_{i=1}^{n-1} L(f^i(\mathbf{a}), t + id)$$
$$+ R_2(n, d, x_t, \varepsilon_t),$$

where the $R_i(n, d, x_t, \varepsilon_t)$*'s denote the remainders which satisfy*

$$E|R_i(n, d, x_t, \varepsilon_t)| < M$$

with M *being a positive real number depending only on* n, d *and* $E|\varepsilon_t|$.

The implication of the above lemma is as follows. Given $x_{t-d+1}, \dots, x_t$, there is a certain situation in which the values of the x_{t+i}'s could go to infinity without proper conditions on ϕ_1 and ϕ_2; for the other situations, the expectations of the absolute values of the x_{t+i}'s are bounded by a constant that only depends on i, d and $E|\varepsilon_t|$. Consequently, we need only consider the first situation in order to understand the ergodicity of x_t.

PROOF OF LEMMA 4.4. For simplicity, we shall only prove the lemma for $n = 2$, but the same method applies to the general n. For any element $\mathbf{a} = (a_1, \dots, a_d)$ of Ω, define

$$(12)\quad a(0) = 0, a(i, 0) = 0, a(i) = \sum_{k=1}^{i} a_k \quad \text{and} \quad a(i, j) = \sum_{k=i-j+1}^{i} a_k,$$

where $i = 1, \dots, d$ and $1 \le j \le i$. By (9), we have $a(d) = \beta(\mathbf{a})$ and $d - a(d) = \alpha(\mathbf{a})$. Given $x_{t-d+1}, \dots, x_t$, and letting $\mathbf{a} = \gamma(x_{t-d+1}, \dots, x_t)$, some straightforward algebra shows that

$$x_{t+i} = \phi_1^{i-a(i)} \phi_2^{a(i)} x_t + \sum_{j=0}^{i-1} \phi_1^{j-a(i,j)} \phi_2^{a(i,j)} \varepsilon_{t+i-j} \quad \text{for } i = 1, 2, \dots, d.$$

In particular, we have

$$x_{t+d} = \phi_1^{\alpha(\mathbf{a})} \phi_2^{\beta(\mathbf{a})} x_t + \sum_{j=0}^{d-1} \phi_1^{j-a(d,j)} \phi_2^{a(d,j)} \varepsilon_{t+d-j}.$$

Similarly, given $\mathbf{c} = \gamma(x_{t+1}, \dots, x_{t+d})$, we have

$$x_{t+2d} = \phi_1^{\alpha(\mathbf{c})} \phi_2^{\beta(\mathbf{c})} x_{t+d} + \sum_{j=0}^{d-1} \phi_1^{j-c(d,j)} \phi_2^{c(d,j)} \varepsilon_{t+2d-j}.$$

Therefore, given $\mathbf{a} = \gamma(x_{t-d+1}, \dots, x_t)$ and using the notation in (10), we have

$$\begin{aligned}
x_{t+2d} &= \phi_1^{\alpha(\mathbf{a})}\phi_2^{\beta(\mathbf{a})} \sum_{\mathbf{c}\in\Omega} \phi_1^{\alpha(\mathbf{c})}\phi_2^{\beta(\mathbf{c})} x_t L(\mathbf{c}, t+d) \\
&\quad + \sum_{\mathbf{c}\in\Omega}\Bigg[\sum_{j=0}^{d-1} \phi_1^{\alpha(\mathbf{c})+j-a(d,j)}\phi_2^{\beta(\mathbf{c})+a(d,j)}\varepsilon_{t+d-j} \\
&\qquad\qquad + \sum_{j=0}^{d-1}\phi_1^{j-c(d,j)}\phi_2^{c(d,j)}\varepsilon_{t+2d-j}\Bigg] L(\mathbf{c}, t+d) \\
&= \phi_1^{\alpha(\mathbf{a})+\alpha(f(\mathbf{a}))}\phi_2^{\beta(\mathbf{a})+\beta(f(\mathbf{a}))} x_t L(f(\mathbf{a}), t+d) \\
&\quad + \phi_1^{\alpha(\mathbf{a})}\phi_2^{\beta(\mathbf{a})} \sum_{\mathbf{c}\in\Omega,\, \mathbf{c}\neq f(\mathbf{a})} \phi_1^{\alpha(\mathbf{c})}\phi_2^{\beta(\mathbf{c})} x_t L(\mathbf{c}, t+d) \qquad (13)\\
&\quad + \sum_{\mathbf{c}\in\Omega}\Bigg[\sum_{j=0}^{d-1} \phi_1^{\alpha(\mathbf{c})+j-a(d,j)}\phi_2^{\beta(\mathbf{c})+a(d,j)}\varepsilon_{t+d-j} \\
&\qquad\qquad + \sum_{j=0}^{d-1}\phi_1^{j-c(d,j)}\phi_2^{c(d,j)}\varepsilon_{t+2d-j}\Bigg] L(\mathbf{c}, t+d) \\
&\equiv \phi_1^{\alpha(\mathbf{a})+\alpha(f(\mathbf{a}))}\phi_2^{\beta(\mathbf{a})+\beta(f(\mathbf{a}))} x_t L(f(\mathbf{a}), t+d) + R_2(2, d, x_t, \varepsilon_t).
\end{aligned}$$

Alternatively, (13) can also be written as

$$\begin{aligned}
x_{t+2d} &= \phi_1^{\alpha(\mathbf{a})+\alpha(f(\mathbf{a}))}\phi_2^{\beta(\mathbf{a})+\beta(f(\mathbf{a}))} x_t \\
&\quad + \phi_1^{\alpha(\mathbf{a})}\phi_2^{\beta(\mathbf{a})} \sum_{\mathbf{c}\in\Omega,\, \mathbf{c}\neq f(\mathbf{a})} \left(\phi_1^{\alpha(\mathbf{c})}\phi_2^{\beta(\mathbf{c})} - \phi_1^{\alpha(f(\mathbf{a}))}\phi_2^{\beta(f(\mathbf{a}))}\right) x_t L(\mathbf{c}, t+d) \\
&\quad + \sum_{\mathbf{c}\in\Omega}\Bigg[\sum_{j=0}^{d-1} \phi_1^{\alpha(\mathbf{c})+j-a(d,j)}\phi_2^{\beta(\mathbf{c})+a(d,j)}\varepsilon_{t+d-j} \\
&\qquad\qquad + \sum_{j=0}^{d-1}\phi_1^{j-c(d,j)}\phi_2^{c(d,j)}\varepsilon_{t+2d-j}\Bigg] L(\mathbf{c}, t+d) \\
&\equiv \phi_1^{\alpha(\mathbf{a})+\alpha(f(\mathbf{a}))}\phi_2^{\beta(\mathbf{a})+\beta(f(\mathbf{a}))} x_t + R_1(2, d, x_t, \varepsilon_t).
\end{aligned}$$

Next we prove that $E|R_1(2, d, x_t, \varepsilon_t)|$ is bounded, depending only on d and $E|\varepsilon_t|$. Write $\mathbf{b} = (b_1, \dots, b_d) = f(\mathbf{a})$ and let

$$\begin{aligned}
A(i) &= \sum_{\mathbf{c}\in\Omega,\, c_i\neq b_i,\, c_k = b_k,\, k=1,\dots,i-1} \phi_1^{\alpha(\mathbf{a})}\phi_2^{\beta(\mathbf{a})} \\
&\qquad \times\left(\phi_1^{\alpha(\mathbf{c})}\phi_2^{\beta(\mathbf{c})} - \phi_1^{\alpha(\mathbf{b})}\phi_2^{\beta(\mathbf{b})}\right) x_t L(\mathbf{c}, t+d), \\
Q(\varepsilon) &= \sum_{\mathbf{c}\in\Omega}\Bigg[\sum_{j=1}^{d} \phi_1^{\alpha(\mathbf{c})+j-a(d,j)}\phi_2^{\beta(\mathbf{c})+a(d,j)}\varepsilon_{t+d-j} \\
&\qquad\qquad + \sum_{j=1}^{d}\phi_1^{j-c(d,j)}\phi_2^{c(d,j)}\varepsilon_{t+2d-j}\Bigg] L(\mathbf{c}, t+d) + \varepsilon_{t+2d}.
\end{aligned}$$

Then

$$R_1(2, d, x_t, \varepsilon_t) = \sum_{i=1}^{d} A(i) + Q(\varepsilon).$$

Letting $Z(\mathbf{c}) = \phi_1^{\alpha(\mathbf{c})}\phi_2^{\beta(\mathbf{c})} - \phi_1^{\alpha(\mathbf{b})}\phi_2^{\beta(\mathbf{b})}$, since $\beta(\mathbf{a}) = a(d, d-i) + a(i)$ and $\alpha(\mathbf{a}) = d - \beta(a)$ we have

$$\begin{aligned}
A(i) &= \sum \phi_1^{(d-i)-a(d,d-i)}\phi_2^{a(d,d-i)} Z(\mathbf{c})\big(\phi_1^{i-a(i)}\phi_2^{a(i)}x_t\big)L(\mathbf{c}, t+d) \\
&= \sum \phi_1^{(d-i)-a(d,d-i)}\phi_2^{a(d,d-i)} Z(\mathbf{c}) x_{t+i} L(\mathbf{c}, t+d) \\
&\quad - \sum \phi_1^{(d-i)-a(d,d-i)}\phi_2^{a(d,d-i)} Z(\mathbf{c}) \\
&\quad \times \left(\sum_{j=0}^{i} \phi_1^{j-a(i,j)}\phi_2^{a(i,j)}\varepsilon_{t+i-j} \right) L(\mathbf{c}, t+d),
\end{aligned}$$

where Σ is summing over $\mathbf{c}$ such that $\mathbf{c} \in \Omega$, $c_i \neq b_i$, $c_k = b_k$, $k = 1, \dots, i-1$.

From the definition of $f(\cdot)$ in (6),

$$\begin{aligned}
b_i &= \sum_{k=1}^{i} a_i + a_d \pmod 2 \\
&= b_{i-1} + a_i \pmod 2,
\end{aligned}$$

so that $c_i \neq b_i$ has four possible cases:

(i) $a_i = 0$, $b_{i-1} = 0$ and $c_i = 1$.
(ii) $a_i = 0$, $b_{i-1} = 1$ and $c_i = 0$.
(iii) $a_i = 1$, $b_{i-1} = 0$ and $c_i = 0$.
(iv) $a_i = 1$, $b_{i-1} = 1$ and $c_i = 1$.

For case (i), since $a_i = 0$, we have $x_{t-d+i} \leq 0$ and, hence,

$$x_{t+i} = \phi_1 x_{t+i-1} + \varepsilon_{t+i}.$$

From $b_{i-1} = 0$, we have $x_{t+i-1} \leq 0$, and from $c_i = 1$, we have $x_{t+i} > 0$. Finally, since $\phi_1 > 0$ and by the assumption of model (1), we obtain

$$\begin{aligned}
&E\big[|x_{t+i}| I(x_{t+i} > 0) I(x_{t+i-1} \leq 0) I(x_{t+i-d} \leq 0)\big] \\
&\leq \int_A |\phi_1 x + \varepsilon| \mu_{t+i}(dx, d\varepsilon) \\
&\leq \int |\varepsilon| \mu_{t+i}(dx, d\varepsilon) \\
&= E|\varepsilon_t|,
\end{aligned}$$

where $A = \{(x, \varepsilon) \colon x < 0,\ \varepsilon > -\phi_1 x\}$ and μ_{t+i} is defined in Section 3.

By using the same argument, we can prove the bounded result for the cases (ii)–(iv). In other words, we have shown that for any $\mathbf{c} \in \Omega$, if $c_i \neq b_i$, and $c_k = b_k$, $k = 1, \dots, i-1$, then $E|x_{t+i}L(\mathbf{c}, t+d)|$ is bounded. Since $A(i)$ is a linear combination of finite terms of $x_{t+i}L(\mathbf{c}, t+d)$, we obtain that $E|A(i)|$ is bounded. Moreover, it is easy to see that $E|Q(\varepsilon)|$ is bounded because $Q(\varepsilon)$ is a linear combination of finite terms of ε_t's. Hence $E|R_1(2, d, x_t, \varepsilon_t)|$ is bounded. Similarly, we can prove that $E|R_2(2, d, x_t, \varepsilon_t)|$ is bounded. □

LEMMA 4.5. *Given* $x_{t-d+1}, \dots, x_t$, *define* $\mathbf{a} = \gamma(x_{t-d+1}, \dots, x_t)$ *by* (11) *and write* $l = l(\mathbf{a})$ *of* (7). *For* $1 \le k < ld$, *write* $k = id + k_1$ *with* $0 \le k_1 < d$ *and define* $\mathbf{b}^{(k)} = (a_{k_1+1}^{(i)}, \dots, a_d^{(i)}, a_1^{(i+1)}, \dots, a_{k_1}^{(i+1)})$ *if* $k_1 \ne 0$ *and* $\mathbf{b}^{(k)} = \mathbf{a}^{(i)}$ *if* $k_1 = 0$, *where* $\mathbf{a}^{(i)} = f^i(\mathbf{a})$ *and* $\mathbf{a}^{(0)} = \mathbf{a}$. *Then for* $1 \le k < ld$,

$$x_{t+k+ld} = \phi_1^{\mathscr{A}(\mathbf{a})}\phi_2^{\mathscr{B}(\mathbf{a})}x_{t+k}\prod_{i=1}^{l-1} L(f^i(\mathbf{b}), t + k + id) + R^*(k, d, x_t, \varepsilon_t),$$

where $\mathscr{A}(\mathbf{a})$ *and* $\mathscr{B}(\mathbf{a})$ *are defined in* (9) *and the remainder* $R^*(k, d, x_t, \varepsilon_t)$ *has a bounded absolute expectation, and* $f^l(\mathbf{b}^{(k)}) = \mathbf{b}^{(k)}$.

PROOF. Since we can treat $t + k - 1$ as t, it suffices to prove the lemma for $k = 1$. Let $x_{t-d+1}, \dots, x_t$ be given, and $\mathbf{a} = \gamma(x_{t-d+1}, \dots, x_t)$. By Lemma 4.4 and since $L(\mathbf{a}, t) = 1$ for given $x_{t-d+1}, \dots, x_t$,

$$\begin{aligned} x_{t+ld} &= \phi_1^{\mathscr{A}(\mathbf{a})}\phi_2^{\mathscr{B}(\mathbf{a})}x_t\prod_{i=1}^{l-1} L(f^i(\mathbf{a}), t + id) + R_2(l, d, x_t, \varepsilon_t) \\ &= \phi_1^{\mathscr{A}(\mathbf{a})}\phi_2^{\mathscr{B}(\mathbf{a})}x_t\prod_{i=0}^{l-1} L(f^i(\mathbf{a}), t + id) + R_2(l, d, x_t, \varepsilon_t) \end{aligned} \tag{14}$$

and by the model in (1),

$$x_{t+1+ld} = \phi_1 x_{t+ld} I(x_{t+1+(l-1)d} \le 0) + \phi_2 x_{t+ld} I(x_{t+1+(l-1)d} > 0) + \varepsilon_{t+1+ld}.$$

Let $c_1 = I(x_{t+1+(l-1)d} > 0)$ and $\omega_t = t + 1 + (l - 1)d$. The above equation can be written as

$$x_{t+1+ld} = \left[\phi_1^{c_1}\phi_2^{1-c_1}J(1 - c_1, \omega_t) + \phi_1^{1-c_1}\phi_2^{c_1}J(c_1, \omega_t)\right]x_{t+ld} + \varepsilon_{t+1+ld}.$$

We prove next that one of the first two terms on the right-hand side of the above equation has a bounded absolute expectation.

Since one of c_1 and $1 - c_1$ is not equal to a_1, and c_1 and $1 - c_1$ play a symmetric role in determining the dynamic of the process, we can assume, without loss of generality, that $c_1 \ne a_1$. We then have

$$x_{t+1+ld} = \left[\phi_1^{1-a_1}\phi_2^{a_1}J(a_1, \omega_t) + \phi_1^{1-c_1}\phi_2^{c_1}J(c_1, \omega_t)\right]x_{t+ld} + \varepsilon_{t+1+ld}. \tag{15}$$

Further, we can expand $x_{t+ld}J(c_1, \omega_t)$ as a linear combination of finite terms in the form of

$$x_{t+(l-1)d}J(c_1, \omega_t)J\left(a_d^{(l-1)}, \omega_t - 1\right)J\left(a_1^{(l-1)}, \omega_t - d\right)$$

plus a remainder that has a bounded absolute expectation. Using the same argument as in the proof of Lemma 4.4, we can prove that

$$E|x_{t+(l-1)d}J(c_1, \omega_t)J\left(a_d^{(l-1)}, \omega_t - 1\right)J\left(a_1^{(l-1)}, \omega_t - d\right)|$$

is bounded under the condition that $c_1 \ne a_d^{(l-1)} + a_1^{(l-1)}$. Consequently,

$E|x_{t+ld}J(c_1, \omega_t)|$ is bounded. Let

$$R^+ = \phi_1^{1-c_1}\phi_2^{c_1}x_{t+ld}J(c_1, \omega_t) + \varepsilon_{t+1+ld}.$$

The above results show that $E|R^+|$ is bounded. By (14) and (15),

$$\begin{aligned} x_{t+1+ld} &= \phi_1^{1-a_1}\phi_1^{a_1}x_{t+ld}J(a_1, \omega_t) + R^+ \\ &= \phi_1^{1-a_1}\phi_2^{a_1}\phi_1^{\mathscr{A}(\mathbf{a})}\phi_2^{\mathscr{B}(\mathbf{a})}x_tJ(a_1, \omega_t)\prod_{i=0}^{l-1}L(f^i(\mathbf{a}), t + id) + R^{**}, \end{aligned}$$

where

$$R^{**} = \phi_1^{1-a_1}\phi_2^{a_1}R_2(l, d, \varepsilon_t)J(a_1, \omega_t) + R^+,$$

which, based on the results shown above, satisfies $E|R^{**}| < \infty$. Next, from

$$x_{t+1} = \phi_1^{1-a_1}\phi_2^{a_1}x_tJ(a_1, t - d + 1) + \varepsilon_{t+1},$$

and by (10), we have

$$\begin{aligned} x_{t+ld+1} = {} & \phi_1^{\mathscr{A}(\mathbf{a})}\phi_2^{\mathscr{B}(\mathbf{a})}x_{t+1}\prod_{j=2}^{d}J(a_j, t - d + j)\prod_{i=1}^{l-1}L(f^i(\mathbf{a}), t + id)J(a_1, \omega_t) \\ & + R^*(1, d, x_t, \varepsilon_t). \end{aligned}$$

Note that $a_d^{(l-1)} + a_1^{(l-1)} = a_1^l = a_1$. By (10) and rearranging $J(a_j^{(i)}, \cdot\,)$ terms, we obtain

$$\begin{aligned} &\prod_{j=2}^{d}J(a_j, t - d + j)\prod_{i=1}^{l-1}L(f^i(\mathbf{a}), t + id)J(a_1, \omega_t) \\ &\quad = \prod_{j=2}^{d}J(a_j, t - d + j)\left[\prod_{i=1}^{l-1}\prod_{j=1}^{d}J\left(a_j^{(i)}, t - (i - 1)d + j\right)\right]J(a_1, \omega_t) \\ &\quad = \left[\left(\prod_{j=2}^{d}J(a_j, t - d + j)\right)J\left(a_1^{(1)}, t + 1\right)\right]\cdots \\ &\quad\quad \times\left[\left(\prod_{j=2}^{d}J\left(a_j^{(l-1)}, t - (l - 2)d + j\right)\right)J\left(a_1^{(l)}, \omega_t\right)\right] \\ &\quad = \prod_{i=1}^{l-1}L(f^i(\mathbf{b}), t + 1 + id), \end{aligned}$$

where $\mathbf{b} = (a_2, \ldots, a_d, a_1^{(1)})$. So

$$x_{t+ld+1} = \phi_1^{\mathscr{A}(\mathbf{a})}\phi_2^{\mathscr{B}(\mathbf{a})}x_{t+1}\prod_{i=1}^{l-1}L(f^i(\mathbf{b}), t + 1 + id) + R^*(1, d, x_t, \varepsilon_t)$$

and

$$R^*(1, d, x_t, \varepsilon_t) = R^{**} - \phi_1^{\mathscr{A}(\mathbf{a})}\phi_2^{\mathscr{B}(\mathbf{a})}\varepsilon_{t+1}J(a_1, \omega_t)\prod_{i=1}^{l-1}L(f^i(\mathbf{a}), t + id),$$

which has a bounded expectation. □

Using the previous lemmas, we establish a theorem.

THEOREM 2. *Consider the* TAR(1) *process in* (1). *Let n be the number of cycles in Ω and $\mathbf{a}_1, \dots, \mathbf{a}_n$ be a set of representatives of the n cycles. Also, let m be the smallest positive integer satisfying*

$$\frac{\mathscr{A}(\mathbf{a}_m)}{\mathscr{B}(\mathbf{a}_m)} = \min_{i=1,\dots,n} \left\{ \frac{\mathscr{A}(\mathbf{a}_i)}{\mathscr{B}(\mathbf{a}_i)} \right\}.$$

Then the process x_t of (1) *with $\phi_1 > 0$ and $\phi_2 < 0$ is geometrically ergodic if and only if*

$$\phi_1 < 1 \quad and \quad \phi_1^{\mathscr{A}(\mathbf{a}_m)} \phi_2^{\mathscr{B}(\mathbf{a}_m)} < 1.$$

PROOF. We begin with the sufficiency. To simplify the notation, write $C(i) = C(\mathbf{a}_i)$, $l(i) = l(\mathbf{a}_i)$, $\mathscr{A}(i) = \mathscr{A}(\mathbf{a}_i)$, $\mathscr{B}(i) = \mathscr{B}(\mathbf{a}_i)$. Let l be the least common multiplier of $(l(1), \dots, l(n))$. Then, we have

$$f^l(\mathbf{a}) = \mathbf{a}, \qquad \forall\, \mathbf{a} \in \Omega.$$

Given $x_{t-d+1}, \dots, x_t$, let $\mathbf{a} \in \Omega$ such that $\mathbf{a} = \gamma(x_{t-d+1}, \dots, x_t)$ defined in (11). Suppose $\mathbf{a} \in C(i)$, then by Lemma 4.4, there exists an $M > 0$ such that

$$x_{t+ld} = \phi_1^{c_i \mathscr{A}(i)} \phi_2^{c_i \mathscr{B}(i)} x_t \prod_{j=1}^{l-1} L(f^j(\mathbf{a}), t + jd) + R_2(l, d, x_t, \varepsilon_t),$$

where $c_i = l/l(i) \geq 1$ and $E|R_2(l, d, x_t, \varepsilon_t)| < M$. By Lemma 4.3, $\mathscr{B}(i)$ is an even number, and since $\phi_1 > 0$, we have

$$E|x_{t+ld}| \leq \phi_1^{c_i \mathscr{A}(i)} \phi_2^{c_i \mathscr{B}(i)} |x_t| + M.$$

By the definition of m and $0 < \phi_1 < 1$, we obtain that, if $\phi_1^{\mathscr{A}(m)} \phi_2^{\mathscr{B}(m)} < 1$, then $\phi_1^{\mathscr{A}(i)} \phi_2^{\mathscr{B}(i)} < 1$ for all i. Let $\eta = \max_{1 \leq i \leq n} \{\phi_1^{\mathscr{A}(i)} \phi_2^{\mathscr{B}(i)}\}$; then $\phi_1^{\mathscr{A}(m)} \phi_2^{\mathscr{B}(m)} < 1$ implies that $\eta < 1$ and, hence, $\phi_1^{c_i \mathscr{A}(i)} \phi_2^{c_i \mathscr{B}(i)} < \eta^{c_i} < \eta < 1$. Hence, for $|x_t|$ large enough,

$$E|x_{t+ld}| \leq (\eta + \varepsilon)|x_t|,$$

where $\eta + \varepsilon < 1$. Consequently, given $x_{t-2d+1}, \dots, x_{t-d}, x_{t-d+1}, \dots, x_t$ and $x_{t-d+1}, \dots, x_t$ large enough, we have

$$E|x_{t-i+ld}| \leq (\eta + \varepsilon)|x_{t-i}| \quad \text{for } i = 0, \dots, d-1.$$

Thus

$$E\|\mathbf{X}_{t+ld}\|^2 < (\eta + \varepsilon)\|\mathbf{X}_t\|^2,$$

where $\mathbf{X}_t = (x_t, \dots, x_{t-d+1})$. By letting $g(X_t) = \|X_t\|$ and using Lemma 1.1, $\mathbf{X}_{t+ld}$ is geometrically ergodic, and by Lemma 1.2, $\mathbf{X}_t$ is also geometrically ergodic.

Next, we turn to the necessary condition. If $\phi_1^{\mathscr{A}(m)} \phi_2^{\mathscr{B}(m)} > \eta > 1$, choose $x_{t-d+1}, \dots, x_t$ such that

$$\gamma(x_{t-d+1}, \dots, x_t) = \mathbf{a} \in C(m).$$

Writing $l = l(m)$, by Lemma 4.4 we have

$$\begin{aligned} x_{t+ld} &= \phi_1^{\alpha(f(\mathbf{a}))+\sum_{i=1}^{l-1}\alpha(f^i(\mathbf{a}))}\phi_2^{\beta(f(\mathbf{a}))+\sum_{i=1}^{l-1}\beta(f^i(\mathbf{a}))}x_t + R_1(l,d,x_t,\varepsilon_t) \\ &= \phi_1^{\mathscr{A}(m)}\phi_2^{\mathscr{B}(m)}x_t + R_1(l,d,x_t,\varepsilon_t), \end{aligned}$$

where $E|R_1(l,d,x_t,\varepsilon_t)| < G$ with G being a constant depending only on l, d and $E|\varepsilon_t|$. Therefore,

$$\begin{aligned} &P\left((-1)^{1-a_d}x_{t+ld} \le \frac{1+\eta}{2}(-1)^{1-a_d}x_t \,\middle|\, x_i, i = t-d+1,\dots,t\right) \\ &\quad\le P\Big((-1)^{1-a_d}\big(\phi_1^{\mathscr{A}(m)}\phi_2^{\mathscr{B}(m)}x_t + R_1(l,d,x_t,\varepsilon_t)\big) \\ &\qquad\qquad \le \frac{1+\eta}{2}(-1)^{1-a_d}x_t \,\Big|\, x_i, i = t-d+1,\dots,t\Big) \\ &\quad= P\Big((-1)^{a_d}R_1(l,d,x_t,\varepsilon_t) \\ &\qquad\qquad \ge \left(\phi_1^{\mathscr{A}(m)}\phi_2^{\mathscr{B}(m)} - \frac{1+\eta}{2}\right)(-1)^{1-a_d}x_t \,\Big|\, x_i, i = t-d+1,\dots,t\Big) \\ &\quad\le \frac{2E|R_1(l,d,x_t,\varepsilon_t)|}{(1-\eta)|x_t|}. \end{aligned}$$

This shows that there exists an $M_0^{(0)} > 0$ such that $c_0 = 2E|R_1(l,d,x_t,\varepsilon_t)|/((1-\eta)M_0^{(0)}) < 1$, and whenever $|x_t| > M_0^{(0)}$ and $\gamma(x_{t-d+1},\dots,x_t) = \mathbf{a}$ is given,

$$P\left((-1)^{1-a_d}x_{t+ld} > \frac{1+\eta}{2}(-1)^{1-a_d}x_t \,\middle|\, x_t,\dots,x_{t-d+1}\right) > 1 - c_0.$$

Note that in the above equation with $a_d = I(x_t > 0)$, we have $(-1)^{1-a_d}x_t > 0$. Next, by Lemma 4.5, for $1 \le k < ld$,

$$x_{t+k+ld} = \phi_1^{\mathscr{A}(\mathbf{a})}\phi_2^{\mathscr{B}(\mathbf{a})}x_{t+k}\prod_{i=1}^{l-1}L\big(f^i(\mathbf{b}^{(k)}), t+k+id\big) + R^*(k,d,x_t,\varepsilon_t),$$

where $E|R^*(k,d,x_t,\varepsilon_t)| < M_k$ with M_k being finite and $f^l(\mathbf{b}^{(k)}) = \mathbf{b}^{(k)}$. By the same argument as before, we have that, whenever $|x_{t+k}| > M_0^{(k)} > 0$ and $\gamma(x_{t+k-d+1},\dots,x_{t+k}) = \mathbf{b}^{(k)}$ is given,

$$P\left((-1)^{1-b_d^{(k)}}x_{t+k+ld} > \frac{1+\eta}{2}(-1)^{1-b_d^{(k)}}x_{t+k} \,\middle|\, x_{t+k-1},\dots,x_{gt+k-d}\right) > 1 - c_k$$

for $k = 1,\dots,ld-1$.

By the definition of $\mathbf{b}^{(k)}$ in Lemma 4.5, for $k = id + k_1$ where $0 \le k_1 < d$,

$$\begin{cases} b_d^{(k)} = a_{k_1}^{(i+1)}, & \text{if } k_1 \ne 0, \\ b_d^{(k)} = a_d^{(i)}, & \text{if } k_1 = 0. \end{cases}$$

So, $(b_d^{(id+1)}, \dots, b_d^{(id+d)}) = (a_1^{(i+1)}, \dots, a_d^{(i+1)})$. Hence, given $(x_{t+1}, \dots, x_{t+ld})$ such that

$$\gamma(x_{t+1}, \dots, x_{t+ld}) = (f(\mathbf{a}), \dots, f^l(\mathbf{a}))$$

it is easy to show that

$$\mathbf{b}^{(k)} = \gamma(x_{t+k-d+1}, \dots, x_{t+k}).$$

Let $M = \max(M_0^{(0)}, \dots, M_0^{(ld-1)})$; by the property of $\mathbf{b}^{(k)}$ shown above, we have that, given $|x_{t+k}| > M$, $k = 1, \dots, ld$ and $\gamma(x_{t+1}, \dots, x_{t+ld}) = (f(\mathbf{a}), \dots, f^l(\mathbf{a}))$,

$$P\left((-1)^{1-b_d^k} x_{t+k+ld} > \frac{1+\eta}{2}(-1)^{1-b_d^k} x_{t+k}, k = 1, \dots, ld \middle| x_{t+1}, \dots, x_{t+ld}\right)$$
$$> \prod_{k=1}^{l} (1 - c_k).$$

Let $1 - c = \prod_{k=1}^{l}(1 - c_k)$, so that $0 < c < 1$, and let $\beta = 1/(1 + \eta)$. Notice that if $k = id + k_1$, where $0 \le k_1 < d$ and

$$\gamma(x_{t+1}, \dots, x_{t+ld}) = (f(\mathbf{a}), \dots, f^l(\mathbf{a})),$$

then $I(x_{t+k} > 0) = a_{k_1}^{(i+1)} = b_d^{(k)}$, so $(-1)^{1-b_d^{(k)}} x_{t+k} = |x_{t+k}| \ge 0$ which says that if $(-1)^{1-b_d^{(k)}} x_{t+ld+k} > (-1)^{1-b_d^{(k)}} x_{t+k}$, then x_{t+ld+k} has the same sign as x_{t+k}. Hence, if

$$(-1)^{1-b_d^{(k)}} x_{t+ld+k} > (-1)^{1-b_d^{(k)}} x_{t+k}$$

holds for $k = 1, \dots, ld$, then

$$\gamma(x_{t+ld+1}, \dots, x_{t+ld+ld}) = \gamma(x_{t+1}, \dots, x_{t+ld})$$
$$= (f(\mathbf{a}), \dots, f^l(\mathbf{a})).$$

By induction, we obtain

$$P\Big((-1)^{1-b_d^{(k)}} x_{d+k+ild} > \frac{1+\eta}{2}(-1)^{1-b_d^{(k)}} x_{d+k+(i-1)ld},$$
$$k = 0, \dots, l-1, i = 1, \dots, s | x_{ld}, \dots, x_1\Big)$$
$$> \prod_{i=1}^{s} (1 - c\beta^{i-1}) > (1 - c)^{1/(1-\beta)}.$$

Therefore, for any $x_j \in \mathfrak{R}^1$, $j = -d + 1, \dots, 0$,

$$P(|x_t| \to \infty | x_{-d+1}, \dots, x_0)$$
$$\ge (1 - c)^{1/(1-\beta)} P(|x_k| > M, k = 1, \dots, ld; \gamma(x_1, \dots, x_{ld})$$
$$= (f(\mathbf{a}), \dots, f^l(\mathbf{a})) | x_{-d+1}, \dots, x_0)$$
$$> 0,$$

and, hence, $\{x_t\}$ is not geometrically ergodic.

For the case of $0 < \phi_1 < 1$, $\phi_2 < 0$ and $\phi_1^{\mathscr{A}(m)}\phi_2^{\mathscr{B}(m)} = 1$, we can use the same techniques as those of Section 3 to show that x_t is not ergodic. For simplicity, we consider here the case of $d = 2$, but the same method applies to general d. For $d = 2$, the condition is $\phi_1\phi_2^2 = 1$. Given $x_t \geq 0$, the model implies that

$$x_{t+1} = \phi_1 x_t I(x_{t-1} < 0) + \phi_2 x_t I(x_{t-1} \geq 0) + \varepsilon_{t+1}, \tag{16}$$

$$x_{t+2} = \phi_1\phi_2 x_t I(x_{t-1} < 0) + \phi_2^2 x_t I(x_{t-1} \geq 0) + \phi_2\varepsilon_{t+1} + \varepsilon_{t+2}, \tag{17}$$

$$x_{t+3} = \phi_1 x_{t+2} I(x_{t+1} < 0) + \phi_2 x_{t+2} I(x_{t+1} \geq 0) + \varepsilon_{t+3}. \tag{18}$$

By (16), we have

$$x_{t+1} I(x_{t-1} < 0) = (\phi_1 x_t + \varepsilon_{t+1}) I(x_{t-1} < 0), \tag{19}$$

$$x_{t+1} I(x_{t-1} \geq 0) = (\phi_2 x_t + \varepsilon_{t+1}) I(x_{t-1} \geq 0). \tag{20}$$

Since $\phi_1\phi_2 < 0$, it is easily seen that

$$\phi_1\phi_2 x_{t+1} I(x_{t+1} < 0) \geq 0 \geq \phi_2^2 x_{t+1} I(x_{t+1} < 0),$$

which, by (19), implies that

$$\begin{aligned} &\phi_1\phi_2(\phi_1 x_t + \varepsilon_{t+1}) I(x_{t-1} < 0) I(x_{t+1} < 0) \\ &\quad \geq \phi_2^2(\phi_1 x_t + \varepsilon_{t+1}) I(x_{t-1} < 0) I(x_{t+1} < 0). \end{aligned} \tag{21}$$

Since $\phi_2^2 > 0$ and $\phi_1\phi_2 < 0$, we have

$$\phi_2^2 x_{t+1} I(x_{t+1} \geq 0) \geq 0 \geq \phi_1\phi_2 x_{t+1} I(x_{t+1} \geq 0),$$

which, by (20), shows that

$$\begin{aligned} &\phi_2^2(\phi_2 x_t + \varepsilon_{t+1}) I(x_{t+1} \geq 0) I(x_{t-1} \geq 0) \\ &\quad \geq \phi_1\phi_2(\phi_2 x_t + \varepsilon_{t+1}) I(x_{t+1} \geq 0) I(x_{t-1} \geq 0). \end{aligned} \tag{22}$$

Using (17), the fact that $\phi_1\phi_2^2 = 1$ and (21), we have

$$\begin{aligned} I_1 &\equiv \phi_1 x_{t+2} I(x_{t+1} < 0) \\ &= \phi_1\left[\phi_1\phi_2 x_t I(x_{t-1} < 0) + \phi_2^2 x_t I(x_{t-1} \geq 0) + \phi_2\varepsilon_{t+1} + \varepsilon_{t+2}\right] I(x_{t+1} < 0) \\ &= \phi_1^2\phi_2 x_t I(x_{t-1} < 0) I(x_{t+1} < 0) + \phi_1\phi_2^2 x_t I(x_{t-1} \geq 0) I(x_{t+1} < 0) \\ &\quad + \phi_1\phi_2\varepsilon_{t+1} I(x_{t+1} < 0) + \phi_1\varepsilon_{t+2} I(x_{t+1} < 0) \\ &= \phi_1\phi_2(\phi_1 x_t + \varepsilon_{t+1}) I(x_{t-1} < 0) I(x_{t+1} < 0) + x_t I(x_{t-1} \geq 0) I(x_{t+1} < 0) \\ &\quad + \phi_1\phi_2\varepsilon_{t+1} I(x_{t+1} < 0)(1 - I(x_{t-1} < 0)) + \phi_1\varepsilon_{t+2} I(x_{t+1} < 0) \\ &= \phi_1\phi_2(\phi_1 x_t + \varepsilon_{t+1}) I(x_{t-1} < 0) I(x_{t+1} < 0) + x_t I(x_{t-1} \geq 0) I(x_{t+1} < 0) \\ &\quad + \phi_1\phi_2\varepsilon_{t+1} I(x_{t+1} < 0) I(x_{t-1} \geq 0) + \phi_1\varepsilon_{t+2} I(x_{t+1} < 0) \\ &\geq \phi_2^2(\phi_1 x_t + \varepsilon_{t+1}) I(x_{t-1} < 0) I(x_{t+1} < 0) + x_t I(x_{t-1} \geq 0) I(x_{t+1} < 0) \\ &\quad + \phi_1\phi_2\varepsilon_{t+1} I(x_{t+1} < 0) I(x_{t-1} \geq 0) + \phi_1\varepsilon_{t+2} I(x_{t+1} < 0) \\ &= x_t I(x_{t-1} < 0) I(x_{t+1} < 0) + x_t I(x_{t-1} \geq 0) I(x_{t+1} < 0) \\ &\quad + \phi_1\varepsilon_{t+2} I(x_{t+1} < 0) + \phi_1\phi_2\varepsilon_{t+1} I(x_{t+1} < 0) I(x_{t-1} \geq 0) \\ &\quad + \phi_2^2\varepsilon_{t+1} I(x_{t+1} < 0) I(x_{t-1} < 0) \\ &= x_t I(x_{t+1} < 0) + \phi_1\phi_2\varepsilon_{t+1} I(x_{t+1} < 0) I(x_{t-1} \geq 0) \\ &\quad + \phi_2^2\varepsilon_{t+1} I(x_{t+1} < 0) I(x_{t-1} < 0) + \phi_1\varepsilon_{t+2} I(x_{t+1} < 0). \end{aligned}$$

Similarly, by (17), the fact that $\phi_1\phi_2^2 = 1$ and (22), we can show that

$$\begin{aligned}I_2 &\equiv \phi_2 x_{t+2} I(x_{t+1} \geq 0)\\ &\geq x_t I(x_{t+1} \geq 0) + \phi_2^2 \varepsilon_{t+1} I(x_{t+1} \geq 0) I(x_{t-1} < 0)\\ &\quad + \phi_1\phi_2\varepsilon_{t+1} I(x_{t+1} \geq 0) I(x_{t-1} \geq 0) + \phi_2 \varepsilon_{t+2} I(x_{t+1} \geq 0).\end{aligned}$$

Now, using I_1, I_2, (18) and the above two inequalities, we obtain

$$\begin{aligned}x_{t+3} &= I_1 + I_2 + \varepsilon_{t+3}\\ &\geq x_t + \varepsilon_{t+3} + \varepsilon_{t+2}\left[\phi_1 I(x_{t+1} < 0) + \phi_2 I(x_{t+1} \geq 0)\right]\\ &\quad + \varepsilon_{t+1}\left[\phi_2^2 I(x_{t-1} < 0) + \phi_1\phi_2 I(x_{t-1} \geq 0)\right].\end{aligned} \tag{23}$$

From (23), we define $y_1 = x_3$ and

$$y_t = y_{t-1} + \eta_t \quad \text{for } t > 1,$$

where $\eta_t = \varepsilon_{3t} + \varepsilon_{3t-1} z_t^{(1)} + \varepsilon_{3t-2} z_t^{(2)}$ with

$$z_t^{(1)} = \phi_1 I(x_{3t-1} < 0) + \phi_2 I(x_{3t-1} \geq 0)$$

and $z_t^{(2)} = \phi_2^2 I(x_{3t-4} < 0) + \phi_1\phi_2 I(x_{3t-4} \geq 0)$. For $d = 2$, since ε_{3t}, ε_{3t-1} and ε_{3t-2} are independent of y_{t-i} for $i \geq 1$, ε_{3t-1} and $z_{t-i}^{(1)}$ are independent for $i \geq 0$, and ε_{3t-2} and $z_{t-i}^{(2)}$ are independent for $i \geq 0$, it is easy to show, by using a similar argument to that of Section 3, that $E[y_t | y_{t-1}, \dots, y_1] = y_{t-1}$, that is, $\{y_t\}$ is a martingale. Let

$$T_2 = \inf\{t | t > 1, y_t < 0\}, \qquad T_1 = \inf\{t | t > 1, x_{3t} < 0\}.$$

For $t < T_2$,

$$\begin{aligned}&E\left[|y_{t+1} - y_t| | y_1, \dots, y_t\right]\\ &\quad = E\left[|\eta_t| | y_1, \dots, y_t\right]\\ &\quad \leq E|\varepsilon_{3t+3}| + E|\varepsilon_{3t+2}| E\left[|\phi_1 I(x_{3t+1} < 0) + \phi_2 I(x_{3t+1} \geq 0)| | y_1, \dots, y_t\right]\\ &\qquad + E|\varepsilon_{3t+1}| E\left[|\phi_2^2 I(x_{3t-1} < 0) + \phi_1\phi_2 I(x_{3t-1} \geq 0)| | y_1, \dots, y_t\right]\\ &\quad \leq E|\varepsilon_{3t+3}| + \left(|\phi_1| + |\phi_2|\right) E|\varepsilon_{3t+2}| + \left(|\phi_2^2| + |\phi_1\phi_2|\right) E|\varepsilon_{3t-1}|\\ &\quad \leq K < \infty.\end{aligned}$$

Therefore, by the same argument as that in Section 3, $\{x_{3t}\}$ is not ergodic, and neither is $\{x_t\}$. Finally, for the case of $\phi_1 > 1$, $\phi_2 < 0$, the proof used in Section 2 can be employed to show that the process is not geometrically ergodic. □

By the symmetry between ϕ_1 and ϕ_2 in model (1), a similar result to Theorem 2 holds for the case of $\phi_1 < 0$ and $\phi_2 > 0$.

For any given d, Theorem 2 provides two numbers, namely, $\mathscr{A}(m)$ and $\mathscr{B}(m)$. We conclude this paper by defining $s(d)$ and $t(d)$ of Theorem 1 to be these two numbers, respectively, for the general TAR(1) model in (1).

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