The Annals of Applied Probability
1993, Vol. 3, No. 3, 746–762

FIRST PASSAGE PERCOLATION FOR RANDOM COLORINGS OF $\mathbb{Z}^d$

BY LUIZ FONTES AND CHARLES M. NEWMAN[1]

Istituto de Matemática e Estatística-USP and Courant Institute of Mathematical Sciences

Random colorings (independent or dependent) of $\mathbb{Z}^d$ give rise to dependent first-passage percolation in which the passage time along a path is the number of color changes. Under certain conditions, we prove strict positivity of the time constant (and a corresponding asymptotic shape result) by means of a theorem of Cox, Gandolfi, Griffin and Kesten about "greedy" lattice animals. Of particular interest are i.i.d. colorings and the $d = 2$ Ising model. We also apply the greedy lattice animal theorem to prove a result on the omnipresence of the infinite cluster in high density independent bond percolation.

1. Introduction and some results. Let $\mathscr{S}$ be a finite (or countably infinite) state space whose elements are regarded as colors. Then a family $\{X_v: v \in \mathbb{Z}^d\}$ of $\mathscr{S}$-valued random variables may be regarded as a random coloring of $\mathbb{Z}^d$. A special situation of interest to us is the case of i.i.d. colorings.

We wish to study certain features of the "color clusters;" these are the connected components of the random graph with vertex set $\mathbb{Z}^d$ and edge set consisting of all nearest-neighbor pairs of vertices $\{v, v'\}$, with $X_v = X_{v'}$. If the coloring is regarded as representing a map (in the geographical sense), then each individual color cluster represents a single country. Let us define for $n = 0, 1, 2, \ldots,$ the subset $\tilde{B}(n)$ of $\mathbb{Z}^d$ consisting of all vertices in $\mathbb{Z}^d$ that can be reached from the origin by some (nearest-neighbor) path along which the color changes n or fewer times. $\tilde{B}(n)$ represents the region attainable from the origin (without resorting to air travel) while crossing no more than n international borders. The main focus of this paper is the asymptotic behavior of $\tilde{B}(n)$ as $n \to \infty$.

Clearly $\tilde{B}(n)$ can grow no slower than linearly in n. It can, however, grow superlinearly and will clearly do so if color percolation occurs; that is, if some color cluster is infinite. Our major interest is in deriving conditions that distinguish between linear and superlinear growth of $\tilde{B}(n)$. In particular, one of our results (see Theorem 2) is that in the i.i.d. case, the boundary between the two growth regimes occurs exactly at the color-percolation threshold. This result is an analogue of a theorem of Kesten (1986) for independent first-passage percolation, as we will explain. We remark that neither the connection between first-passage percolation and color-cluster borders nor Theorem

Received March 1992; revised October 1992.
[1]Supported in part by NSF Grants DMS-89-02156 and DMS-91-96086.
AMS 1991 *subject classifications*. Primary 60K35, 82A43; secondary 60G60, 82A68.
Key words and phrases. First-passage percolation, percolation, random colorings, Ising model.

2 as a conjecture are particularly new. Our own interest originated several years ago in certain physical applications of the two-dimensional, two-color case [see Abraham and Newman (1988, 1989, 1991)]. In a recent paper (which we received in September 1992) the $d = 2$ case of Theorem 2 was proved by Chayes and Winfield (1992) using arguments rather different from those used here.

A first-passage percolation model [see Smythe and Wierman (1978) and Kesten (1987) for reviews] begins with a family $\{t(e): e \in \mathscr{B}^d\}$ of nonnegative random variables indexed by $\mathscr{B}^d$, the set of nearest-neighbor edges in $\mathbb{Z}^d$. Each $t(e)$ represents the passage time through the individual edge e. One then defines the passage time $T(r)$ of a path r, consisting of the edges $e_1, e_2, \dots, e_n$, as $T(r) = \sum_i t(e_i)$. The travel time $T(u,v)$ between two vertices is then defined as

$$T(u,v) = \inf\{T(r): r \text{ is a path from } u \text{ to } v\}. \tag{1.1}$$

The region attainable from the origin in time t or less is

$$\tilde{B}(t) = \{v \in \mathbb{Z}^d: T(0,v) \le t\}. \tag{1.2}$$

The classic model of Hammersley and Welsh (1965) takes the $t(e)$'s as i.i.d. variables, and the special case of an exponential distribution leads to Eden's (1961) growth model.

Random coloring gives rise to a first-passage model by defining

$$t(\{u,v\}) = \begin{cases} 1, & \text{if } X_u \neq X_v, \\ 0, & \text{if } X_u = X_v; \end{cases} \tag{1.3}$$

with this choice, our two definitions of $\tilde{B}$ are clearly consistent. Note though that even i.i.d. X_v's yield *dependent* $t(e)$'s. There are, however, some results, originally obtained for the classic case of independent first-passage percolation, that immediately extend to the general context of translation-invariant, ergodic $t(e)$'s, and hence to translation-invariant ergodic X_v's. This is the case for Richardson's (1973) "shape theorem" [improved by Cox and Durrett (1981)], which relates the asymptotic behavior of $\tilde{B}(t)$ as $t \to \infty$ to strict positivity of the *time constant*.

The time constant μ is defined by

$$\lim_{n\to\infty} \frac{T(0, nf_1)}{n} = \mu \quad \text{a.s. and in } L^1, \tag{1.4}$$

where f_1 is the vector $(1, 0, \dots, 0)$. If $E(t(e)) < \infty$, then the limit exists and is nonrandom because the subadditivity property,

$$T(u,v) \le T(u,w) + T(w,v) \quad \text{for all } u, v, w \in \mathbb{Z}^d, \tag{1.5}$$

permits the application of Kingman's (1968) subadditive ergodic theorem. μ is always finite but it can be zero. When discussing the asymptotic behavior of $\tilde{B}(t)$, it is convenient to replace it by the subset of $\mathbb{R}^d$,

$$B(t) = \{v + \overline{U}: v \in \tilde{B}(t)\} \tag{1.6}$$

where $\overline{U}$ is the unit cube,

$$\overline{U} = \{(x_1, \dots, x_d) : |x_i| \le \tfrac{1}{2} \text{ for each } i\}. \tag{1.7}$$

The following theorem, due to Y. Derrienic [as reported on page 259 of Kesten (1986)], is an extension of Richardson's (1973) shape theorem. The hypothesis that $t(e)$ is bounded is automatically satisfied in our random coloring context, where $t(e)$ only takes the values 0 and 1. In the independent first-passage context, optimal assumptions on the distribution of $t(e)$ were determined by Cox and Durrett (1981). In the general translation-invariant context, an improved version of Theorem 1 not requiring boundedness of the $t(e)$s was obtained by Boivin (1990).

THEOREM 1 (Derrienic). *Let $\{t(e): e \in \mathscr{B}^d\}$ be a translation-invariant, ergodic family of nonnegative, bounded random variables. Define the time constant μ and the subset $B(t)$ of $\mathbb{R}^d$ for $t \ge 0$ as above. If $\mu > 0$, then there exists a nonrandom, compact, convex subset B_0 in $\mathbb{R}^d$ (with nonempty interior) such that, almost surely,*

$$\forall \varepsilon > 0, \qquad (1-\varepsilon)B_0 \subset \frac{1}{t}B(t) \subset (1+\varepsilon)B_0 \quad \textit{for all large } t. \tag{1.8}$$

If $\mu = 0$, then, almost surely,

$$\forall \textit{ bounded } K \textit{ in } \mathbb{R}^d, \qquad K \subset \frac{1}{t}B(t) \quad \textit{for all large } t. \tag{1.9}$$

The proof of Theorem 1 is essentially the same as for independent first-passage percolation [e.g., as in Cox and Durrett (1981)] except for the final "filling in process," which is easier. We just mention a few features. B_0 may be defined as the closure of the set of $x = \lambda v$, with v in $\mathbb{Z}^d$ and $\lambda > 0$, such that

$$\lambda < \left[\lim_{n\to\infty} \frac{T(0, nv)}{n}\right]^{-1}. \tag{1.10}$$

To show, for example, the first inclusion of (1.8), one may begin by choosing an ε' in $(0, \varepsilon)$ such that, for any $\delta > 0$, almost surely, every point in $(1-\varepsilon)B_0$ is within distance δ of some point in $(1/t)\tilde{B}((1-\varepsilon')t)$ for all large t. To "fill in" the rest of $(1-\varepsilon)B_0$, we rely on the boundedness of $t(e)$. If C denotes the bound on $t(e)$, then every point in $\tilde{B}((1-\varepsilon')t)$ generates an l_1-sphere of radius approximately $\varepsilon' t/C$ about it in $B(t)$. Thus $(1/t)B(t)$ fills in $(1-\varepsilon)B_0$, providing $\delta < \varepsilon'/C$.

In general, one cannot determine either the value of μ or the shape of the region B_0. However, in the case of i.i.d. $t(e)$s, one can at least determine precisely when $\mu > 0$. This result, due to Kesten (1986), states that $\mu > 0$ if and only if

$$P(t(e) = 0) < p_c(\mathbb{Z}^d, \text{bond}), \tag{1.11}$$

where $p_c(\mathbb{Z}^d$, bond) denotes the critical value for independent nearest-neighbor bond percolation on $\mathbb{Z}^d$. [See Kesten (1987) and Grimmett (1989) for reviews of percolation.] The following theorem gives a natural analogue of (1.11) for random colorings. As previously mentioned, the $d = 2$ case has also been obtained by Chayes and Winfield (1992).

THEOREM 2. *Let $\{X_v: v \in \mathbb{Z}^d\}$ be i.i.d. random variables taking values in the finite (or countably infinite) state space $\mathscr{S}$. Define (dependent) passage times $\{t(e): e \in \mathscr{B}^d\}$ by (1.3). Then the time constant μ is strictly positive if and only if*

$$\text{(1.12)} \qquad \text{for each } s \in \mathscr{S}, \qquad P(X_v = s) < p_c(\mathbb{Z}^d, \text{site}),$$

where $p_c(\mathbb{Z}^d$, site) denotes the critical value for independent nearest-neighbor site percolation on $\mathbb{Z}^d$.

PROOF. To demonstrate the necessity of condition (1.12), we will suppose that for some s, $p_s \equiv P(X_v = s) \geq p_c \equiv p_c(\mathbb{Z}^d, \text{site})$ and then show that $\mu = 0$. Let us define a modified time constant μ_s by replacing $T(0, nf_1)$ in the left-hand side of (1.4) by $T_s(0, nf_1)$, the minimal number of sites v, for which $X_v \neq s$, in any path from 0 to nf_1. Because $t(\{u, v\}) = 0$ if both $X_u = s$ and $X_v = s$, we see that $T(0, nf_1)/2 \leq T_s(0, nf_1)$ and so $\mu \leq 2\mu_s$. The desired vanishing of μ then follows from the fact that $\mu_s = 0$ if $p_s \geq p_c$. This fact is an analogue of the necessity of condition (1.11) (for strict positivity of the time constant in independent first-passage percolation), but with bond percolation replaced by site percolation. The proof for site percolation is essentially the same as that used for bond percolation by Kesten (1986) and is based on the exponentially small probability of large deviations of $T_s(0, nf_1)/n$ from μ_s [see Grimmett and Kesten (1984) and Kesten (1986)].

Before proceeding with the sufficiency of (1.12), we remark that without loss of generality, $\mathscr{S}$ may be assumed finite; if not, then lump together all but finitely many colors (with the sum of the lumped probabilities below p_c) and regard the lump as one new color.

To begin the proof of sufficiency, we first note that for any (site self-avoiding) path r from 0 to nf_1,

$$\text{(1.13)} \qquad \begin{aligned} 1 + T(r) &= 1 + \text{number of color changes along } r \\ &\geq \text{number of distinct color clusters touched by } r \\ &= \sum_{v \in r} |\mathscr{C}_v \cap r|^{-1}, \end{aligned}$$

where $\mathscr{C}_v$ denotes the color cluster of vertex v. Here, we regard r and $\mathscr{C}_v$ as collections of vertices and $|\mathscr{C}|$ denotes the number of vertices in $\mathscr{C}$. By Jensen's inequality,

$$\text{(1.14)} \qquad \frac{1}{|r|} \sum_{v \in r} |\mathscr{C}_v \cap r|^{-1} \geq \left[\frac{1}{|r|} \sum_{v \in r} |\mathscr{C}_v \cap r| \right]^{-1}$$

and thus

$$\frac{1+T(r)}{n} \ge \frac{1+T(r)}{|r|} \ge \left[\frac{1}{|r|}\sum_{v\in r}|\mathscr{C}_v \cap r|\right]^{-1} \ge \left[\frac{1}{|r|}\sum_{v\in r}|\mathscr{C}_v|\right]^{-1}. \tag{1.15}$$

Consequently,

$$\mu \ge \left[\limsup_{m\to\infty}\sup_{|r|=m}\frac{1}{m}\sum_{v\in r}|\mathscr{C}_v|\right]^{-1}, \tag{1.16}$$

where the sup is over all (site self-avoiding) paths r starting from the origin and containing exactly m sites.

For each color $s \in \mathscr{S}$, let us define the color-s cluster at vertex v by

$$\mathscr{C}_v^s = \begin{cases} \mathscr{C}_v, & \text{if } X_v = s, \\ \varnothing, & \text{if } X_v \ne s. \end{cases} \tag{1.17}$$

Because $|\mathscr{C}_v|$ is the sum over s of $|\mathscr{C}_v^s|$, we see from (1.16) that to prove $\mu > 0$, it suffices to have for each s,

$$\limsup_{m\to\infty}\sup_{|r|=m}\frac{1}{m}\sum_{v\in r}|\mathscr{C}_v^s| < \infty \quad \text{a.s.} \tag{1.18}$$

For a fixed s, the color-s clusters are just the clusters of a standard *subcritical* site percolation model. (1.18) will be an immediate consequence of Theorem 4. □

To prove (1.18), we find it necessary to prove a stronger result, Theorem 4, in which the paths r from the origin are replaced by more general *lattice animals* Γ. By a lattice animal, we will mean a finite connected (by nearest-neighbor edges) subset of $\mathbb{Z}^d$ containing the origin. (We will also regard the empty set as a lattice animal.) Even though the underlying coloring is i.i.d., the clusters $\mathscr{C}_v^s$ for different v's are of course dependent. Nevertheless, our proof of Theorem 4 (and hence of Theorem 2) is based on a recent theorem of Cox, Gandolfi, Griffin and Kesten (1992), in which the $|\mathscr{C}_v^s|$'s are replaced by i.i.d. variables, W_v. Because the theorem concerns lattice animals Γ of a given size that maximize $\sum_{v\in\Gamma} W_v$, it is said to be about "greedy" lattice animals.

THEOREM 3 (Cox–Gandolfi–Griffin–Kesten). *Let* $\{W_v\colon v \in \mathbb{Z}^d\}$ *be i.i.d. nonnegative random variables. If*

$$E\left(W_v^{d+\varepsilon}\right) < \infty \quad \textit{for some } \varepsilon > 0, \tag{1.19}$$

then (almost surely)

$$\limsup_{n\to\infty} \sup_{|\Gamma|=n} \frac{1}{n} \sum_{v\in\Gamma} W_v < \infty, \tag{1.20}$$

where the sup *is over all lattice animals with n vertices.*

REMARK. In Cox, Gandolfi, Griffin and Kesten (1992), the hypothesis (1.19) is weakened to the assumption that the expectation of $W_v^d(\log^+ W_v)^{d+\varepsilon}$ is finite. The left-hand side of (1.20) [and of (1.21) and (1.22)] is a constant (a.s.) by the Kolmogorov zero–one law. Gandolfi and Kesten (1992) have shown that the lim sup in (1.20) may be replaced by an ordinary limit.

The proof of the next theorem is based on Theorem 3; it will be given in Section 2.

THEOREM 4. *Consider independent nearest-neighbor site percolation on $\mathbb{Z}^d$ with probability p for any site to be occupied. Let C_v denote the occupied cluster of vertex $v \in \mathbb{Z}^d$. If $p < p_c(\mathbb{Z}^d$, site), then (almost surely)*

$$K_p \equiv \limsup_{n\to\infty} \sup_{|\Gamma|=n} \frac{1}{n} \sum_{v\in\Gamma} |C_v| < \infty. \tag{1.21}$$

The final theorem of this section is analogous to Theorem 4; its proof, given in Section 3, is also based on Theorem 3. In this theorem, the usual percolation clusters of Theorem 4 are replaced by somewhat more complicated objects. For *bond* percolation in the supercritical regime, we consider the random graph whose vertex set is the set of all vertices *not* belonging to the infinite open cluster and whose edge set consists of all the edges (open or closed) in $\mathscr{B}^d$ between those vertices. The connected component, in this random graph, of a vertex v will be denoted $\hat{C}_v$. The number of vertices in $\hat{C}_v$ is denoted $|\hat{C}_v|$; it is zero if v belongs to the infinite open cluster.

The motivation for the next theorem concerns certain extensions of the results of Newman and Stein (1990) about domain structure in Ising-like models. This application will be pursued in a future paper. The theorem should be regarded as providing a quantitative measure of the omnipresence of the infinite open cluster in terms of the smallness and sparseness of the $\hat{C}_v$'s.

THEOREM 5. *consider independent nearest-neighbor bond percolation on $\mathbb{Z}^d$ with probability p for any edge to be open. If $d \geq 2$ and p is sufficiently close to* 1, *then (almost surely)*

$$\hat{K}_p \equiv \limsup_{n\to\infty} \sup_{|\Gamma|=n} \frac{1}{n} \sum_{v\in\Gamma} |\hat{C}_v| < \infty; \tag{1.22}$$

moreover, $\hat{K}_p \to 0$ as $p \to 1$.

The remainder of this paper is organized as follows. Section 2 begins with Theorem 6, which provides a sufficient condition for strict positivity of the time constant for dependent random colorings. This condition is a natural extension of the sufficiency condition (1.12) for independent colorings, and like (1.12), is a consequence of Theorem 4. Then the proof of Theorem 4 is given, modulo certain graph-theoretical lemmas, which are presented in the Appendix. The rest of Section 2 concentrates on a specific two-parameter family of random two-colorings of $\mathbb{Z}^2$, the two-dimensional Ising model. Two results giving sufficient conditions for strict positivity of the time constant are given: one (Theorem 7) based on the very general Theorem 6 and a second (Theorem 8) that utilizes extra properties of the Ising model. As explained in the remark at the end of Section 2, recent results of Higuchi (1992a, b) show that the conditions of Theorem 8 are essentially optimal. Applications of these Ising model results to the random surface model of Abraham and Newman (1988, 1989, 1991) will be pursued elsewhere. In Section 3, the proof of Theorem 5 is given. Finally, in the Appendix, several nonprobabilistic graphical lemmas are stated and proved; these are needed for the proofs of Theorems 4 and 5.

2. Strict positivity of the time constant. In this section we consider translation-invariant, ergodic colorings of $\mathbb{Z}^d$ and the corresponding first-passage models. Theorem 1 is applicable in this case and so we are interested in conditions sufficient to guarantee $\mu > 0$. The first theorem gives a simple extension of condition (1.12) to dependent colorings. We define $\mathscr{F}_v$ to be the σ-field generated by $\{X_u: u \in \mathbb{Z}^d, u \neq v\}$.

THEOREM 6. *Let $\{X_v: v \in \mathbb{Z}^d\}$ be a translation-invariant, ergodic family of $\mathscr{S}$-valued random variables with $\mathscr{S}$ finite, and let $\{t(e): e \in \mathscr{B}^d\}$ be the related passage times given by* (1.3). *To guarantee that the time constant μ is strictly positive, it suffices that for some $\varepsilon > 0$, the following condition be valid* (*almost surely*):

$$\text{(2.1)} \qquad \textit{for each } s \in \mathscr{S}, \qquad P(X_v = s | \mathscr{F}_v) \leq p_c(\mathbb{Z}^d, \textit{site}) - \varepsilon.$$

PROOF. First we note that exactly the same arguments as used in the proof of Theorem 2 [see the inequalities (1.13)–(1.16)] also show here that it suffices to prove (1.18) valid for each s. But now the color-s clusters $\mathscr{C}_u^s$ are *not* the clusters of an independent percolation model.

However, if we define color-s occupation variables by

$$\text{(2.2)} \qquad N_v = \begin{cases} 1, & \text{if } X_v = s, \\ 0, & \text{if } X_v \neq s, \end{cases}$$

then we claim (as will be explained) that hypothesis (2.1) implies the stochastic domination

$$\text{(2.3)} \qquad \{N_v: v \in \mathbb{Z}^d\} \ll \{N_v^p: v \in \mathbb{Z}^d\},$$

where $\{N_v^p\}$ are i.i.d. 0- or 1-valued variables with

$$P(N_v^p = 1) = p \equiv p_c(\mathbb{Z}^d, \text{site}) - \varepsilon. \tag{2.4}$$

(2.3) means that increasing events (in $\{0,1\}^{\mathscr{B}^d}$) are assigned probability by the joint distribution of $\{N_v\}$, which is less than or equal to the probability assigned by the standard independent percolation distribution of $\{N_v^p\}$. It follows that *for a given s*, (1.18) will be valid for the color-s clusters (of our *dependent* coloring model) if it is valid with $\mathscr{C}_v^s$ replaced by C_v, the occupied clusters of a density-p *independent* site percolation model. The validity of (1.18) for the C_v's follows from Theorem 4, whose proof is given immediately following this proof.

It only remains to justify the claim of stochastic domination. This, in fact, is a simple (more or less standard) application of Harris' (1960) original version of the FKG inequalities, because (2.1) implies that the Radon–Nikodym derivative of the distribution of $\{N_v\}$ with respect to that of $\{N_v^p\}$ is a decreasing function. □

PROOF OF THEOREM 4. In order to prove (1.21) by means of the corresponding result (1.20) for independent variables, we need to relate the dependent $|C_v|$'s to some independent variables. Toward this end we consider an i.i.d. family $\{\tilde{\Gamma}_v: v \in \mathbb{Z}^d\}$ of random lattice animals, which are equidistributed with C_0, the occupied cluster of the origin. In order to compare the percolation clusters, C_v, to the independent random subsets, $\tilde{C}_v \equiv v + \tilde{\Gamma}_v$, we define random variables

$$\tilde{U}_v = \sup\left\{|\tilde{C}_u|: u \in \mathbb{Z}^d, v \in \tilde{C}_u\right\}, \tag{2.5}$$

where the sup of an empty set is taken to be zero. We then claim the stochastic domination inequality

$$\left\{|C_v|: v \in \mathbb{Z}^d\right\} \ll \left\{\tilde{U}_v: v \in \mathbb{Z}^d\right\}, \tag{2.6}$$

which we proceed to prove by an algorithmic construction of the percolation clusters, C_v.

Let $v_1, v_2, \dots$ be some (deterministic) ordering of $\mathbb{Z}^d$. First take $C_{v_1} = \tilde{C}_{v_1}$ and proceed inductively. Given $C_{v_1}, \dots, C_{v_n}$, take $C_{v_{n+1}} = C_{v_j}$ if $v_{n+1} \in C_{v_j}$ for some $j \leq n$. Let B_n denote the union of C_{v_j} for $j \leq n$ and let ∂B_n denote its boundary; that is, the vertices not in B_n which are nearest neighbors of some vertex in B_n. If $v_{n+1} \notin B_n$ but $v_{n+1} \in \partial B_n$, then take $C_{v_{n+1}}$ to be the empty set. If $v_{n+1} \notin \overline{B}_n$, the union of B_n and ∂B_n, then the conditional distribution of $C_{v_{n+1}}$ (given $C_{v_1}, \dots, C_{v_n}$) is that of the percolation cluster of v_{n+1} in a site percolation model where $\mathbb{Z}^d$ is replaced by $\mathbb{Z}^d \setminus \overline{B}_n$; thus $C_{v_{n+1}}$ may be regarded as (in fact, taken as) a subset of $\tilde{C}_{v_{n+1}}$. It follows that each C_v may be regarded as the subset of some $\tilde{C}_u$ (which contains v); this immediately implies the desired domination inequality (2.6).

Now, the nonprobabilistic, graph theoretical Lemma 2 of the Appendix (with $\tilde{G}_v$ taken as the cluster $\tilde{C}_v$) states that (for each ω and) for any (nonempty) lattice animal Γ,

$$(2.7)\qquad \frac{1}{|\Gamma|}\sum_{v\in\Gamma}\tilde{U}_v \le 2\sup_{\Gamma'}\frac{1}{|\Gamma'|}\sum_{v\in\Gamma'}|\tilde{C}_v|^2,$$

where the sup is over lattice animals Γ' containing Γ. Combining this with the stochastic domination and the definition of $\tilde{C}_v$ yields

$$(2.8)\qquad K_p \le 2\limsup_{n\to\infty}\sup_{|\Gamma'|=n}\frac{1}{n}\sum_{v\in\Gamma'}|\tilde{\Gamma}_v|^2.$$

Because the $|\tilde{\Gamma}_v|^2$ are i.i.d. and have moments of all orders because $p < p_c$ [by the result of Menshikov (1986) and of Aizenman and Barsky (1987)], we can now apply Theorem 3 to conclude the proof. □

REMARKS. If the random variables W_v of Theorem 3 have a finite moment-generating function, then the conclusion (1.20) follows easily by a simple large deviation argument. We note that $|\tilde{\Gamma}_v|$ has an exponential tail, but $|\tilde{\Gamma}_v|^2$ does not and hence the special case of Theorem 3 just mentioned is not sufficient to prove Theorem 4. We also recall that the proofs of Theorems 2 and 6 only require a weakened version of (1.21) in which the lattice animals are restricted to paths starting from the origin. Nevertheless, the proof of Lemma 2 of the Appendix shows that even if the Γ in (2.7) is a path, the sup on the right-hand side cannot be restricted to paths; thus a weakened version of (1.20), in which the sup is only over paths, would not suffice to prove Theorem 2 or 6.

For the remainder of this section we focus on the standard two-dimensional Ising ferromagnet. This is a two-parameter family of random colorings of $\mathbb{Z}^2$ with $\mathscr{S} = \{-1, +1\}$. The two parameters are a coupling constant $J \in [0, \infty)$, and an external field $h \in (-\infty, \infty)$. (The temperature parameter has been absorbed into J and h.) The (formal) Hamiltonian for a configuration $\{s_v: v \in \mathbb{Z}^2\} \in \mathscr{S}^{\mathbb{Z}^2}$ is

$$(2.9)\qquad H = -\frac{J}{2}\sum_{\{u,v\}} s_u s_v - h\sum_v s_v,$$

where the first sum is over edges in $\mathscr{B}^2$ (i.e., nearest-neighbor pairs) and the second is over vertices in $\mathbb{Z}^2$. For given J and h, we take $\{X_v: v \in \mathbb{Z}^2\}$ to have as its joint distribution the Gibbs distribution for this Hamiltonian, obtained as an infinite volume limit with free boundary conditions. [For more information on Ising models and Gibbs distributions, see Georgii (1988), especially Section 6.2 and pages 450–454.] We remark that the theorems that follow will automatically be inapplicable to values of J and h where multiple Gibbs distributions occur; thus boundary conditions will not matter and $\{X_v\}$ will be ergodic.

Our first result about Ising models is a corollary of Theorem 6. Combined with the result of Higuchi (1982) that $p_c(\mathbb{Z}^2, \text{site}) > \frac{1}{2}$, it shows that if both J and $|h|$ are sufficiently small, then $\mu > 0$.

THEOREM 7. *Let $\{X_v: v \in \mathbb{Z}^2\}$ be a (free boundary condition) standard two-dimensional Ising ferromagnet with Hamiltonian* (2.9) *and let $\{t(e): e \in B^2\}$ be the related passage times given by* (1.3). *A sufficient condition to guarantee a strictly positive time constant is*

$$[1 + \exp(-4J - 2|h|)]^{-1} < p_c(\mathbb{Z}^2, site). \tag{2.10}$$

PROOF. We need to show that (2.1) is valid. From (2.9), it follows that for $s = +1$ or -1,

$$\frac{P(X_v = s|\mathscr{F}_v)}{P(X_v = -s|\mathscr{F}_v)} = \exp\left[s\left(2h + J\sum_{u \in \mathscr{N}(v)} X_u\right)\right], \tag{2.11}$$

where $\mathscr{N}(v)$ denotes the set of the four nearest neighbors of v. Equivalently,

$$P(X_v = s|\mathscr{F}_v) = \left(1 + \exp\left[-s\left(2h + J\sum_{u \in \mathscr{N}(v)} X_u\right)\right]\right)^{-1}. \tag{2.12}$$

This last expression is maximized, for a given s, by taking $X_u = s$ for each nearest neighbor u of v. With that choice of the X_u's, it is then maximized (if $h \neq 0$) for the case $s = \operatorname{sgn}(h)$. Thus (2.10) implies the validity of (2.1), as desired. □

REMARK. Theorem 6 can easily be applied to the Gibbs distributions of other statistical mechanics models besides the Ising model. In cases such as the Potts models, where q, the number of colors, exceeds 2, there is no need to restrict the spatial dimension to $d = 2$. In particular, we note that the analogue of (2.10) is nonvacuous for a given d, providing only that q be big enough for $p_c(\mathbb{Z}^d, \text{site})$ to exceed $1/q$.

Theorem 7 is a less than optimal result for the $d = 2$ Ising model. For example, when $h = 0$, (2.10) becomes

$$J < \tfrac{1}{4}\ln[p_c/(1 - p_c)] \approx 0.09, \tag{2.13}$$

where we have used the numerical value $p_c \approx 0.59$ reported by Essam (1972). On the other hand, it was proved by Coniglio, Nappi, Peruggi and Russo (1976) that (when $h = 0$) there are no infinite color clusters until J exceeds the critical value J_c whose exact value [Kramers and Wannier (1941); Onsager (1944)] is

$$J_c = \operatorname{arcsinh}(1) \approx 0.88. \tag{2.14}$$

It is a natural conjecture, in the spirit of Theorem 2, that in the interior of the region of the $J - h$ plane where color percolation does not occur, the time constant should be strictly positive; in particular this should be the case for $h = 0$ and $J < J_c$. Our second and last result about Ising models, when combined with recent results of Higuchi (1992a, b), proves this conjecture, as we explain in a remark at the end of this section.

THEOREM 8. *A sufficient condition to guarantee a strictly positive time constant for the $d = 2$ Ising ferromagnet (as in Theorem 7) is*

$$E\left(|\mathscr{C}_v|^{4+\delta}\right) < \infty \quad \textit{for some } \delta > 0, \tag{2.15}$$

where $\mathscr{C}_v$ is the color cluster (usually called the parallel spin cluster in Ising model terminology) of vertex v.

PROOF. We follow the proof of Theorem 2 through (1.18). To verify (1.18) for a *fixed* $s \in \{-1, +1\}$, we follow the proof of Theorem 4. In particular, we claim that the following analogue of (2.6) is valid:

$$\left\{|\mathscr{C}_v^s|: v \in \mathbb{Z}^d\right\} \ll \left\{\tilde{U}_v^s: v \in \mathbb{Z}^d\right\}, \tag{2.16}$$

where

$$\tilde{U}_v^s = \sup\left\{|u + \tilde{\Gamma}_u^s|: u \in \mathbb{Z}^d, v \in u + \tilde{\Gamma}_u^s\right\} \tag{2.17}$$

and $\{\tilde{\Gamma}_v^s: v \in \mathbb{Z}^d\}$ are i.i.d. random lattice animals, equidistributed with $\mathscr{C}_0^s$. As in the proof of Theorem 2, Lemma 2 of the Appendix then leads to a bound on the left-hand side of (1.18) by the left-hand side of (1.20) with $W_v = |\tilde{\Gamma}_v^s|^2$. This bound is finite, according to Theorem 3, if $E(|\mathscr{C}_v^s|^{4+2\varepsilon}) < \infty$. The finiteness of this moment for both $s = +1$ and $s = -1$ is equivalent to (2.15).

It remains to justify the claimed stochastic domination, (2.16). We proceed by a similar algorithmic construction as used in the proof of Theorem 4. It is crucial now to keep in mind that we are constructing color-s clusters for fixed s (say $s = +1$) so that $\mathscr{C}_v^{+1}$ is empty if $X_v = -1$. The key point here is that when $v_{n+1} \notin \overline{B}_n$, the conditional distribution of $\mathscr{C}_{v_{n+1}}^{+1}$ (given $\mathscr{C}_{v_1}^{+1}, \dots, \mathscr{C}_{v_n}^{+1}$) is stochastically dominated by (and thus may be regarded as a subset of) $v_{n+1} + \tilde{\Gamma}_{v_{n+1}}$. This is because $\mathscr{C}_{v_{n+1}}^{+1}$ is (conditionally) the plus cluster of an Ising model in $\mathbb{Z}^d \setminus \overline{B}_n$ with *minus* boundary conditions on the vertices of $\partial(\mathbb{Z}^d \setminus \overline{B}_n)$ (which is just ∂B_n). By a standard application of the FKG inequalities [Fortuin, Kasteleyn and Ginibre (1971)], this plus cluster is stochastically dominated by the plus cluster of an Ising model with no conditions on ∂B_n. This completes the proof. □

REMARK. The nature of color percolation in the $d = 2$ Ising model has been extensively investigated in two recent papers of Higuchi (1992a, b). His results (combined with previous ones) imply the following. Color percolation only occurs when $J > J_c$, or when $J \le J_c$ and $|h| > h_c(J)$, where $h_c(J) > 0$ for $J < J_c$ and $h_c(J_c) = 0$. There is no color percolation for $J \le J_c$ and $|h| \le h_c(J)$. In the *interior* of the nonpercolating regime (i.e., when $J < J_c$ and $|h| < h_c$) the color connectivity function decays exponentially:

$$P(u \in \mathscr{C}_v) \le \exp\left[-\kappa\left(|u_1 - v_1| + \cdots + |u_d - v_d|\right)\right] \tag{2.18}$$

with $\kappa = \kappa(J, h) > 0$. It is an easy consequence of (2.18) that all moments of $|\mathscr{C}_v|$ are finite and thus Theorem 8 implies $\mu > 0$ in all the interior of the nonpercolating regime.

3. Proof of Theorem 5. As in the proof of Theorem 4, we need to relate the dependent $|\hat{C}_v|$'s to independent random variables in order to apply Theorem 3. This will require a series of arguments. A key idea is that the boundary of $\hat{C}_v$ is a surface of *closed* edges (or, equivalently, of dual plaquettes) that separates $\hat{C}_v$ from the infinite open cluster and that $|\hat{C}_v| = O((D_v)^d)$, where D_v is the maximum distance of that surface from v. (Distance between vertices v and u in $\mathbb{Z}^d$ will be the l_1 distance, denoted $\|u - v\|$; distance between edges in $\mathscr{B}^d$, or between an edge and a vertex, will be a maximum of the l_1 distance between endpoints.) The edges (or plaquettes) forming this surface are connected, if we choose some appropriate notion of connectedness. To avoid detailed considerations of discrete geometry, we will choose a conservative notion by declaring two edges as neighboring if they are within distance M of each other (M will be chosen sufficiently large, but fixed for a given dimension d). In other words, we consider the independent site percolation model with site occupation density $1 - p$, on the d-dimensional lattice $\mathbb{L}$ with *vertex* set $\mathscr{B}^d$ and edges between any two elements of $\mathscr{B}^d$ within distance M of each other. We will call the occupied clusters of this percolation model $\mathbb{L}$-*clusters*.

For $e \in \mathscr{B}^d$, we denote by R_e the maximum distance from e to any e' in its $\mathbb{L}$-cluster and we define for $u \in \mathbb{Z}^d$,

$$R_u = \sup\{R_e\colon e = \{u, u + f_j\} \text{ for } j = 1 \text{ or } 2 \text{ or } \cdots d\}, \tag{3.1}$$

where $f_1, \dots, f_d$ are the standard unit basis vectors. [We remark that $R_e = 0$ if the $\mathbb{L}$-cluster of e is empty (i.e., if e is an open bond) and otherwise $R_e \geq 1$.] Then

$$D_v \leq Y_v \equiv \sup\{R_u\colon u \in \mathbb{Z}^d, \|v - u\| \leq R_u\}. \tag{3.2}$$

The key idea mentioned previously has been absorbed into the fact that $|\hat{C}_v| = O((Y_v)^d)$.

We now consider i.i.d. variables $\{\tilde{R}_e\colon e \in \mathscr{B}^d\}$ such that each $\tilde{R}_e$ is equidistributed with R_e and we define $\{\tilde{R}_u\}$ and $\{\tilde{Y}_v\}$ in terms of $\{\tilde{R}_e\}$, as in (3.1) and (3.2). Note that the $\tilde{R}_u$s are also i.i.d. Then by an algorithmic construction of the $\mathbb{L}$-clusters, analogous to the algorithmic construction used in the proof of (2.6), one has the stochastic domination

$$\{Y_v\colon v \in \mathbb{Z}^d\} \ll \{\tilde{Y}_v\colon v \in \mathbb{Z}^d\}. \tag{3.3}$$

Thus the quantity $\hat{K}_p$ of (1.22) is bounded by

$$\hat{K}_p \leq c_d \limsup_{n \to \infty} \sup_{|\Gamma| = n} \frac{1}{n} \sum_{v \in \Gamma} \left(\tilde{Y}_v\right)^d, \tag{3.4}$$

where c_d is a constant depending only on d (and on M, which depends only on d).

Now, it follows from Lemma 3 of the Appendix [because the volume $|B(v, \rho)|$ considered there, in the case of the graph $(\mathbb{Z}^d, \mathscr{B}^d)$ and integer ρ, has

$|B(v,\rho)|/\rho^d$ bounded away from 0 and ∞ by d-dependent constants] that (for each ω and) for any (nonempty) lattice animal Γ,

$$\frac{1}{|\Gamma|}\sum_{v\in\Gamma}\tilde{Y}_v^d \le c_d' \sup_{\Gamma'}\frac{1}{|\Gamma'|}\sum_{v\in\Gamma'}\left(\tilde{R}_v\right)^{2d}, \tag{3.5}$$

where the sup is over lattice animals Γ' containing Γ, and c_d' is another constant depending only on d. Thus we have

$$\hat{K}_p \le c_d'' \limsup_{n\to\infty}\sup_{|\Gamma|=n}\frac{1}{n}\sum_{v\in\Gamma}W_v, \tag{3.6}$$

where the W_v's are i.i.d. variables, equidistributed with $(\tilde{R}_v)^{2d}$, and $c_d'' = c_d\cdot c_d'$. Because the percolation model on $\mathbb{L}$ is a finite-range, independent model on a regular d-dimensional lattice, it follows by standard arguments that for $1-p$ sufficiently small, the distribution of R_e has an exponentially decreasing tail, and hence the same is true for $\tilde{R}_e$ and for $\tilde{R}_v$. Furthermore, as $1-p$ decreases to zero, $\tilde{R}_v$ stochastically decreases to zero. Thus for any $k>0$,

$$E\left(\left(\tilde{R}_v\right)^k\right)\to 0 \quad \text{as } p\to 1. \tag{3.7}$$

Choosing $k>2d^2$, we see that the W_v's in (3.6) satisfy $E(W_v^{d+\varepsilon})\to 0$ as $p\to 1$. The following corollary of Theorem 3 then implies that $\hat{K}_p\to 0$ as $p\to 1$, as desired.

COROLLARY TO THEOREM 3 (Cox–Gandolfi–Griffin–Kesten). *For each $j=1,2,\dots,$ let $\{W_v^{(j)}: v\in\mathbb{Z}^d\}$ be i.i.d. nonnegative random variables and let*

$$K^{(j)} = \limsup_{n\to\infty}\sup_{|\Gamma|=n}\frac{1}{n}\sum_{v\in\Gamma}W_v^{(j)}. \tag{3.8}$$

If for some $\varepsilon>0$,

$$E\left(\left(W_v^{(j)}\right)^{d+\varepsilon}\right)\to 0 \quad \text{as } j\to\infty, \tag{3.9}$$

then $K^{(j)}\to 0$ as $j\to\infty$.

PROOF OF COROLLARY. By Chebyshev's inequality,

$$P\left(W_v^{(j)}\ge w\right)\le\frac{\delta_j}{w^{d+\varepsilon}} \quad \text{for } w\ge\left(\delta_j\right)^{1/(d+\varepsilon)}, \tag{3.10}$$

where $\delta_j = E((W_v^{(j)})^{d+\varepsilon})\to 0$ as $j\to\infty$. Let $\{W_v\}$ be i.i.d. with

$$P(W_v\ge w) = \frac{1}{w^{d+\varepsilon}} \quad \text{for } w\ge 1 \tag{3.11}$$

and let K denote the left-hand side of (1.20) for this choice of $\{W_v\}$. Then

$$\left\{W_v^{(j)}: v\in\mathbb{Z}^d\right\}\ll\left\{\left(\delta_j\right)^{1/(d+\varepsilon)}W_v: v\in\mathbb{Z}^d\right\}, \tag{3.12}$$

and so

$$K^{(j)} \le (\delta_j)^{1/(d+\varepsilon)} K \quad \text{for all } j. \tag{3.13}$$

Because $E(W_v^{d+\varepsilon'}) < \infty$ for $0 < \varepsilon' < \varepsilon$, it follows from Theorem 3 that $K < \infty$ and thus (3.13) implies $K^{(j)} \to 0$ as desired. □

REMARK. The arguments of Cox, Gandolfi, Griffin and Kesten (1992) show that the moment in (3.9) can be replaced by the expectation of $(W_v^{(j)})^d (\log^+ W_v^{(j)})^{d+\varepsilon}$.

APPENDIX

Some graph-theoretical lemmas. In this Appendix we give several (nonprobabilistic) lemmas concerning any single (finite or infinite) simple graph G with vertex set $\mathscr{V}$ and edge set $\mathscr{E}$ and with a distinguished vertex v_0. The particular graph to which these lemmas are applied in the body of the paper has $\mathscr{V} = \mathbb{Z}^d$, $\mathscr{E} = \mathscr{B}^d$ = {nearest neighbor pairs in $\mathbb{Z}^d$} and v_0 = origin of $\mathbb{Z}^d$. For a subgraph G' of G, we write $[G']$ to denote the *set* of vertices of G' and $|G'|$ to denote the *number* of vertices in G'. We define a G-animal to be a finite subset Γ of $\mathscr{V}$ containing v_0 such that the subgraph induced by Γ is connected. (The induced subgraph has vertex set Γ and edge set consisting of all $\mathscr{E} = \{v, u\}$ in $\mathscr{E}$ such that both v and u are in Γ.)

LEMMA 1. *For each v in $\mathscr{V}$, let $\tilde{G}_v$ be either a connected subgraph of G, which contains v in its vertex set, or else the empty subgraph of G. Suppose that for each v in $\mathscr{V}$, there is a vertex $z(v)$ that is either v itself or else is such that $v \in [\tilde{G}_{z(v)}]$. Then for any G-animal Γ, there exists some G-animal Γ' containing Γ such that*

$$\frac{1}{|\Gamma|} \sum_{v \in \Gamma} |\tilde{G}_{z(v)}| \le \frac{2}{|\Gamma'|} \sum_{v' \in \Gamma'} |\tilde{G}_{v'}|^2. \tag{A.1}$$

PROOF. We assume that $\tilde{G}_v$ is finite for each v in $\mathscr{V}$. [Even if not, the proof we give will work unless $\tilde{G}_{z(v)}$ is infinite for some $v \in \Gamma$; in that case, simply choose Γ' to include that $z(v)$.] The left-hand side of (A.1) is the mean of the numbers $|\tilde{G}_{z(v)}|$, for v in the G-animal Γ; we denote this mean by A. We will construct our new G-animal Γ' by enlarging Γ to include those $z(v)$s for which $|\tilde{G}_{z(v)}|$ is relatively large:

$$\Gamma' = \Gamma \cup \left(\bigcup_{v \in \Gamma_1} [\tilde{G}_{z(v)}] \right), \tag{A.2}$$

where

$$\Gamma_1 = \left\{ v \in \Gamma : |\tilde{G}_{z(v)}| \ge A/2 \right\}. \tag{A.3}$$

Because each (nonempty) $\tilde{G}_{z(v)}$ is connected and contains v, it follows that Γ' is a G-animal. For each $u \in (\Gamma' \setminus \Gamma)$ we can choose $z' = z'(u) \in \Gamma'$ with

$|\tilde{G}_{z'}| \geq A/2$ and such that $u \in \tilde{G}_{z'}$ [z' equals $z(v)$ for some $v \in \Gamma_1$]; for $u \in \Gamma_1$, we choose $z'(u) = z(u)$. We then have the following inequalities, which yield (A.1):

$$\begin{aligned}\sum_{v' \in \Gamma'} |\tilde{G}_{v'}|^2 &\geq \sum_{u \in \Gamma_1 \cup (\Gamma' \setminus \Gamma)} |\tilde{G}_{z'(u)}| \geq \sum_{v \in \Gamma_1} |\tilde{G}_{z(u)}| + \sum_{u \in \Gamma' \setminus \Gamma} \left(\frac{A}{2}\right) \\ &= A|\Gamma| - \sum_{u \in \Gamma \setminus \Gamma_1} |\tilde{G}_{z(u)}| + \frac{A}{2}|\Gamma' \setminus \Gamma| \\ &\geq A|\Gamma| - \frac{A}{2}|\Gamma| + \frac{A}{2}(|\Gamma'| - |\Gamma|) = \frac{A}{2}|\Gamma'|.\end{aligned} \tag{A.4}$$

To obtain the first inequality, group together those summands in the right-hand side with a fixed $z'(u) = v'$; because each such u belongs to $\tilde{G}_{v'}$, there can be at most $|\tilde{G}_{v'}|$ such summands (each of value $|\tilde{G}_{v'}|$). The second inequality is valid because $|\tilde{G}_{z'}| \geq A/2$ and the third is valid because for $u \in \Gamma \setminus \Gamma_1$, $|\tilde{G}_{z(u)}| < A/2$. □

The next two lemmas are corollaries of Lemma 1.

LEMMA 2. *Let $\tilde{G}_v$ be as in Lemma 1 and let*

$$\tilde{U}_v = \sup\left\{|\tilde{G}_u|: u \in \mathscr{V} \text{ and } v \in \tilde{G}_u\right\}. \tag{A.5}$$

(The sum over the empty set is taken to be zero.) Then for any G-animal Γ,

$$\frac{1}{|\Gamma|} \sum_{v \in \Gamma} \tilde{U}_v \leq 2 \sup_{\Gamma'} \frac{1}{|\Gamma'|} \sum_{v' \in \Gamma'} |\tilde{G}_{v'}|^2, \tag{A.6}$$

where the sup *is over G-animals Γ' that contain Γ.*

PROOF. By Lemma 1, we have for *any* allowed mapping $v \to z(v)$,

$$\frac{1}{|\Gamma|} \sum_{v \in \Gamma} |\tilde{G}_{z(v)}| \leq 2 \sup_{\Gamma'} \frac{1}{|\Gamma'|} \sum_{v' \in \Gamma'} |\tilde{G}_{v'}|^2. \tag{A.7}$$

Then take the supremum of this inequality over all allowed mappings. □

For the next lemma, we need a few definitions. The graph distance, $\text{dist}(u, v)$, between two vertices is the minimum length (number of edges) of any path on G connecting u and v. The (open) ball about v of radius $\rho \in [0, \infty)$, denoted $B(v, \rho)$, is the subgraph whose vertex set consists of all u with $\text{dist}(u, v) < \rho$ and whose edge set consists of all edges in $\mathscr{E}$ between pairs of such vertices. Clearly $B(v, \rho)$ is a connected subgraph containing v (or else it is empty, when $\rho = 0$). The volume of this ball is just $|B(v, \rho)|$, the number of vertices within distance ρ of v.

LEMMA 3. *Suppose $\tilde{R}_v \geq 0$ for each v in $\mathscr{V}$ and define*

$$\hat{U}_v = \sup\left\{|B(u, \tilde{R}_u)|: u \in \mathbb{Z}^d \text{ and } \operatorname{dist}(u, v) \leq \tilde{R}_u\right\}. \tag{A.8}$$

Then for any G-animal Γ,

$$\frac{1}{|\Gamma|} \sum_{v \in \Gamma} |\hat{U}_v| \leq 2 \sup_{\Gamma'} \frac{1}{|\Gamma'|} \sum_{v \in \Gamma'} |B(v', \tilde{R}_{v'})|^2,$$

where the sup *is over G-animals* Γ' *that contain* Γ.

PROOF. This is an immediate consequence of Lemma 2, by taking $\tilde{G}_v = B(v, \tilde{R}_v)$. □

NOTE ADDED IN PROOF: An alternative demonstration of the positivity of μ in the $d = 2$ Ising ferromagnet appears in a recent preprint of L. Chayes.

Acknowledgments. We thank Harry Kesten for very useful discussions about condition (1.11) and about first-passage percolation in general. We also thank Alberto Gandolfi and Harry Kesten for explanations of their results about greedy lattice animals. The original motivation for obtaining Theorems 5, 7 and 8 arose from joint work and discussions of one of the authors (C.M.N.) with Douglas Abraham, Jean Bricmont, Alan Sokal and Dan Stein.

REFERENCES

ABRAHAM, D. B. and NEWMAN, C. M. (1988). Wetting in a three-dimensional system: An exact solution. *Phys. Rev. Lett.* **61** 1969–1972.

ABRAHAM, D. B. and NEWMAN, C. M. (1989). Surfaces and Peierls contours: 3-d wetting and 2-d Ising percolation. *Comm. Math. Phys.* **125** 181–200.

ABRAHAM, D. B. and NEWMAN, C. M. (1991). The wetting transition in a random surface model. *J. Statist. Phys.* **63** 1097–1111.

AIZENMAN, M. and BARSKY, D. J. (1987). Sharpness of the phase transition in percolation models. *Comm. Math. Phys.* **108** 489–526.

BOIVIN, D. (1990). First passage percolation: The stationary case. *Probab. Theory Related Fields* **86** 491–499.

CHAYES, L. and WINFIELD, C. (1992). The density of interfaces: A new first passage problem. Preprint, UCLA.

CONIGLIO, A., NAPPI, C. R., PERUGGI, F. and RUSSO, L. (1976). Percolation and phase transition in the Ising model. *Comm. Math. Phys.* **51** 315–323.

COX, J. T. and DURRETT, R. (1981). Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.* **9** 583–603.

COX, J. T., GANDOLFI, A., GRIFFIN, P. and KESTEN, H. (1992). Greedy lattice animals I: Upper bounds. Syracuse–Berkeley–Cornell preprint. *Ann. Appl. Probab.* To appear.

EDEN, M. (1961). A two-dimensional growth process. *Proc. Fourth Berkeley Symp. Math. Statist. Probab.* **4** 223–239. Univ. California Press, Berkeley.

ESSAM, J. W. (1972). Percolation and cluster size. In *Phase Transitions and Critical Phenomena* (C. Domb and M. S. Green, eds.) **2** 197–270. Academic, New York.

FORTUIN, C. M., KASTELEYN, P. W. and GINIBRE, J. (1971). Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.* **22** 89–103.

GANDOLFI, A. and KESTEN, H. (1992). Greedy lattice animals II: Linear growth. Berkeley–Cornell preprint. *Ann. Appl. Probab.* To appear.

GEORGII, H.-O. (1988). *Gibbs Measures and Phase Transitions*. de Gruyter, Berlin.

GRIMMETT, G. (1989). *Percolation*. Springer, New York.

GRIMMETT, G. and KESTEN, H. (1984). First-passage percolation, network flows and electrical resistances. *Z. Wahrsch. Verw. Gebiete* **66** 335–366.

HAMMERSLEY, J. M. and WELSH, D. J. A. (1965). First-passage percolation, subadditive processes, stochastic networks and generalized renewal theory. In *Bernoulli, Bayes, Laplace Anniversary Volume* (J. Neyman and L. M. LeCam, eds.) 61–110. Springer, New York.

HARRIS, T. E. (1960). A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc.* **56** 13–20.

HIGUCHI, Y. (1982). Coexistence of the infinite (*) clusters: A remark on the square lattice site percolation. *Z. Wahrsch. Verw. Gebiete* **61** 75–81.

HIGUCHI, Y. (1992a). Coexistence of infinite (*)-clusters II: Ising percolation in two dimensions. Preprint, Kobe Univ.

HIGUCHI, Y. (1992b). A sharp transition for the two-dimensional Ising percolation. Preprint, Kobe Univ.

KESTEN, H. (1986). *Aspects of First-Passage Percolation. Lecture Notes in Math.* **1180** 125–264. Springer, New York.

KESTEN, H. (1987). Percolation theory and first-passage percolation. *Ann. Probab.* **15** 1231–1271.

KINGMAN, J. F. C. (1968). The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. B* **30** 499–510.

KRAMERS, H. A. and WANNIER, G. K. (1941). Statistics of the two-dimensional ferromagnet I, II. *Phys. Rev.* **60** 252–276.

MENSHIKOV, M. V. (1986). Coincidence of critical points in percolation problems. *Sov. Math. Dokl.* **33** 856–859.

NEWMAN, C. M. and STEIN, D. L. (1990). Broken symmetry and domain structure in Ising-like systems. *Phys. Rev. Lett.* **65** 460–463.

ONSAGER, L. (1944). Crystal statistics I. A two-dimensional model with an order–disorder transition. *Phys. Rev.* **65** 117–149.

RICHARDSON, D. (1973). Random growth in a tesselation. *Proc. Cambridge Philos. Soc.* **74** 515–528.

SMYTHE, R. T. and WIERMAN, J. C. (1978). *First-Passage Percolation on the Square Lattice. Lecture Notes in Math.* **671**. Springer, New York.

ISTITUTO DE MATEMÁTICA
E ESTATÍSTICA-USP
CAIXA POSTAL 20570
SÃO PAULO, SP, 01498
BRAZIL

COURANT INSTITUTE
OF MATHEMATICAL SCIENCES
251 MERCER STREET
NEW YORK, NEW YORK 10012