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A PROBLEM OF SINGULAR STOCHASTIC CONTROL WITH DISCRETIONARY STOPPING[1]

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In this paper a simple problem of combined singular stochastic control and optimal stopping is formulated and solved. We find that the optimal strategies can take qualitatively different forms, depending on parameter values. We also study a variant on the problem in which the value function is inherently nonconvex. The proofs employ the generalised Itô formula applicable for differences of convex functions.

1. Introduction. Problems of combined continuous control and optimal stopping have been studied by several authors. Krylov [(1980), Section 6.4] gives some general conditions for optimality, while in a recent paper Beneš (1992) gives an explicit solution for an "LQG"-type problem with discretionary stopping. Here we study a variant of this problem in which the continuously acting control takes the form of *singular* control rather than the controlled drift of Beneš (1992). Singular control arises when the control acts additively on the system model and a cost is paid for the total variation of control effort, representing the use of fuel. Several problems of this sort have been solved in recent years; a typical example, taken from Karatzas (1983), is described in Section 2. Discretionary stopping in singular control arises in at least two applications. One is in target tracking problems where one has to decide when one is "sufficiently close" to the target. A second application is in consumption/investment problems of financial economics. It is shown in Davis and Norman (1990) that trading strategies are naturally of singular control type in models including transaction costs, and if an American option is held in a portfolio, then its exercise time—a stopping time—is an additional decision variable.

The purpose of this paper is to analyse a simple model involving both singular control and discretionary stopping, with a view to discovering what the nature of optimal strategies is for such problems. The model is described in Section 3, where the main results of the paper are stated. Briefly, we find that qualitatively different kinds of optimal behaviour can occur, depending on parameter values. Proofs are given in Section 4 and involve the use of a generalised Itô formula applicable to differences of convex functions. A vari-

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ant of the problem, whose value function is inherently nonconvex, is studied in Section 5.

2. An example of singular control. Let $(\Omega, \mathscr{F}, P)$ be a probability space equipped with a filtration $\{\mathscr{F}_t\}_{t \geq 0}$ satisfying the usual conditions of right continuity and augmentation by P-negligible sets and carrying a standard one-dimensional $\{\mathscr{F}_t\}$-Brownian motion $\{w_t\}_{t \geq 0}$. Also, let $\mathscr{C}$ denote the set of $\{\mathscr{F}_t\}$-adapted, right-continuous processes ξ such that with probability 1 the sample path $t \to \xi_t(\omega)$ has bounded variation on any compact subset of $[0, \infty)$, and $\xi_{0-}(\omega) = 0$. A process $\xi \in \mathscr{C}$ will be represented as $\xi_t = \xi_t^+ - \xi_t^-$, $t \geq 0$, where the processes $\xi^{\pm} \in \mathscr{C}$ are nondecreasing and the representation is minimal, so that the total variation $\check{\xi}$ of ξ on the interval $[0, t]$ can be written in the form $\check{\xi}_t = \xi_t^+ + \xi_t^-$. Also, note that throughout this paper, we use the notation "$\int_0^t$" for integrals of the form $\int_{[0,t]}$.

For $\xi \in \mathscr{C}$ and $x \in \mathbb{R}$, define

$$x_t = x + w_t + \xi_t \tag{1}$$

and

$$J_x(\xi) = E\int_0^\infty e^{-\delta t}\left[\lambda x_t^2\, dt + d\check{\xi}_t\right], \tag{2}$$

where $\lambda, \delta > 0$ are given constants. The problem is to minimise $J_x(\xi)$ over $\xi \in \mathscr{C}$. The solution to this problem is described as follows [Karatzas (1983)]. Suppose that an even C^2 function v satisfies, for some $b > 0$,

$$\tfrac{1}{2}v''(x) - \delta v(x) + \lambda x^2 = 0, \quad x \in [-b, b], \tag{3a}$$

$$v(x) = x - b + v(b), \qquad x \geq b, \tag{3b}$$

$$v(x) = -x - b + v(b), \qquad x \leq -b. \tag{3c}$$

The C^2 property implies that $v'(b-) = 1$ and $v''(b-) = 0$. Now define the process $\tilde{\xi}_t := \tilde{\xi}_t^+ - \tilde{\xi}_t^-$, where the processes $\tilde{\xi}_t^{\pm}$ are the unique solution to the pair of functional equations

$$\tilde{\xi}_t^+ = \max\left[0, \max_{0 \leq s \leq t}\left\{-x - w_s + \tilde{\xi}_s^- - b\right\}\right] \tag{4}$$

and

$$\tilde{\xi}_t^- = \max\left[0, \max_{0 \leq s \leq t}\left\{x + w_s + \tilde{\xi}_s^+ - b\right\}\right]. \tag{5}$$

It can be shown by an application of the Doléans–Dade–Meyer change of variables formula that $\tilde{\xi}_t$ is optimal in $\mathscr{C}$ and that $v(x) = J_x(\tilde{\xi})$.

In fact, equations (3a)–(3c) have a unique C^2 solution. The general solution of (3a) is

$$v(x) = A\cosh\sqrt{2\delta}\,x + B\sinh\sqrt{2\delta}\,x + \frac{\lambda}{\delta}x^2 + \frac{\lambda}{\delta^2}. \tag{6}$$

If v is even, then $B = 0$ and the boundary conditions $v'(b) = 1$ and $v''(b) = 0$ provide a pair of transcendental equations for the two remaining parameters

b and A. It is shown in Karatzas (1983) that these equations have a unique solution.

Note from (4) and (5) that the optimal strategy involves three "tactics": *move*, *wait* and *reflect*. If we start from $x > b$ (or $x < -b$) there is an immediate *move* to b (or $-b$). The controller takes no action (i.e., *waits*) while $x_t \in (-b, b)$, whereas at $\pm b$ it takes "minimal action" (*reflect*) to keep x_t inside $[-b, b]$.

3. Discretionary stopping. We now include the possibility of stopping in the above problem, thus adding *stop* to the above repertoire of tactics. Let $\mathscr{T}$ be the class of all $\{\mathscr{F}_t\}$-stopping times. For $\xi \in \mathscr{C}$ and $\tau \in \mathscr{T}$, define

$$J_x(\xi, \tau) = E\left\{\int_0^\tau e^{-\delta t}\left[\lambda x_t^2 \, dt + d\check{\xi}_t\right] + e^{-\delta\tau}\alpha x_\tau^2\right\}, \tag{7}$$

where $\alpha > 0$ is a further parameter. We now seek a pair (ξ, τ) to minimise $J_x(\xi, \tau)$.

Some immediate simplifications can be made to this problem formulation. First, it is clear that the process should be stopped at 0 if 0 is ever reached and that it cannot be optimal to "jump across" 0, that is, to introduce a control ξ_t such that for some t, $x_{t-} > 0$ and $x_t < 0$ or conversely. Also, a simple contradiction argument shows that no control with $d\xi_t^+ > 0$ when $x_t > 0$ or $d\xi_t^- < 0$ when $x_t < 0$ can be optimal. All of these assertions can be checked post facto. For these reasons, we can assume for the rest of this section that $x > 0$ and that the processes x_t evolve on the positive half-line $\mathbb{R}^+$ with monotone control $\xi_t = -\xi_t^-$ (the situation for starting points $x < 0$ being just the mirror image of this).

By convention, $\xi_{0-} = 0$ so that ξ_0 denotes the jump in ξ_t at $t = 0$. Since the stopping cost αx^2 is minimum at $x = 0$, it is natural to envisage that the optimal stopping set is a neighbourhood of zero. One possible strategy is "move-and-stop", that is, for some $a \geq 0$, take $\tilde{\xi}_0 = 0$ if $x \leq a$, $\tilde{\xi}_0 = x - a$ if $x > a$, and $\tilde{\tau} = 0$. The cost of this strategy is

$$J_x(\tilde{\xi}, \tilde{\tau}) = \begin{cases} \alpha x^2, & x \leq a, \\ x - a + \alpha a^2, & x \geq a. \end{cases} \tag{8}$$

Clearly, the best choice of a is the abscissa of the point at which a straight line with slope 1 is tangent to the function αx^2, that is $a = 1/2\alpha$ (see Figure 1). Thus the cost of the best move-and-stop strategy is the C^1 function

$$v(x) = \begin{cases} \alpha x^2, & x \leq 1/2\alpha, \\ x - 1/4\alpha, & x \geq 1/2\alpha. \end{cases} \tag{9}$$

Is move-and-stop the best strategy? Answer: Yes, if $\alpha\delta \leq \lambda$ (see Theorem 1).

Bearing in mind the solution when $\tau \equiv \infty$ of Section 2, another possible strategy is to introduce a stopping barrier at some point $a \geq 0$ and a

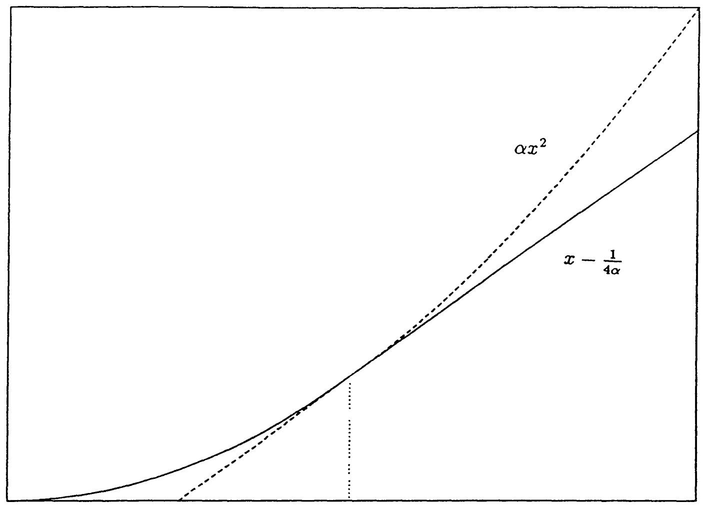

FIG. 1. *Optimal "move-and-stop" policy.*

reflecting barrier at some point $b > a$. The corresponding controlled process is

$$\bar{\xi}_t = \max\Big[0, \max_{0 \le s \le t} \{x + w_s - b\}\Big] \tag{10}$$

and the corresponding stopping time is

$$\bar{\tau} = \inf\{t \ge 0: x_t \in [0, a]\}. \tag{11}$$

By the recurrence property of Brownian motion, $P[\bar{\tau} < \infty] = 1$ for all $x \in \mathbb{R}$.

We conjecture that if this strategy is optimal for some a and b, then the corresponding cost function $v(x) = J_x(\bar{\xi}, \bar{\tau})$ will be C^2 at b (as in Section 2) and C^1 at a (the "smooth pasting" condition of optimal stopping). Since there is no control in the open set (a, b), v will satisfy

$$\tfrac{1}{2}v''(x) - \delta v(x) + \lambda x^2 = 0, \qquad x \in (a, b), \tag{12}$$

and the conjectured boundary conditions are

$$\begin{aligned} v(a) &= \alpha a^2, & v'(b) &= 1, \\ v'(a) &= 2\alpha a, & v''(b) &= 0. \end{aligned} \tag{13}$$

The general solution of (12) is of the form (6), and we therefore have four boundary conditions (13) to settle the four parameters a, b, A and B. Outside (a, b), v is described by

$$v(x) = \alpha x^2, \qquad x \in [0, a], \tag{14}$$

and

$$v(x) = x - b + v(b), \qquad x \ge b. \tag{15}$$

It turns out that a strategy of this form is optimal whenever $\alpha\delta > \lambda$.

THEOREM 1. *For the process x_t given by* (1), *consider the problem of minimising the cost function given by* (7) *over all strategies $\xi \in \mathscr{C}$ and $\tau \in \mathscr{T}$.*

Case 1. *If $\alpha\delta \leq \lambda$, then the move-and-stop strategy, that is, $\tilde{\xi}_0^+ = \max\{0, -x - a\}$, $\tilde{\xi}_0^- = \max\{0, x - a\}$ and $\tilde{\tau} = 0$, is optimal, with stopping barrier $a = 1/(2\alpha)$.*

Case 2. *If $\alpha\delta > \lambda$, then there exist a and b with $0 < a < b < \infty$ such that the strategy $(\bar{\xi}, \bar{\tau})$ described by $\bar{\xi}_t^+ = \max[0, \max_{0 \leq s \leq t}\{-x - w_s - b\}]$, $\bar{\xi}_t^- = \max[0, \max_{0 \leq s \leq t}\{x + w_s - b\}]$ and $\bar{\tau} = \inf\{t \geq 0\colon x_t \in [-a, a]\}$ is optimal with cost $v(x)$ which is an even function of x and satisfies* (12)–(15). *The parameters a and b are uniquely fixed by the requirement that $v(x)$ satisfy* (12), (14) *and* (15) *and be C^1 at a and C^2 at b.*

The proof of Theorem 1 is given in the next section. We have to show (in Case 2) that the free boundary problem (12), (13) has a unique solution and then (in either case) that the conjectured solutions are indeed optimal.

4. Proof of Theorem 1. Denote the infimum of $J_x(\xi, \tau)$ by $v(x)$. It will become clear in the development of the proof of Theorem 1 that the optimal cost $v(x)$ must satisfy the following variational inequalities:

(i) $v(x) \leq \alpha x^2$;

(ii) $|v'(x)| \leq 1$;

(iii) $\frac{1}{2}v''(x) - \delta v(x) + \lambda x^2 \geq 0$;

(iv) $(v(x) - \alpha x^2)(|v'(x)| - 1)(\frac{1}{2}v''(x) - \delta v(x) + \lambda x^2) = 0$.

In the next three lemmas we consider the "reduced" problem where $x \geq 0$. The following lemma describes the C^1 solution to the variational inequalities (i)–(iv) when the move-and-stop strategy is optimal.

LEMMA 2. *The function defined by*

$$v(x) = \begin{cases} \alpha x^2, & 0 \leq x \leq 1/2\alpha, \\ x - 1/(4\alpha), & 1/(2\alpha) \leq x, \end{cases} \tag{16}$$

satisfies inequalities (i) *and* (ii) *for all $x \in \mathbb{R}^+$, and equation* (iv) *for all $x \in \mathbb{R}^+ - \{1/(2\alpha)\}$. Also, it satisfies inequality* (iii) *for any $x \in \mathbb{R}^+ - \{1/(2\alpha)\}$ if and only if $\alpha\delta - \lambda \leq 0$.*

PROOF. It is straightforward to verify the first three assertions. Inequality (iii) is equivalent to the following two inequalities:

$$-(\alpha\delta - \lambda)x^2 + \alpha \geq 0, \qquad 0 \leq x < \frac{1}{2\alpha}, \tag{17}$$

$$\lambda x^2 - \delta x + \frac{\delta}{4\alpha} \geq 0, \qquad \frac{1}{2\alpha} < x. \tag{18}$$

If $\alpha\delta - \lambda \le 0$, then (17) holds trivially, and (18) holds because the quadratic appearing on its left-hand side has no real roots or one double root [since its discriminant is $\Delta = \delta(\alpha\delta - \lambda)/\alpha \le 0$]. On the other hand, assuming that $\alpha\delta - \lambda > 0$, it is easy to verify that $1/(2\alpha)$ is strictly smaller than the largest root of the quadratic on the left-hand side of (18), and so (18) cannot hold. □

The next lemma proves that the free boundary value problem described in Section 3 has actually a unique solution.

LEMMA 3. *The differential equation*

$$\tfrac{1}{2}f''(x) - \delta f(x) + \lambda x^2 = 0 \tag{19}$$

with boundary values

$$\begin{aligned} f(a) &= \alpha a^2, & f'(b) &= 1, \\ f'(a) &= 2\alpha a, & f''(b) &= 0, \end{aligned} \tag{20}$$

has a unique solution with $b > a > 0$ if and only if $\alpha\delta - \lambda > 0$.

PROOF. The general solution of the differential equation (19) is

$$f(x) = A\exp(x\sqrt{2\delta}) + B\exp(-x\sqrt{2\delta}) + \frac{\lambda}{\delta}x^2 + \frac{\lambda}{\delta^2} \tag{21}$$

The boundary values (20) give rise to the following system of transcendental equations:

$$A\exp(a\sqrt{2\delta}) = \frac{1}{2\delta^2}\left[\delta(\alpha\delta - \lambda)a^2 + \sqrt{2\delta}(\alpha\delta - \lambda)a - \lambda\right]; \tag{22}$$

$$B\exp(-a\sqrt{2\delta}) = \frac{1}{2\delta^2}\left[\delta(\alpha\delta - \lambda)a^2 - \sqrt{2\delta}(\alpha\delta - \lambda)a - \lambda\right]; \tag{23}$$

$$A\exp(b\sqrt{2\delta}) = \frac{1}{4\delta^2}\left[-2\lambda - \sqrt{2\delta}(2\lambda b - \delta)\right]; \tag{24}$$

$$B\exp(-b\sqrt{2\delta}) = \frac{1}{4\delta^2}\left[-2\lambda + \sqrt{2\delta}(2\lambda b - \delta)\right]. \tag{25}$$

If $A \ge 0$, then (24) implies $-2\lambda \ge \sqrt{2\delta}(2\lambda b - \delta)$ and so $B < 0$ by (25). On the other hand, if $B \ge 0$, then (25) holds if $\sqrt{2\delta}(2\lambda b - \delta) \ge 2\lambda$ and hence $A < 0$ by (24). Consequently, at least one of A, B must be negative.

Either of the quadratic forms which appear on the right-hand side of equations (22) and (23) has one positive and one negative root. Hence, since at least one of A, B is negative, a must be smaller than the largest of their roots

$$a < \frac{\sqrt{2\delta}(\alpha\delta - \lambda) + \sqrt{2\delta(\alpha^2\delta^2 - \lambda^2)}}{2\delta(\alpha\delta - \lambda)} =: \rho_1. \tag{26}$$

Consequently, if (19) and (20) have a solution, then $a \in [0, \rho_1)$ and $B < 0$.

Dividing (24) by (22), and (23) by (25) yields

$$\exp\left[\sqrt{2\delta}(b-a)\right] = \frac{-2\lambda - \sqrt{2\delta}(2\lambda b - \delta)}{2\left[\delta(\alpha\delta-\lambda)a^2 + \sqrt{2\delta}(\alpha\delta-\lambda)a - \lambda\right]} \tag{27}$$

and

$$\exp\left[\sqrt{2\delta}(b-a)\right] = \frac{2\left[\delta(\alpha\delta-\lambda)a^2 - \sqrt{2\delta}(\alpha\delta-\lambda)a - \lambda\right]}{-2\lambda + \sqrt{2\delta}(2\lambda b - \delta)}. \tag{28}$$

Equating the right-hand sides of (27) and (28) leads to

$$2\delta(2\lambda b - \delta)^2 = 4\delta^2(\alpha\delta-\lambda)\left[2\alpha - (\alpha\delta-\lambda)a^2\right]a^2. \tag{29}$$

If $\alpha\delta - \lambda < 0$, the right-hand side of this equation is negative, and so there can be no solution. On the other hand, if $\alpha\delta - \lambda = 0$, (29) and (22)–(25) imply $a = b$. Hence, (19) and (20) do not accept a solution with $b > a \geq 0$ for $\alpha\delta - \lambda \leq 0$. Note also that it is straightforward to verify that $\rho_1^2 < 2\alpha/(\alpha\delta - \lambda)$.

Equation (29) gives rise to two possible cases.

Case 1. $\sqrt{2\delta}(2\lambda b - \delta) \leq 0$. Assume that the system of equations (27) [or (28)] and (29) has (at least) one solution. Since $B < 0$, $b > a$ implies $B\exp\{-b\sqrt{2\delta}\} > B\exp\{-a\sqrt{2\delta}\}$, and, using (23) and (25),

$$-\sqrt{2\delta}(2\lambda b - \delta) < -2\delta(\alpha\delta-\lambda)a^2 + 2\sqrt{2\delta}(\alpha\delta-\lambda)a. \tag{30}$$

The left-hand side of this inequality is nonnegative by assumption, and so the inequality is equivalent to the one obtained by squaring its two sides. Hence, using (29) and after some simple calculations, (30) is shown to be equivalent to

$$\delta(\alpha\delta-\lambda)a^2 - \sqrt{2\delta}(\alpha\delta-\lambda)a - \lambda > 0,$$

but then $B > 0$ by (23), which is a contradiction.

Case 2. $\sqrt{2\delta}(2\lambda b - \delta) > 0$. In this case, equation (29) gives

$$b = \frac{\delta}{2\lambda} + \frac{\sqrt{2\delta}(\alpha\delta-\lambda)}{2\lambda}a\sqrt{\frac{2\alpha}{\alpha\delta-\lambda} - a^2}. \tag{31}$$

Substitution into equation (27) yields

$$\begin{aligned} g_1(a) &:= \left[-\lambda - \delta(\alpha\delta-\lambda)a\sqrt{\frac{2\alpha}{\alpha\delta-\lambda} - a^2}\right] \\ &\quad\times\left[\delta(\alpha\delta-\lambda)a^2 + \sqrt{2\delta}(\alpha\delta-\lambda)a - \lambda\right]^{-1} \\ &= \exp\left\{\frac{\delta\sqrt{2\delta}}{2\lambda} + \frac{\delta(\alpha\delta-\lambda)}{\lambda}a\sqrt{\frac{2\alpha}{\alpha\delta-\lambda} - a^2} - \sqrt{2\delta}a\right\} \\ &=: g_2(a) \end{aligned} \tag{32}$$

The numerator of the function g_1 is bounded and negative in the interval $[0, \rho_1)$. The denominator of g_1 has a root in $[0, \rho_1)$—denote it by ρ_2—and is negative in the interval $[0, \rho_2)$ and positive in the interval (ρ_2, ρ_1). Consequently, since $g_1(0) = 1$, the function g_1 is increasing from 1 to $+\infty$ as a moves from 0 to ρ_2 and is negative in the interval (ρ_2, ρ_1). On the other hand, the function g_2 is bounded and positive in $[0, \rho_1)$. Hence, since $g_2(0) = \exp\{\delta\sqrt{2\delta}/2\lambda\} > 1$, equation (32) has at least one solution which lies in $(0, \rho_2)$. It is a tedious but straightforward exercise to verify that $d \ln g_1(a)/da > d \ln g_2(a)/da$ for any a in the interval of interest. Hence the equation $\ln g_1(a) = \ln g_2(a)$ which is equivalent to (32) has a unique solution. Also, note that since the unique solution of (32) lies in $(0, \rho_2)$, we have A, $B < 0$.

In order to see that $b > a$, observe that since $A < 0$, $b > a$ if and only if $A \exp\{b\sqrt{2\delta}\} < A \exp\{a\sqrt{2\delta}\}$. Using (22) and (24), this is equivalent to

$$(33) \qquad -\sqrt{2\delta}(2\lambda b - \delta) < 2\delta(\alpha\delta - \lambda)a^2 + 2\sqrt{2\delta}(\alpha\delta - \lambda)a.$$

However, this inequality holds identically since its left-hand side is negative by assumption and its right-hand side is positive. □

REMARK 1. In order to obtain some qualitative feeling about how big the "gap" $b - a$ can be, fix α and λ and parametrise a and b by $\delta > \lambda/\alpha$. With reference to inequality (26), observe that $\lim_{\delta \to \infty} \rho_1(\delta) = 0$, and hence $\lim_{\delta \to \infty} a(\delta) = 0$. On the other hand, (31) implies that $\lim_{\delta \to \infty} b(\delta) = \infty$. We conclude that the "gap" $b - a$ can be arbitrarily large, depending on how large δ is. As an example, in Figure 2 we plot the barriers a and b as

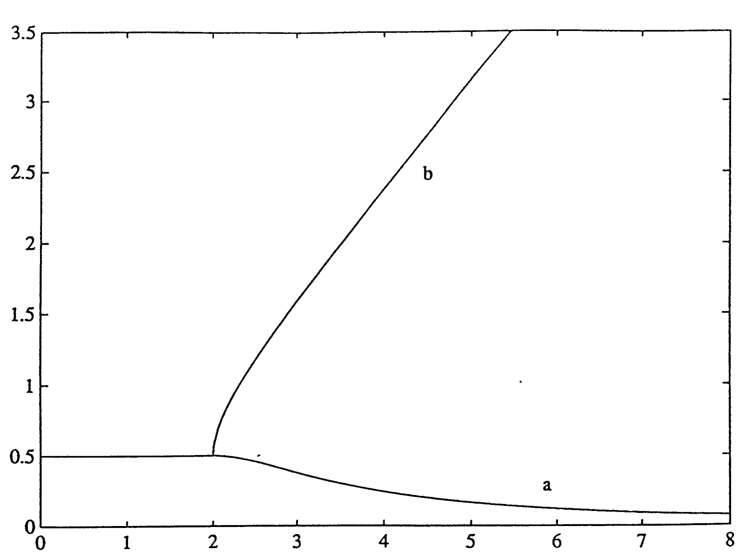

FIG. 2. *Barriers a and b as functions of δ with fixed $\alpha = 1$, $\lambda = 2$.*

functions of δ for $\alpha = 1$, $\lambda = 2$. Similar observations hold if we parametrise a and b by α or λ. □

The following lemma describes the C^1 solution to the variational inequalities (i)–(iv) for Case 2 of Theorem 1.

LEMMA 4. *Let $\alpha\delta - \lambda > 0$, and let the positive real numbers a and b and the function f: $[a, b] \to \mathbb{R}^+$ solve the problem stated in Lemma 3. The function defined by*

$$(34)\qquad v(x) = \begin{cases} \alpha x^2, & 0 \le x \le a, \\ f(x), & a \le x \le b, \\ x - b + \lambda b^2/\delta, & b \le x, \end{cases}$$

satisfies inequalities (i) *and* (ii) *for all $x \in \mathbb{R}^+$, and inequality* (iii) *and equation* (iv) *for any $x \in \mathbb{R}^+ - \{a\}$.*

PROOF. The first step is to prove that v is convex. Clearly, $v''(x) \ge 0$ for $x \in [0, a) \cup [b, \infty)$. For $x \in (a, b)$, use equations (22) and (23) to obtain

$$(35)\qquad v'''(x) = \frac{2\sqrt{2\delta}}{\delta}\Big\{\big[\delta(\alpha\delta - \lambda)a^2 - \lambda\big]\sinh\sqrt{2\delta}(x-a) + \sqrt{2\delta}(\alpha\delta - \lambda)a\cosh\sqrt{2\delta}(x-a)\Big\}.$$

Equation $v'''(x) = 0$ is then equivalent to

$$(36)\qquad \tanh\sqrt{2\delta}(x-a) = -\frac{\sqrt{2\delta}(\alpha\delta - \lambda)a}{\delta(\alpha\delta - \lambda)a^2 - \lambda}.$$

The right-hand side of (36) is positive and smaller than 1, and so (36) has a unique solution, denoted r. Since $v''(a+) = 2(\alpha\delta - \lambda)a^2 > 0$, $v''(b-) = 0$ and $v'''(a+) = 4(\alpha\delta - \lambda)a > 0$ by (35), $r \in (a, b)$ and $v''(x)$ increases (strictly) from some positive constant as x moves from a to r and then decreases (strictly) to zero as x moves from r to b. Consequently, $v''(x) > 0$ for any x in (a, b). Hence, the function v is convex.

Since $\alpha\delta - \lambda > 0$ is equivalent to $2\lambda/\delta < 2\alpha$, and the parameters A and B appearing in the proof of Lemma 3 are both negative, $v''(x) < 2\lambda/\delta < 2\alpha$, $\forall x \in (a, b)$. Consequently, $v''(x) \le 2\alpha$ for any x, and so $v(x) \le \alpha x^2$ for all x [since $v(0) = 0$ and $v'(0) = 0$].

For $x \ge b$, $v'(x) = 1$. For $0 \le x \le b$, $0 \le v'(x) \le 1$ because the function v' is increasing, since v is convex. Hence, v satisfies inequality (ii) for any $x \ge 0$.

The facts that $v''(x) \le 2\alpha$ for any x, $v''(a+) = 2(\alpha\delta - \lambda)a^2$ and that v'' is strictly increasing at $x = a+$ imply

$$(37)\qquad (\alpha\delta - \lambda)a^2 < \alpha.$$

For $x \in [0, a)$,

(38) $\frac{1}{2}v''(x) - \delta v(x) + \lambda x^2 = \alpha - (\alpha\delta - \lambda)x^2 \geq \alpha - (\alpha\delta - \lambda)a^2 > 0,$

with the last inequality holding because of (37). For $x \in (a, b)$, $\frac{1}{2}v''(x) - \delta v(x) + \lambda x^2 = 0$ by construction. For $x \in [b, \infty)$,

$$\begin{aligned} \tfrac{1}{2}v''(x) - \delta v(x) + \lambda x^2 &= (x - b)[\lambda(x + b) - \delta] \\ &\geq (x - b)(2\lambda b - \delta) > 0, \end{aligned} \tag{39}$$

with the last inequality holding because of Lemma 3 (see Case 2, in which the solution is attained). Consequently, inequality (iii) holds for any $x \in \mathbb{R}^+ - \{a\}$.

Finally, equation (iv) holds trivially for all $x \in \mathbb{R}^+ - \{a\}$ by the construction of the function v. □

Proof of Theorem 1. First, extend the functions v appearing in Lemmas 2 and 4 in the whole real line by even symmetry, and note that the "extended" functions are C^2 at 0. In the case that $\alpha\delta - \lambda \leq 0$, consider the function v described in Lemma 2, and in the case that $\alpha\delta - \lambda > 0$, consider the function v described in Lemma 4.

Since the function v is C^1 and its second derivative is piecewise continuous, it can be written as the difference of the convex functions

$$v_1(x) = v(0) + xv'(0) + \int_0^x \int_0^y [f''(r)]^+ \, dr\, dy$$

and

$$v_2(x) = \int_0^x \int_0^y [f''(r)]^- \, dr\, dy.$$

Hence, the Meyer–Itô formula [Protter (1992), Theorem IV.51, or Meyer (1976), VI.II] can be applied to yield

$$\begin{aligned} v(x_t) - v(x_{0-}) = {} & \int_0^t v'(x_{s-})\, dx_s + \tfrac{1}{2}\int_{-\infty}^{\infty} L_t^a \mu(da) \\ & + \sum_{0 \leq s \leq t} [v(x_s) - v(x_{s-}) - v'(x_{s-})\,\Delta x_s], \end{aligned} \tag{40}$$

where x_t is given by (1) for arbitrary $\xi \in \mathscr{C}$, $x_{0-} = x$ and L_t^a is the local time of the semimartingale x_t at a; μ is the signed measure which is the second derivative of v in the generalised function sense. However, $v \in C^1(\mathbb{R})$ implies that v'' exists for Lebesgue almost all x, and so

$$\begin{aligned} \int_{-\infty}^{\infty} L_t^a \mu(da) &= \int_{-\infty}^{\infty} L_t^a v''(a)\, da \\ &= \int_0^t v''(x_{s-})\, ds, \end{aligned} \tag{41}$$

the second equality following from Corollary 1 to Theorem IV.51 in Protter (1992). Now consider the finite variation, previsible process $y_t = e^{-\delta t}$ and the semimartingale $z_t = v(x_t)$, and recall the integration by parts formula [Meyer (1976), IV.38]

$$y_\tau z_\tau - y_{0-} z_{0-} = \int_0^\tau y_t \, dz_t + \int_0^\tau z_{t-} \, dy_t. \tag{42}$$

Equations (42), (40) and (41) give

$$\begin{aligned} & e^{-\delta\tau} v(x_\tau) - v(x) \\ &\quad = \int_0^\tau e^{-\delta t} v'(x_{t-}) \, dw_t + \int_0^\tau e^{-\delta t}\left[\tfrac{1}{2} v''(x_t) - \delta v(x_t)\right] dt \\ &\qquad + \int_0^\tau e^{-\delta t} v'(x_{t-}) \, d\xi_t \\ &\qquad + \sum_{0 \le t \le \tau} e^{-\delta t}\left[v(x_t) - v(x_{t-}) - v'(x_{t-}) \, \Delta x_t\right]. \end{aligned} \tag{43}$$

Adding

$$\int_0^\tau e^{-\delta t}\left[\lambda x_t^2 \, dt + d\check{\xi}_t\right] + e^{-\delta\tau} \alpha x_\tau^2$$

to both sides of (43), rearranging terms and using the identities (valid, since $\Delta x_t = \Delta \xi_t$)

$$\xi_t = [\xi^c]_t^+ - [\xi^c]_t^- + \sum_{0 \le s \le t} \Delta x_t, \tag{44}$$

$$\check{\xi}_t = [\xi^c]_t^+ + [\xi^c]_t^- + \sum_{0 \le s \le t} |\Delta x_t|, \tag{45}$$

where ξ^c is the continuous part of ξ with $\xi_0^c = 0$, gives

$$\begin{aligned} & \int_0^\tau e^{-\delta t}\left[\lambda x_t^2 \, dt + d\check{\xi}_t\right] + e^{-\delta\tau} \alpha x_\tau^2 \\ &\quad = v(x) + e^{-\delta\tau}\left[\alpha x_\tau^2 - v(x_\tau)\right] + \int_0^\tau e^{-\delta t} v'(x_{t-}) \, dw_t \\ &\qquad + \int_0^\tau e^{-\delta t}\left[\tfrac{1}{2} v''(x_t) - \delta v(x_t) + \lambda x_t^2\right] dt \\ &\qquad + \int_0^\tau e^{-\delta t}\left\{[1 + v'(x_{t-})] \, d[\xi^c]_t^+ + [1 - v'(x_{t-})] \, d[\xi^c]_t^-\right\} \\ &\qquad + \sum_{0 \le t \le \tau} e^{-\delta t}\left[v(x_t) - v(x_{t-}) + |\Delta x_t|\right]. \end{aligned} \tag{46}$$

Now let any admissible strategy $\xi \in \mathscr{C}$ and $\tau \in \mathscr{T}$. Since the function v satisfies the variational inequalities (i)–(iii) (see Lemma 2 or 4 according to the case), it is clear that except for the stochastic integral, all the terms on the right-hand side of (46) are non-negative, and so

$$\int_0^\tau e^{-\delta t}\left[\lambda x_t^2 \, dt + d\check{\xi}_t\right] + e^{-\delta\tau} \alpha x_\tau^2 \ge v(x) + \int_0^\tau e^{-\delta t} v'(x_{t-}) \, dw_t. \tag{47}$$

Taking expectations, and noting that the expectation of the stochastic integral is zero, the above inequality yields

$$E\left\{\int_0^{\tau} e^{-\delta t}\left[\lambda x_t^2\, dt + d\check{\xi}_t\right] + e^{-\delta\tau}\alpha x_\tau^2\right\} \geq v(x). \tag{48}$$

In order to complete the proof, a strategy with cost given by v must be constructed, and this is possible because the functions v satisfy equality (iv).

If $\alpha\delta - \lambda \leq 0$, then the move-and-stop strategy $\tilde{\xi}$ and $\tilde{\tau}$ clearly yields a cost which is equal to $v(x)$.

Hence, consider $\alpha\delta - \lambda > 0$, and let $x > 0$. A careful inspection of equation (46) reveals that inequality (48) will hold with equality if the stopping strategy $\bar{\tau}$ is equal to the hitting time of the set $[-a, a]$, and the control strategy $\bar{\xi} = \bar{\xi}^+ - \bar{\xi}^-$ is constructed so that $\bar{\xi}^+ = 0$, and

$$x_t = x + w_t - \bar{\xi}_t^- \leq b, \tag{49a}$$

$$\bar{\xi}^- \text{ is nondecreasing} \tag{49b}$$

and

$$\bar{\xi}^- \text{ is flat off } \{t \geq 0\colon x_t \leq b\}, \text{ that is, } \int_0^{\tau} I_{\{x_s < b\}}\, d\bar{\xi}_t^- = 0. \tag{49c}$$

However, (49a)–(49c) is a Skorohod equation and it admits the unique, continuous solution [see, e.g., Karatzas and Shreve (1988), Lemma 3.6.14]

$$\bar{\xi}_t^- = \max\left[0, \max_{0 \leq s \leq t}\{x + w_s - b\}\right]. \tag{50}$$

Finally, the situation when $x < 0$ is treated similarly (it is the mirror image of the situation when $x > 0$). □

Remark 2. Note that the proof of Theorem 1 essentially contains a new verification theorem for one-dimensional problems [e.g., it is not a special case of Theorem VIII.4.1 in Fleming and Soner (1993) because the optimal cost function v is not required to be twice continuously differentiable on the set $\{x\colon |v'(x)| < 1\}$]. To see this, consider some general functions $h(x)$ and $g(x)$ (satisfying certain smoothness assumptions) in place of the functions λx^2 and αx^2, respectively. If a C^1 function w satisfies the inequalities (i)–(iii), then it can be shown as until equation (48) that $w(x)$ is smaller than or equal to the optimal cost $v(x)$. On the other hand, satisfaction of equality (iv) suggests that w is indeed the optimal cost function, with optimal strategy which "switches" among the tactics described in Sections 2 and 3 according to whether inequality (i), (ii) or (iii) holds with equality. Note also that convexity of the value function was established in the proof of Lemma 4 only in order to show that $|v'(x)| \leq 1$. Convexity plays no role in the "verification" argument given above.

5. A problem with inherently nonconvex value function. In this section we sketch the solution to a further problem involving singular control and discretionary stopping. Again, we find qualitative dependence of the optimal strategy on parameter values, but the real interest lies in the fact that this example exhibits a genuinely nonconvex character.

Everything else being the same as before, consider the minimisation of the cost

$$(51)\qquad I_x(\xi,\tau)=E\left\{\int_0^\tau e^{-\delta t}\left[\lambda\,dt+d\check{\xi}_t\right]+e^{-\delta\tau}\alpha x_\tau^2\right\}.$$

By considering the cost associated with the two extreme strategies, namely, stop immediately and do nothing, we can see that in this problem the optimal cost function v must satisfy, for all x,

$$(52)\qquad v(x)\le\min\left\{\alpha x^2,\ \frac{\lambda}{\delta}\right\}.$$

Hence, being bounded, equal to 0 at $x=0$ and positive elsewhere, the optimal cost function cannot be convex.

The variational inequalities for this problem are inequalities (i) and (ii) of the previous section and

(v) $\frac{1}{2}v''(x)-\delta v(x)+\lambda\ge 0$,

(vi) $(v(x)-\alpha x^2)(|v'(x)|-1)(\frac{1}{2}v''(x)-\delta v(x)+\lambda)=0$.

As in Section 3, consider (without real loss of generality) $x\ge 0$. One possible strategy is to "do nothing" while being inside the set (a,∞) for some $a>0$, and to stop as soon as the process hits the set $[0,a]$. Noting that the solution of the differential equation $\frac{1}{2}f''(x)-\delta f(x)+\lambda=0$ is

$$(53)\qquad f(x)=A\exp(x\sqrt{2\delta})+B\exp(-x\sqrt{2\delta})+\frac{\lambda}{\delta},$$

the cost of this strategy is given by the function

$$(54)\qquad v(x)=\begin{cases}\alpha x^2, & 0\le x\le a,\\ B\exp(-x\sqrt{2\delta})+\lambda/\delta, & a<x,\end{cases}$$

where we have used $A=0$ since v has to be bounded. Assuming C^1 fit at the point a, we obtain a system of two equations for the unknowns a and B. The solution to this is

$$(55)\qquad a=\frac{-\alpha+\sqrt{\alpha(\alpha+2\lambda)}}{\alpha\sqrt{2\delta}},\qquad B=-\frac{2\alpha a}{\sqrt{2\delta}}\exp(a\sqrt{2\delta}).$$

Using rather simple analytic arguments, we can show that the function v defined by (54) and (55) satisfies the variational inequalities (i), (ii), (v) and (vi) if and only if

$$(56)\qquad \lambda\le\frac{\delta+2\alpha\sqrt{2\delta}}{4\alpha}.$$

Because of this condition, we have to look for a further possible strategy. The natural generalisation of the strategy just described is to consider two barrier points $0 < a < b$. Whenever the process is inside the set (b, ∞), the controller takes no action at all. If the process is inside the set $(a, b]$, the controller moves it immediately to the point a. Finally, the process is stopped as soon as it hits the set $[0, a]$. The cost associated with this strategy is then given by

$$v(x) = \begin{cases} \alpha x^2, & 0 \le x \le a, \\ x - a + \alpha a^2, & a < x \le b, \\ B \exp(-x\sqrt{2\delta}) + \lambda/\delta, & b < x. \end{cases} \tag{57}$$

Again, assuming C^1 fit at both points a and b, we obtain a system of three equations for the three unknowns a, b and B. The solution to this system is

$$a = \frac{1}{2\alpha}, \qquad b = \frac{1}{4\alpha} + \frac{\lambda}{\delta} - \frac{1}{\sqrt{2\delta}}, \qquad B = -\frac{1}{\sqrt{2\delta}} \exp(b\sqrt{2\delta}). \tag{58}$$

From (58) we can easily calculate that $a < b$ if and only if

$$\lambda > \frac{\delta + 2\alpha\sqrt{2\delta}}{4\alpha}, \tag{59}$$

which is the logical complement of condition (56). Again, we can easily show that in this case the function defined by (57) and (58) satisfies the variational inequalities (i), (ii), (v) and (vi) [using also inequality (59)].

Finally, the verification that the strategies described above are indeed optimal [according to whether (56) or (59) holds] can be done in exactly the same way as in the proof of Theorem 1 (see also Remark 2).

At this point, note that $b \to \infty$ as $\delta \to 0$, showing that the "move-and-stop" strategy as described in Section 3 is always optimal for the cost function (51) with no discounting ($\delta = 0$). This has the same form as the cost function considered by Beneš (1992).

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