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FINITE MOMENTS FOR INVENTORY PROCESSES

BY KARL SIGMAN[1] AND DAVID D. YAO[2]

Columbia University

We study a continuous time inventory process that is a reflection mapping of a semimartingale *netput* process. Inventory processes of this type include the workload process in queues, dam and storage processes (with perhaps pure jump Lévy input), as well as processes arising in fluid models. We establish sufficient conditions on the netput ensuring that the steady-state inventory has *finite* moments of order $k \geq 1$, and derive explicit bounds for these moments. The sufficient conditions require that the netput have a negative (local) drift and that the (conditional) $(k+1)$th moment of its increments be bounded.

1. Introduction. The purpose of the present paper is to establish sufficient conditions that ensure finite moments of any desired order for the steady-state distribution of inventory processes. Let $Z(t)$ denote the amount of inventory at time t, satisfying the *inventory equation*

$$Z(t) = X(t) + L(t), \tag{1}$$

where $\{X(t): t \geq 0\}$, $X(0) = 0$, is a given *netput* with negative drift [refer to the condition given in (2) below] and $L(t)$ is defined as

$$L(t) := \sup_{0 \leq s \leq t} X^-(s), \quad \text{with } X^-(s) := -\min\{0, X(s)\}.$$

The setup here, in particular the term "netput," is borrowed from Harrison ([9], pages 18 and 19], where X is a Brownian motion. The mapping in (1), which takes a path of X and maps it to a path of Z, is sometimes called the *reflection mapping* and is of fundamental importance in a wide variety of applications both deterministic and stochastic. See, for example, Borovkov ([3], Chapter 1, Section 6), Chung and Williams ([4], Section 8.2), El Karoui and Chaleyat-Maurel [6], Glynn [8] and Prabhu [13], in addition to [9] cited above. Recent applications to inventory processes also appeared in Bardhan and Sigman [1, 2].

We study two cases: (a) the netput X has a bounded variation and (b) X is a semimartingale (martingale plus a process with bounded variation). Although (a) is a special case of (b), we choose to focus on (a) first so as to bring out the basic ingredients of our approach, which then generalizes easily to case (b) through incorporating certain martingale inequalities.

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In the case of bounded variation, $X(t)$ is of the form $A(t) - B(t)$, where $A = \{A(t): t \geq 0\}$ and $B = \{B(t): t \geq 0\}$ are two nonnegative and nondecreasing processes representing the input and the *potential* output, respectively. For example, in a standard single-server queueing model with $Z(t)$ denoting total workload, we have $B(t) = t$ and $A(t) = \sum_{n=1}^{N(t)} S_n$, where $\{N(t)\}$ denotes the counting process for arrivals and $\{S_n\}$ denotes customer service times. In this case, $L(t)$ is precisely the total idle time of the server during $[0, t]$. On the other hand, in an inventory model the input $A(t)$ need not be generated by a point process of arrivals; it can be a continuous flow process (e.g., have continuously differentiable sample paths) or a pure jump process (such as a pure jump Lévy process with infinite Lévy measure).

We derive sufficient conditions (not necessary in general) ensuring that the steady-state inventory has finite kth moments, $k \geq 1$, and obtain explicit bounds for these moments. The sufficient conditions require that the netput have a negative local drift and that the conditional $(k + 1)$th moments of its increments be bounded. In applications, finite moments are often useful performance measures. For instance, to guarantee the quality of service in communication systems, it is usually required that the expected delay be bounded by a given constant. Our conditions for finite moments and the computable bounds provide a practical means to support system design without having to make unnecessary independence and distributional assumptions on the underlying processes. Other possible contexts for applying our results include single-processor models such as fluid models (storage and dam models) as well as single-server queueing models (workload and queue length). On the other hand, we note that our stationary semimartingale framework, while quite general, does not cover certain processes that have recently found useful applications, for instance, fractional Brownian motion, which is not a semimartingale.

Our results are motivated by the classical finite moment conditions of Kiefer and Wolfowitz [11] for queues with renewal input and i.i.d. services, as well as the more recent works of Wolff [14] and Daley and Rolski [5]. By adopting the setup in (1), our results, even in the bounded variation case, are considerably more general than those for queueing models. Our approach, on the other hand, is largely influenced by the recent work of Meyn and Down [12], which studied the first moment of delay in a generalized Jackson network.

In Section 2 we present in detail the results for the case of bounded variation, followed by several examples in Section 3. Section 4 extends the results to the semimartingale case.

2. The case of bounded variation.

All stochastic processes that follow are assumed to have sample paths that are right continuous with left limits (e.g., in the space $\mathscr{D}[0, \infty)$ endowed with the Skorohod topology; see, e.g., [7]). Let the netput $X = \{X(t): t \geq 0\}$, $X(0) = 0$, be a stochastic process on an underlying probability space, $(\Omega, \mathscr{F}, P)$, with stationary and ergodic increments and *negative drift*:

$$\lim_{t\to\infty} X(t)/t = \mathsf{E}[X(1)] = -\alpha < 0 \quad \text{a.s.,} \tag{2}$$

where $\alpha > 0$ is a constant. (Note: the negative drift is a *stability* condition.) The inventory process Z is then constructed via (1) with $Z(0) = 0$. This ensures the existence of a unique proper steady-state distribution for Z, as well as the ergodicity of Z; refer to [3] (Chapter 1, Section 6) and [1] for details. Hence, in particular, for any nonnegative Borel measurable function $g: \Re_+ \mapsto \Re_+$, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t g(Z(s))\,ds = \mathsf{E}\big[g(\hat{Z})\big] \quad \text{a.s.,} \tag{3}$$

where $\hat{Z}$ denotes a random variable having the steady-state distribution:

$$\mathsf{P}(\hat{Z} \le x) = \lim_{t\to\infty} \mathsf{P}(Z(t) \le x). \tag{4}$$

Let $\mathscr{F}_t = \sigma\{X(s): 0 \le s \le t\}$ be our filtration. From (1), clearly $Z(t) \in \mathscr{F}_t$ for all t. Denote $\mathsf{E}_{\mathscr{F}_t}[X] := \mathsf{E}[X|\mathscr{F}_t]$. Assume that the netput is of the form $X(t) = A(t) - B(t)$, where A and B are both nonnegative and nondecreasing processes, $A(0) = 0$, $B(0) = 0$, with jointly stationary ergodic increments. Let

$$A(t,h) := A(t+h) - A(t) \quad \text{and} \quad B(t,h) := B(t+h) - B(t)$$

denote the increments, and similarly, $X(t,h) := X(t+h) - X(t)$.

The key conditions for our main result (Theorem 1) are as follows:

A1. There exist constants $\hat{\alpha} > 0$ and $l \ge 0$ such that for all $t \ge 0$ and $h > 0$,

$$\mathsf{E}_{\mathscr{F}_t}[X(t,h)] \le -\hat{\alpha}h + l.$$

A2 (k) ($k \ge 1$ is an integer). There exist functions $a_k: \Re_+ \mapsto \Re_+$ and $b_k: \Re_+ \mapsto \Re_+$, such that for all $t \ge 0$ and $h > 0$, $\int_0^h a_k(s)\,ds < \infty$,

$$\mathsf{E}_{\mathscr{F}_t}\big[A^{k+1}(t,h)\big] \le a_k(h) \quad \text{and} \quad \mathsf{E}_{\mathscr{F}_t}\big[B^{k+1}(t,h)\big] \le b_k(h).$$

THEOREM 1. *Suppose for a given $k \ge 1$, the netput $X = A - B$ satisfies* A1 *and* A2 (k). *Then there exists a constant $c_k < \infty$, computable in terms of a_k and b_k, such that*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathsf{E}\big[Z^k(s)\big]\,ds \le c_k \tag{5}$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Z^k(s)\,ds = \mathsf{E}[\hat{Z}^k] \le c_k \quad a.s., \tag{6}$$

where $\hat{Z}$ follows the steady-state distribution of Z in (4).

Note that A1 is a *local* drift condition, ensuring that the input does not overpower the output in small periods of time, and A2(k) ensures that the $(k+1)$th conditional moment of the input and the output in small intervals is not too large. It is easily seen that under A1 it is necessary that $\alpha \geq \hat{\alpha}$, since

$$\mathsf{E}[X(t,h)] = \mathsf{E}[X(h)] = -\alpha h$$

follows from the stationary increments and (2), while A1 implies (via taking expectation on both sides of the inequality)

$$\mathsf{E}[X(t,h)] \leq -\hat{\alpha} h + l.$$

Hence, $-\alpha h \leq -\hat{\alpha} h + l$ for all h, implying $\alpha \geq \hat{\alpha} > 0$. [In other words, A1 implies the negative drift condition in (2).]

To prove Theorem 1, we need several lemmas. The first one, an elementary result, is in the same spirit as (but simpler than) Lemma 2.3 of [12].

LEMMA 1. *Let f, g and c be finite functions on $\Re_+$, with f and g nonnegative and Lebesgue measurable. If for some fixed $h > 0$,*

$$f(s+h) \leq f(s) - g(s) + c(h) \qquad \forall s \in \Re_+,$$

then

$$\frac{1}{t}\int_0^t g(s)\,ds \leq c(h) + \frac{1}{t}\int_0^h f(s)\,ds \qquad \forall t > 0.$$

PROOF. Integrating over the first inequality,

$$\frac{1}{t}\int_0^t g(s)\,ds \leq c(h) + \frac{1}{t}\int_0^t [f(s) - f(s+h)]\,ds,$$

immediately leads to the second one by observing that

$$\int_0^t [f(s) - f(s+h)]\,ds = \left[\int_0^t - \int_h^{t+h}\right] f(s)\,ds \leq \left[\int_0^t - \int_h^t\right] f(s)\,ds$$
$$= \int_0^h f(s)\,ds. \qquad \square$$

To present the next two lemmas, we need more notation. From (1), it follows that Z satisfies the recursion

$$Z(t+h) = Z(t) + X(t,h) + L(t,h), \tag{7}$$

where $L(t,h) := L(t+h) - L(t)$, and it is known ([4], page 149) that

$$L(t,h) = \sup_{0 \leq u \leq h} [Z(t) + X(t,u)]^{-}. \tag{8}$$

In addition, define

$$D(t) := A(t) - Z(t) = B(t) - L(t) \quad \text{and} \quad D(t,h) := B(t,h) - L(t,h).$$

Intuitively, $D = \{D(t)\colon t \geq 0\}$ is the *throughput* process: $D(t)$ is the amount of input that has been processed during $[0,t]$ and $D(t,h)$ denotes the amount of input that is processed during $[t, t+h]$. Finally, let

$$\Delta(t,h) := A(t,h) - D(t,h) = X(t,h) + L(t,h)$$

and rewrite (7) as

$$Z(t+h) = Z(t) + \Delta(t,h). \tag{9}$$

It is useful to keep in mind the following facts: $A(t)$ and $B(t)$ are both nondecreasing by definition, and so is $L(t)$. Hence, $A(t,h)$, $B(t,h)$ and $L(t,h)$ are all nonnegative and nondecreasing in h.

LEMMA 2. *For any nonnegative t and h*:

(i) *$L(t,h) \le B(t,h)$, and hence $D(t,h) \ge 0$.*
(ii) *$Z(t)L(t,h) \le B^2(t,h)$.*

PROOF. (i) From (8), we have

$$\begin{aligned} L(t,h) &= \sup_{0 \le u \le h} [-\min\{0, Z(t) + X(t,u)\}] \\ &= \sup_{0 \le u \le h} [\max\{0, -Z(t) - A(t,u) + B(t,u)\}] \\ &\le \sup_{0 \le u \le h} [\max\{0, B(t,u)\}] = B(t,h), \end{aligned}$$

where the inequality follows from $Z(t) \ge 0$ and $A(t,u) \ge 0$, and the last equality follows from the nondecreasing property of $B(t,u)$ in u.

(ii) If $Z(t) \le B(t,h)$, the desired result follows immediately from (i). On the other hand, $Z(t) > B(t,h)$ implies

$$L(t,h) = \sup_{0 \le u \le h} [\max\{0, -Z(t) - A(t,u) + B(t,u)\}] = 0,$$

which also leads to the desired result. □

REMARK 1. A well-known property of the reflection mapping is that $L(t)$ remains a constant when $Z(t) > 0$ (see, e.g., [9], page 20). In other words, $Z(t) > 0$ implies $L(t,\varepsilon) = 0$ for a sufficiently small $\varepsilon > 0$. This property is now strengthened by the fact, brought up in the proof of Lemma 2(ii), that $Z(t) > B(t,h) \ge 0$ implies $L(t,h) = 0$ for *any* positive h.

LEMMA 3. (i) *Suppose* A1 *and* A2(1) *hold. Then, for any $t \ge 0$ and $h > 0$,*

$$Z(t)\mathsf{E}_{\mathscr{F}_t}[\Delta(t,h)] \le -(\hat{\alpha}h - l)Z(t) + b_1(h).$$

(ii) *Suppose* A2(j) *holds for some $j \ge 1$. Then, for any $t \ge 0$ and $h > 0$,*

$$\mathsf{E}_{\mathscr{F}_t}\left[|\Delta(t,h)|^{j+1}\right] \le a_j(h) + b_j(h).$$

PROOF. (i) Since $Z(t) \in \mathscr{F}_t$, we have

$$\begin{aligned} Z(t)\mathsf{E}_{\mathscr{F}_t}[\Delta(t,h)] &= \mathsf{E}_{\mathscr{F}_t}[Z(t)(X(t,h) + L(t,h))] \\ &\le Z(t)\mathsf{E}_{\mathscr{F}_t}[X(t,h)] + \mathsf{E}_{\mathscr{F}_t}[B^2(t,h)] \\ &\le Z(t)(-\hat{\alpha}h + l) + b_1(h), \end{aligned}$$

where the first inequality follows from Lemma 2(ii) and the second one makes use of A1 and A2(1).

(ii) Recall $\Delta(t,h) := A(t,h) - D(t,h)$ and $D(t,h) := B(t,h) - L(t,h) \le B(t,h)$. Since $A(t,h)$ and $D(t,h)$ are nonnegative [for the latter, refer to Lemma 2(i)], we have

$$|\Delta(t,h)| \le \max\{A(t,h), D(t,h)\} \le \max\{A(t,h), B(t,h)\}$$

and

$$|\Delta(t,h)|^{j+1} \le \max\{A^{j+1}(t,h), B^{j+1}(t,h)\} \le A^{j+1}(t,h) + B^{j+1}(t,h).$$

Applying $\mathsf{E}_{\mathscr{F}_t}$ on both sides and making use of A2(j) yields the desired result. □

We are now ready to prove Theorem 1.

PROOF OF THEOREM 1. First note that (6) follows directly from (5). Applying Fatou's lemma, we have

$$\mathsf{E}\left[\liminf \frac{1}{t}\int_0^t Z^k(s)\,ds\right] \le \liminf \frac{1}{t}\int_0^t \mathsf{E}[Z^k(s)]\,ds$$
$$\le \limsup \frac{1}{t}\int_0^t \mathsf{E}[Z^k(s)]\,ds \le c_k,$$

which implies that $\liminf(1/t)\int_0^t Z^k(s)\,ds < \infty$, almost surely. However, by the ergodicity of Z [via (3) with $g(x) = x^k$], the limit itself exists and is a constant. Hence,

$$\mathsf{E}[\hat{Z}^k] = \lim \frac{1}{t}\int_0^t Z^k(s)\,ds = \liminf \frac{1}{t}\int_0^t Z^k(s)\,ds \le c_k \quad \text{a.s.},$$

which is (6).

To prove (5), use induction on k. For $k = 1$, we proceed in a way that is similar to the proof of Theorem 2.1 of [12]. Squaring both sides of (9), taking conditional expectation and making use of Lemma 3(ii), we have

$$\mathsf{E}_{\mathscr{F}_t}[Z^2(t+h)] \le Z^2(t) + 2Z(t)\mathsf{E}_{\mathscr{F}_t}[\Delta(t,h)] + a_1(h) + b_1(h).$$

Applying Lemma 3(i) yields

$$\mathsf{E}_{\mathscr{F}_t}[Z^2(t+h)] \le Z^2(t) - 2(\hat{\alpha}h - l)Z(t) + c(h),$$

where $c(h) := a_1(h) + 3b_1(h)$. Taking expectation on both sides yields

$$\mathsf{E}[Z^2(t+h)] \le \mathsf{E}[Z^2(t)] - 2(\hat{\alpha}h - l)\mathsf{E}[Z(t)] + c(h).$$

Pick any $h > l/\hat{\alpha}$ (to ensure that $\hat{\alpha}h - l > 0$). Applying Lemma 1, we have

$$\frac{1}{t}\int_0^t 2(\hat{\alpha}h - l)\mathsf{E}[Z(s)]\,ds \le c(h) + \frac{1}{t}\int_0^h \mathsf{E}[Z^2(s)]\,ds$$
$$\le c(h) + \frac{1}{t}\int_0^h a_1(s)\,ds,$$

where the second inequality is due to $Z(s) \leq A(s)$ and A2(1). Hence, taking lim sup and letting

$$c_1 := \frac{a_1(h) + 3b_1(h)}{2(\hat{\alpha}h - l)}$$

yields (5). [Note that $\int_0^h a_1(s)\,ds < \infty$ as assumed.]

Now suppose that (5) holds for all $1 \leq k \leq n - 1$. Consider the case of $k = n$. First observe that condition A2(n) implies A2(j), for all $1 \leq j \leq n - 1$, via Hölder's inequality, with the functions a_j and b_j specified as

$$a_j(h) := [a_n(h)]^{j/n}, \qquad b_j(h) := [b_n(h)]^{j/n}. \tag{10}$$

Hence, with the netput X satisfying A1 and A2(n), we know that (5) holds for $1 \leq k \leq n - 1$ following the induction hypothesis. Taking the $(n + 1)$th power on (9), we have

$$\begin{aligned} Z^{n+1}(t+h) &= Z^{n+1}(t) + (n+1)Z^n(t)\Delta(t,h) \\ &\quad + \sum_{j=0}^{n-1} \binom{n+1}{j} Z^j(t)\Delta^{n+1-j}(t,h). \end{aligned} \tag{11}$$

Applying Lemma 3(ii), we have

$$\begin{aligned} \mathsf{E}_{\mathscr{F}_t}\left[\sum_{j=0}^{n-1}\binom{n+1}{j} Z^j(t)\Delta^{n+1-j}(t,h)\right] &\leq \sum_{j=0}^{n-1}\binom{n+1}{j} Z^j(t)\mathsf{E}_{\mathscr{F}_t}\left[|\Delta(t,h)|^{n+1-j}\right] \\ &\leq \sum_{j=0}^{n-1}\binom{n+1}{j} Z^j(t)d_{n-j}(h), \end{aligned}$$

where

$$d_j(h) := a_j(h) + b_j(h).$$

Taking conditional expectation on both sides of (11), making use of the above inequality and applying Lemma 3(i), we have

$$\begin{aligned} \mathsf{E}_{\mathscr{F}_t}[Z^{n+1}(t+h)] &\leq Z^{n+1}(t) + (n+1)Z^n(t)\mathsf{E}_{\mathscr{F}_t}[\Delta(t,h)] \\ &\quad + \sum_{j=0}^{n-1}\binom{n+1}{j} Z^j(t)d_{n-j}(h) \\ &\leq Z^{n+1}(t) - (n+1)(\hat{\alpha}h - l)Z^n(t) + Q(Z(t),h), \end{aligned}$$

where

$$Q(Z(t),h) := \sum_{j=0}^{n-1}\binom{n+1}{j} Z^j(t)d_{n-j}(h) + (n+1)Z^{n-1}(t)b_1(h).$$

Taking expectations on the last inequality yields

$$\begin{aligned} \mathsf{E}[Z^{n+1}(t+h)] &\leq \mathsf{E}[Z^{n+1}(t)] - (n+1)(\hat{\alpha}h - l)\mathsf{E}[Z^n(t)] \\ &\quad + \mathsf{E}[Q(Z(t),h)]. \end{aligned}$$

Pick any $h > l/\hat{\alpha}$ as before. Applying Lemma 1, we have

$$(12)\qquad \begin{aligned} &\frac{1}{t}\int_0^t (n+1)(\hat{\alpha}h - l)\mathsf{E}[Z^n(s)]\,ds \\ &\quad \le \frac{1}{t}\int_0^t \mathsf{E}[Q(Z(s),h)]\,ds + \frac{1}{t}\int_0^h \mathsf{E}[Z^{n+1}(s)]\,ds. \end{aligned}$$

Now,

$$\begin{aligned} \frac{1}{t}\int_0^t \mathsf{E}[Q(Z(s),h)]\,ds &= \sum_{j=0}^{n-1}\binom{n+1}{j} d_{n-j}(h)\frac{1}{t}\int_0^t \mathsf{E}[Z^j(s)]\,ds \\ &\quad + (n+1)b_1(h)\frac{1}{t}\int_0^t \mathsf{E}[Z^{n-1}(s)]\,ds. \end{aligned}$$

Taking lim sup and making use of the induction hypothesis, we have

$$\begin{aligned} \limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathsf{E}[Q(Z(s),h)]\,ds &\le \sum_{j=0}^{n-1}\binom{n+1}{j} c_j d_{n-j}(h) \\ &\quad + (n+1)c_{n-1}b_1(h). \end{aligned}$$

Hence, taking lim sup on both sides of (12) and letting

$$(13)\qquad c_n := (\hat{\alpha}h - l)^{-1}\left[\frac{1}{n+1}\sum_{j=0}^{n-1}\binom{n+1}{j} c_j d_{n-j}(h) + c_{n-1}b_1(h)\right]$$

leads to the desired inequality in (5) for $k = n$. [Again, note that the second integral on the right side of (12) is dominated by $\int_0^h a_n(s)\,ds$, which is finite as assumed.]

This completes the induction and hence the proof. □

From the induction step of the above proof, we can strengthen the conclusion of Theorem 1 as follows:

COROLLARY 1. *Under the conditions of Theorem* 1, *for each* $j = 1,\dots,k$, *there exists a* $c_j < \infty$, *such that*

$$(14)\qquad \limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathsf{E}[Z^j(s)]\,ds \le c_j$$

and

$$(15)\qquad \lim_{t\to\infty}\frac{1}{t}\int_0^t Z^j(s)\,ds = \mathsf{E}[\hat{Z}^j] \le c_j \quad a.s.$$

REMARK 2. Suppose $B(t,h) \ge H(t,h)$ for all nonnegative t and h, where H is a nonnegative and nondecreasing process. Then, if A and H (rather than B) satisfy the conditions of Theorem 1, then the conclusions there still hold. This follows since $Z(t) \le Y(t)$ for all t where Y is the inventory process corresponding to the netput $A - H$. This illustrates the fact that the moment condition on B is certainly not a necessary one.

REMARK 3. Our assumption that $Z(0) = 0$ is not needed; all the results hold for $Z(0) \geq 0$ arbitrary as long as $\mathsf{E}[Z^{k+1}(0)] < \infty$. We must, however, include $Z(0)$ in the filtration $\mathscr{F}_t = \sigma\{X(u): 0 \leq u \leq t\}$ and verify that A1 and A2(k) still hold. The most convenient setup is to allow an arbitrary $X(0) \geq 0$ and define Z using (1) exactly as before. Then $Z(0) = X(0)$. A special case is when $Z(0)$ is independent of X and $\mathsf{E}[Z^{k+1}(0)] < \infty$.

REMARK 4. If the increments of X are stationary but not ergodic, then Theorem 1 still holds, except that (6) must be changed to $\mathsf{E}[\hat{Z}^k] \leq c_k$. Letting $\mathscr{I}$ denote the invariant σ-field for the increments of X, we have

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t Z^k(s)\, ds = \mathsf{E}_{\mathscr{I}}[\hat{Z}^k] < \infty \quad \text{a.s.}$$

However, $\mathsf{E}_{\mathscr{I}}[\hat{Z}^k]$ is a random variable, which in general is not bounded by c_k. Furthermore, our framework only requires that X be *asymptotically stationary*, which means that for $X_s := (X(s+t) - X(s): t \geq 0\}$ denoting the shifted increments (by time s), the distributions $P(X_s \in \cdot)$ on the Borel sets of $\mathscr{D}[0, \infty)$ must converge in total variation in mean, as $s \to \infty$, to a probability measure (see [1] for details). This includes a wide variety of netput such as those with increments having some kind of positive recurrent regenerative structure. Finally, observe that to obtain (5), even an asymptotically stationary X is not required; only A1 and A2(k) are needed.

We close this section by illustrating the use of (13) for computing the bounds $\{c_k\}$. Suppose A1 and A2(n) hold. Then, $a_n(h)$, $b_n(h)$, $\hat{\alpha}$ and l are known. For $j = 1, \ldots, n-1$, set $a_j(h)$ and $b_j(h)$ following (10). Since the first moment is perhaps the most important among all moments, pick h so as to minimize c_1:

$$\min_{h > l/\hat{\alpha}} c_1 := \frac{a_1(h) + 3b_1(h)}{2(\hat{\alpha}h - l)}.$$

By definition, $c_0 := 1$. For $k = 2, \ldots, n$, use the recursion

$$c_k = (\hat{\alpha}h - l)^{-1}\left[\frac{1}{k+1}\sum_{j=0}^{k-1}\binom{k+1}{j} c_j\left[a_{k-j}(h) + b_{k-j}(h)\right] + c_{k-1}b_1(h)\right].$$

Evidently the strength of the above bounds lies in their generality. In problem specific applications where distributional information is available about the netput process, one certainly expects that tighter bounds can be constructed using ad hoc approaches.

3. Examples.

EXAMPLE 1. *Workload in a standard single-server queue.* In this case $B(t) = t$, that is, the server processes work (if available) at a constant, unit rate. Let $\{(t_n, S_n): n \geq 1\}$ be a stationary ergodic marked point process of customer arrival times t_n and service times S_n. Let $N(t)$ denote the counting process for arrivals. Then

$$A(t) = \sum_{n=1}^{N(t)} S_j$$

denotes the amount of work arriving in $(0, t]$. The workload process, $Z(t)$, follows (1): $Z(t) = A(t) - t + L(t)$. Note that $B(t) = t$ satisfies A.2(k) for any $k \geq 1$, so if we want to apply Theorem 1 for the kth moment of the workload, we need constants $\hat{\alpha} > 0$ and $l \geq 0$ such that for all $t, h \geq 0$,

$$\mathsf{E}_{\mathcal{F}_t}[A(t,h) - h] \leq -\hat{\alpha} h + l,$$

together with the existence of a function a_k: $\Re_+ \mapsto \Re_+$ such that for all $h \geq 0$,

$$\mathsf{E}_{\mathcal{F}_t}\left[A^{k+1}(t,h)\right] \leq a_k(h).$$

As a special case, if service times are i.i.d. and independent of the arrival times $\{t_n\}$, then the above conditions will hold if

$$\mathsf{E}\left[S_n^{k+1}\right] < \infty$$

and if there exist constants $\gamma > 0$ and $l \geq 0$ such that

$$\mathsf{E}_{\mathcal{F}_t}[N(t,h)] \leq \gamma h + l$$

with

$$\gamma^{-1} > E(S_n).$$

When $k = 1$, these are exactly the conditions required in Theorem 2.1 of [12].

EXAMPLE 2. *Single-server queue with varying service rate.* As in Example 1, Z denotes the workload process in a system with $A(t) = \sum_{n=1}^{N(t)} S_j$. However, here the server serves at rate $r(t)$ at time t, where jointly with A, $\{r(t): t \geq 0\}$ is a stationary ergodic process and A has stationary ergodic increments. Here, $B(t) = \int_0^t r(s)\,ds$ and $B(t,h) = \int_t^{t+h} r(s)\,ds$. If for all $t \geq 0$, $r(t)$ is bounded almost surely, $m \leq r(t) \leq M$ for some constants $0 < m < M < \infty$, then

$$\mathsf{E}_{\mathcal{F}_t}[B(t,h)] \geq mh \quad \text{and} \quad \mathsf{E}_{\mathcal{F}_t}\left[B^k(t,h)\right] \leq (Mh)^k.$$

If additionally, A satisfies A2(k) and also

$$\mathsf{E}_{\mathcal{F}_t}[A(t,h)] \leq \hat{\rho} h \quad \text{with } \hat{\rho} - m < 0,$$

then both A1 and A2(k) are satisfied.

EXAMPLE 3. *Pure jump Lévy input.* Here $B(t) = t$ and $A(t)$ is a pure jump Lévy process with infinite Lévy measure (the number of jumps in any finite interval is infinite; see [13], Section 3.2). Lévy processes have independent as well as stationary increments. In particular, consider the case when $A(t) \sim \Gamma(\lambda, \gamma t)$. Then

$$\mathsf{E}[e^{iuA(t)}] = \left(\frac{\lambda}{\lambda - iu}\right)^{\gamma t},$$

from which we obtain $\mathsf{E}[A(t)] = \gamma t/\lambda$ and $\mathsf{E}[A^2(t)] = \gamma t(\gamma t + 1)/\lambda^2$. In this case, $\mathsf{E}[X(t,h)] = \gamma h/\lambda - h$ and hence $\alpha > 0$ iff $\lambda > \gamma$. By independent increments, we also have A2(1) satisfied with $a_1(t) = \mathsf{E}[A^2(t)] = \gamma t(\gamma t + 1)/\lambda^2$ and $b_1(t) = t^2$. Also by independent increments, A1 is satisfied with $\hat{\alpha} = \alpha$, and thus we see that whenever $\lambda > \gamma$, Theorem 1 applies and $\mathsf{E}[\hat{Z}] < \infty$. The finiteness of higher moments is similarly argued. (This example is meant for illustrative purposes only. Even the more general case, with the netput X being an arbitrary Lévy process with negative drift, is classic and in fact already well understood; see, e.g., [13] and [1], Section 4.)

Finally, we remark that based on an obvious discrete-time analogy of Theorem 1, we can also reproduce some of the known finite-moment results for customer delays in queues, for example, those in [11] and [14].

4. The semimartingale case. Here we assume that the netput process X is a semimartingale. Specifically, X can be expressed as

$$X(t) = M(t) + A(t) - B(t),$$

where $\{M(t)\}$ is a martingale with respect to the filtration $\{\mathscr{F}_t\}$, and $\{A(t)\}$ and $\{B(t)\}$ are two nondecreasing processes as before. Assume all processes have stationary increments and paths in $\mathscr{D}[0,\infty)$. It is known that this model covers a wide range of processes.

Let $M(t,h) := M(t+h) - M(t)$. Other notations are the same as before. Suppose A1 is still in force, while A2 is strengthened as follows:

A2′(k) ($k \geq 1$ is an integer). In addition to A2(k), there also exists a function $m_k\colon \Re_+ \mapsto \Re_+$, such that for all $t \geq 0$ and $h > 0$, $\int_0^h m_k(s)\,ds < \infty$ and

$$\mathsf{E}_{\mathscr{F}_t}\Big[|M(t,h)|^{k+1}\Big] \leq m_k(h).$$

(The above condition is easily verified for Brownian motion and is easy to check for more general Lévy processes.)

We want to show that Theorem 1 still holds. Recall that the two results in Lemma 3 are key to the proof of Theorem 1. Both rely crucially on Lemma 2. So here we start with modifying Lemma 2.

LEMMA 4. *For a semimartingale netput process X and any nonnegative t and h, we have* (i) $L(t,h) \leq \sup_{0\leq u\leq h}|M(t,u)| + B(t,h)$ *and* (ii) $Z(t)L(t,h) \leq [\sup_{0\leq u\leq h}|M(t,u)| + B(t,h)]^2$.

PROOF. Similar to the proof of Lemma 2, we have

$$\begin{aligned} L(t,h) &= \sup_{0\leq u\leq h}\big[\max\{0, -Z(t) - M(t,u) - A(t,u) + B(t,u)\}\big] \\ &\leq \sup_{0\leq u\leq h}\big[-M(t,u) + B(t,u)\big] \\ &\leq \sup_{0\leq u\leq h}|M(t,u)| + B(t,h). \end{aligned}$$

In addition, it is also easy to see that

$$Z(t) > \sup_{0 \le u \le h} |M(t,u)| + B(t,h)$$

implies $Z(t) > -M(t,u) + B(t,u)$ for any $u \le h$, and hence $L(t,h) = 0$. □

To generalize Lemma 3 based on Lemma 4, we need the following adaptation of Doob's L_p inequality:

LEMMA 5. *Let* $\{Y(u),\ u \ge 0\}$ *be a nonnegative submartingale with respect to a filtration* $\{\mathscr{F}_u\}$. *Then, for* $p > 1$ *and* $h > 0$,

$$\mathsf{E}_{\mathscr{F}_0}\Big[\sup_{0 \le u \le h} Y^p(u)\Big] \le \left(\frac{p}{p-1}\right)^p \mathsf{E}_{\mathscr{F}_0}[Y^p(h)] \quad a.s.$$

Note that replacing $\mathsf{E}_{\mathscr{F}_0}$ by E in the above inequality recovers the original Doob L_p inequality (see, e.g., [10], page 14). On the other hand, for any $A \in \mathscr{F}_0$, since $\{(Y(u)1(A)\}$ is still a nonnegative submartingale, applying Doob's L_p inequality, we have

$$\mathsf{E}\Big[\sup_{0 \le u \le h} Y^p(u) 1(\mathbf{A})\Big] \le \left(\frac{p}{p-1}\right)^p \mathsf{E}[Y^p(h) 1(\mathbf{A})]$$

(noticing that $[1(A)]^p = 1(A)$), hence, the desired a.s. inequality in Lemma 5.

LEMMA 6. (i) *Suppose* A1 *and* A2′(1) *hold. Then, for any* $t \ge 0$ *and* $h > 0$,

$$Z(t)\mathsf{E}_{\mathscr{F}_t}[\Delta(t,h)] \le -[\hat{\alpha}h - l]Z(t) + \big[2m_1^{1/2}(h) + b_1^{1/2}(h)\big]^2.$$

(ii) *Suppose* A2′(j) *holds for some* $j \ge 1$. *Then, for any* $t \ge 0$ *and* $h > 0$,

$$\mathsf{E}_{\mathscr{F}_t}\big[|\Delta(t,h)|^{j+1}\big] \le \big[(2 + 1/j)m_j^{1/(j+1)}(h) + a_j^{1/(j+1)}(h)\big]^{j+1} + \big[m_j^{1/(j+1)}(h) + b_j^{1/(j+1)}(h)\big]^{j+1}.$$

PROOF. Let $Y(h) := M(t,h)$ for any given $t \ge 0$. Then, clearly $\{Y(h)\}$ is a martingale with respect to the filtration $\{\mathscr{F}_h'\}$ with $\mathscr{F}_j' := \mathscr{F}_{t+h}$. Hence, $\{|Y(h)|\}$ is a nonnegative submartingale. For any $j \ge 1$, making use of Lemma 5 and A2′(j), we have

$$(16)\quad \begin{aligned} \mathsf{E}_{\mathscr{F}_t}\Big[\sup_{0 \le u \le h} |M(t,u)|^{j+1}\Big] &= \mathsf{E}_{\mathscr{F}_0'}\Big[\sup_{0 \le u \le h} |Y(u)|^{j+1}\Big] \\ &\le (1 + 1/j)^{j+1}\mathsf{E}_{\mathscr{F}_0'}\big[|Y(h)|^{j+1}\big] \\ &= (1 + 1/j)^{j+1}\mathsf{E}_{\mathscr{F}_t}\big[|M(t,h)|^{j+1}\big] \\ &\le (1 + 1/j)^{j+1} m_j(h). \end{aligned}$$

Therefore, to prove (i), modify the proof of Lemma 3(i), taking into account Lemma 4(ii). In particular, we have, making use of Minkowski's inequality and (16),

$$\begin{aligned}\mathsf{E}_{\mathscr{F}_t}[Z(t)L(t,h)] &\le \mathsf{E}_{\mathscr{F}_t}\Big[\sup_{0\le u\le h}|M(t,u)| + B(t,h)\Big]^2\\ &\le \left(\mathsf{E}_{\mathscr{F}_t}^{1/2}\Big[\sup_{0\le u\le h}|M(t,u)|^2\Big] + \mathsf{E}_{\mathscr{F}_t}^{1/2}\big[B^2(t,h)\big]\right)^2\\ &\le \big[2m_1^{1/2}(h) + b_1^{1/2}(h)\big]^2.\end{aligned}$$

To prove (ii), from

$$\Delta(t,h) = X(t,h) + L(t,h) = M(t,h) + A(t,h) - B(t,h) + L(t,h),$$

we have, taking into account Lemma 4(i),

$$\begin{aligned}|\Delta(t,h)| \le \max\Big\{&|M(t,h)| + \sup_{0\le u\le h}|M(t,u)| + A(t,h),\\ &B(t,h) + |M(t,h)|\Big\}.\end{aligned}$$

Hence,

$$\begin{aligned}|\Delta(t,h)|^{j+1} &\le \max\Big\{\Big[|M(t,h)| + \sup_{0\le u\le h}|M(t,u)| + A(t,h)\Big]^{j+1},\\ &\qquad\qquad [B(t,h) + |M(t,h)|]^{j+1}\Big\}\\ &\le \Big[|M(t,h)| + \sup_{0\le u\le h}|M(t,u)| + A(t,h)\Big]^{j+1}\\ &\qquad + [B(t,h) + |M(t,h)|]^{j+1}.\end{aligned}$$

Again, applying Minkowski's inequality and (16) leads to the desired result. □

REMARK 5. In the above proof it is also possible to have

$$|\Delta(t,h)| \le |M(t,h)| + \sup_{0\le u\le h}|M(t,u)| + A(t,h) + B(t,h),$$

which leads to

$$\mathsf{E}_{\mathscr{F}_t}\big[|\Delta(t,h)|^{j+1}\big] \le \big[(2 + 1/j)m_j^{1/(j+1)}(h) + a_j^{1/(j+1)}(h) + b_j^{1/(j+1)}(h)\big]^{j+1}.$$

This bound is not necessarily tighter than the one in Lemma 6(ii). Also, it fails to specialize to the bound in Lemma 3(ii) if we set $m_j(h) \equiv 0$.

With the above lemma, the proof of Theorem 1 is easily adapted. There are only two modifications:

1. Rename $d_j(h)$ as the bound for $\mathsf{E}_{\mathscr{F}_t}[|\Delta(t,h)|^{j+1}]$ in Lemma 6(ii).
2. Since

$$Z(s) = \Delta(0,s) \le |M(0,s)| + \sup_{0\le u\le s} |M(0,u)| + A(0,s),$$

bound the last integral on the right side of (12) by

$$\int_0^h \left[(2+1/n) m_n^{1/(n+1)}(s) + a_n^{1/(n+1)}(s)\right]^{n+1} ds,$$

which is finite, following A2′(n).

THEOREM 2. *When the netput process X is a semimartingale, Theorem* 1 *and Corollary* 1 *still hold, with* A2(k) *strengthened to* A2′(k).

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DEPARTMENT OF INDUSTRIAL ENGINEERING
AND OPERATIONS RESEARCH
COLUMBIA UNIVERSITY
SEELEY W. MUDD BUILDING
NEW YORK, NEW YORK 10027-6699