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LIMIT THEOREM ON OPTION REPLICATION COST WITH TRANSACTION COSTS

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Option replication in a discrete-time framework with transaction costs and its limit is discussed. First the notion of an efficient strategy is introduced, and then it is shown that an efficient strategy is the best strategy if it exists. It is also shown that the limit of the cost for option replication is given by a certain formula when the unit of time Δt tends to zero and the transaction costs tends to zero on the order of $\sqrt{\Delta t}$.

0. Introduction. Several recent papers have discussed option replication in the presence of transaction costs [e.g., Leland (1985), Merton (1990), Davis and Norman (1990). Boyle and Vorst (1992), Henrotte (1991) and Grannan and Swindle (1993)]. In particular, Boyle and Vorst (1992) obtain the "best" strategy to create a European call option in the multiplicative binomial lattice model. However, it appears to be very difficult to find the best strategy or to compute the cost for creating more general contingent claims, even in this simple model. In the present paper, we study the asymptotic behaviour of the replication cost in a setting closely related to that of Boyle and Vorst (1992) and Leland (1985). Also, we show that "hidden preconsistent price systems" play a key role in the proof.

1. Main results. Let $(\Omega, \mathscr{F}, P)$ be a probability space. Let $K \geq 1$ and $\{\mathscr{F}_k\}_{k=0,\dots,K}$ be a family of increasing sub-σ-algebras of $\mathscr{F}$. We assume that $\mathscr{F}_0$ is trivial, that is, $P(B) = 0$ or 1 for any $B \in \mathscr{F}_0$, and that $\#(\mathscr{F}_K) < \infty$. We consider two kinds of securities: bonds and stocks. We think of security 0 as the bond and security 1 as the stock. We assume that the price $P^i(k)$, $k = 0,\dots,K$, of security i, $i = 0, 1$, is an $\mathscr{F}_k$-measurable positive-valued random variable. We assume that the proportional transaction cost on sales of security 1 is c_0, $c_0 \in [0, 1)$, and the proportional transaction cost on purchases of security 1 is c_1, $c_1 \geq 0$. We also assume that there is no transaction cost for trade of security 0. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by

$$f(z) = \begin{cases} (1-c_0)z, & \text{if } z \leq 0, \\ (1+c_1)z, & \text{if } z > 0. \end{cases}$$

Then we obtain the amount $-f(z)P^1(k)$ if we sell security 1 in the amount $-z$, $z \leq 0$, and that it costs $f(z)P^1(k)$ to buy security 1 in the amount z, $z > 0$, at time k.

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Let $\mathscr{I}$ be the family of adapted stochastic processes $\{I(k)\}_{k=0,\dots,K}$. We call an element $I \in \mathscr{I}$ a strategy. Here we think of the situation in which at time k, we buy security 1 in the amount $I(k)$ if $I(k) \geq 0$ and we sell security 1 in the amount of $-I(k)$ if $I(k) < 0$.

Let $\tilde{P}(k;\ \omega)$ denote $P^0(k;\ \omega)^{-1}P^1(k;\ \omega)$, $k = 0, 1, \dots, K$, $\omega \in \Omega$. For each $x = (x^0,\ x^1) \in \mathbb{R}^2$ and $I \in \mathscr{I}$, let $X(k;\ x,\ I) = (X^0(k;\ x,\ I),\ X^1(k;\ x,\ I))$, $k = 0, 1, \dots, K$, be the adapted $\mathbb{R}^2$-valued process defined by

$$X^0(k;x,I) = x^0 - \sum_{l=0}^{k} f(I(l))\tilde{P}(l)$$

and

$$X^1(k;x,I) = x^1 + \sum_{l=0}^{k} I(l).$$

Then $X(k;\ x,\ I)$ is the post-trade portfolio at time k if the initial portfolio is x and the investment strategy is I.

DEFINITION 1.1. For each $Y \in L^\infty(\Omega;\ \mathbb{R}^2,\ \mathscr{F}_K,\ dP)$, the *replication cost* $\pi^*(Y) = \pi^*(Y;\ c_0,\ c_1)$ of Y is defined by

$$\pi^*(Y) = \inf\bigl\{x^0P^0(0);\ x^0 \in \mathbb{R}, \text{there is an } I \in \mathscr{I} \text{ such that } Y^0 \leq X^0(K;(x^0,0),I) \text{ and } Y^1 \leq X^1(K;(x^0,0),I)\ P\text{-a.s.}\bigr\}.$$

The replication cost $\pi^*(Y)$ is the minimum initial cost that one needs to replicate the portfolio Y at time K almost surely.

DEFINITION 1.2. We say that $I \in \mathscr{I}$ is an *efficient strategy* if

$$P\bigl(I(k+1) \geq 0,\ \tilde{P}(k+1) \leq \tilde{P}(k) \mid \mathscr{F}_k\bigr) > 0, \quad P\text{-a.s.},$$

and

$$P\bigl(I(k+1) \leq 0,\ \tilde{P}(k+1) \leq \tilde{P}(k) \mid \mathscr{F}_k\bigr) > 0, \quad P\text{-a.s.}$$

for any $k = 0, 1, \dots, K-1$.

The following theorem is proved in Section 3.

THEOREM 1. *Suppose that $I \in \mathscr{I}$ is an efficient strategy. Then*

$$\pi^*\bigl(X(K;(x^0,0),I)\bigr) = x^0P^0(0), \qquad x^0 \in \mathbb{R}.$$

REMARK. Boyle and Vorst (1992) show that European call options can be realized by an efficient strategy in a multiplicative binomial lattice model. However, our definition of replication cost is somewhat different from that of Boyle and Vorst.

From now on, we think of a special situation. Let $\Omega = \{-1, 1)^{\mathbb{N}}$ and let $\mathcal{F}$ be the Borel σ-algebra on Ω. Let $Z_k\colon \Omega \to \mathbb{R}$, $k \in \mathbb{N}$, be given by $Z_k(\omega) = \omega_k$, $\omega = (\omega_1, \omega_2, \dots) \in \Omega$. Let P be the Bernoulli measure on Ω such that $P(Z_k = -1) = P(Z_k = 1) = \frac{1}{2}$, $k \in \mathbb{N}$, and Z_k, $k = 1, 2, \dots$, are independent under P. Let $\mathcal{F}_0 = \{\phi, \Omega\}$ and $\mathcal{F}_k = \sigma\{Z_1, \dots, Z_k\}$, $k = 1, 2 \dots$.

Now let r, σ, $\mu \in (0, \infty)$, $T > 0$ and c_0, $c_1 \in [0, \infty)$, and fix them throughout this section. Also let r_n, σ_n, μ_n, $c_{1,n} \in (0, \infty)$ and $c_{0,n} \in (0, 1)$, $n = 1, 2, \dots$, be such that

$$\mu_n - \sigma_n < r_n < \mu_n + \sigma_n, \qquad n \in \mathbb{N}, \tag{1.3}$$

$$\lim_{n\to\infty} (n^{-1}T)^{-1} r_n = r, \tag{1.4}$$

$$\lim_{n\to\infty} (n^{-1}T)^{-1} \mu_n = \mu, \tag{1.5}$$

$$\lim_{n\to\infty} (n^{-1}T)^{-1/2} \sigma_n = \sigma, \tag{1.6}$$

$$\lim_{n\to\infty} (n^{-1}T)^{-1/2} c_{0,n} = c_0 \tag{1.7}$$

and

$$\lim_{n\to\infty} (n^{-1}T)^{-1/2} c_{1,n} = c_1. \tag{1.8}$$

For each $n \geq 1$, let

$$P_n^0(k; \omega) = \exp(r_n k) \tag{1.9}$$

and

$$P_n^1(k; \omega) = \exp\left(\sigma_n \sum_{l=1}^{k} Z(l) + \mu_n k\right), \qquad k = 0, \dots, n,\ \omega \in \Omega. \tag{1.10}$$

We consider a model in which the unit of time is $n^{-1}T$, the price of security i, $i = 0, 1$, at time $(k/n)T$ is given by $P_n^i(k; \omega)$ and the maturity is T. This is equivalent to the n-step multiplicative binomial lattice model employed by Cox, Ross and Rubinstein. We consider the case in which the transaction cost coefficients are given by $(c_{0,n}, c_{1,n}) \in (0, 1) \times (0, \infty)$. We are interested in the asymptotic behaviour of the replication cost π_n^* as $n \to \infty$.

Let $W_n\colon \mathbb{R}^{1+n} \to C([0, T]; \mathbb{R})$ be the linear interpolation operator given by

$$\begin{aligned} W_n(\{z(k)\}_{k=0}^n)(t) = &\left(\left[\frac{nt}{T}\right] + 1 - \frac{nt}{T}\right) z\left(\left[\frac{nt}{T}\right]\right) \\ &+ \left(\frac{nt}{T} - \left[\frac{nt}{T}\right]\right) z\left(\left[\frac{nt}{T}\right] + 1\right) \end{aligned} \tag{1.11}$$

for any $t \in [0, T)$, and

$$W_n(\{z(k)\}_{k=0}^n)(T) = z(n), \qquad \{z(k)\}_{k=0}^n \in \mathbb{R}^{1+n}. \tag{1.12}$$

Let $\gamma = c_0 + c_1$ and let $\mathscr{P}_M(\sigma, \gamma, r)$ be the set of probability measures Q on $C([0, T]; \mathbb{R})$ such that $\{e^{-rt}w(t); t \in [0, T]\}$ is a positive martingale under Q, $Q(w(0) = 1) = 1$, and such that the quadratic variation $\langle \log w \rangle_t$ of $\{\log w(t); t \in [0, T]\}$ satisfies

$$(1.13)\quad \sigma(\sigma - \gamma)\,dt \le d\langle \log w \rangle_t \le \sigma(\sigma + \gamma)\,dt, \qquad t \in [0, T] \text{ for } Q\text{-a.s. } w.$$

Then we have the following theorem.

THEOREM 2. *Let F: $C([0, T]; \mathbb{R}) \to \mathbb{R}^2$ be a continuous function such that there are $C \in (0, \infty)$ and $p \in (1, \infty)$ for which*

$$|F(w)| \le C\Big(1 + \max_{t \in [0,T]} |w(t)|\Big)^p, \qquad w \in C([0, T]; \mathbb{R}).$$

Then we have

$$\lim_{n \to \infty} \pi_n^*\Big(F\Big(W_n\big(\{P_n^1(k; \cdot)\}_{k=0}^n\big)\Big)\Big)$$
$$= \sup\big\{E^Q\big[F_0(w) + e^{-rT}w(T)F_1(w)\big]; Q \in \mathscr{P}_M(\sigma, \gamma, r)\big\},$$

where $F(w) = (F_0(w), F_1(w))$, $w \in C([0, T]; \mathbb{R})$.

The proof is found in Section 6.

COROLLARY 1. *Let F: $C([0, T]; \mathbb{R}) \to \mathbb{R}^2$ be a continuous function such that there are $C \in (0, \infty)$ and $p \in (1, \infty)$ such that*

$$|F(w)| \le C\Big(1 + \max_{t \in [0,T]} |w(t)|\Big)^p.$$

Let G: $C([0, T]; \mathbb{R}) \to \mathbb{R}$ be given by $G(w) = F_0(w) + e^{-rT}w(T)F_1(w)$, $w \in C([0, T]; \mathbb{R})$.

(i) *If G: $C([0, T]; \mathbb{R}) \to \mathbb{R}$ is concave, then*

$$\lim_{n \to \infty} \pi_n^*\Big(F\Big(W_n\big(\{P_n^1(k; \cdot)\}_{k=0}^n\big)\Big)\Big) = E^{\overline{Q}}[G(w)].$$

Here $\overline{Q}$ is a probability law of $\{\exp(\bar{\sigma}B(t) + (r - \bar{\sigma}^2/2)t); t \in [0, T]\}$, $\bar{\sigma} = (\sigma(\sigma + \gamma))^{1/2}$ and $\{B(t); t \in [0, T]\}$ is a standard Brownian motion.

(ii) *If there is a convex function $\tilde{g}$: $\mathbb{R} \to \mathbb{R}$ such that $G(w) = \tilde{g}(w(T))$, $w \in C([0, T]; \mathbb{R})$, then*

$$\lim_{n \to \infty} \pi_n^*\Big(F\Big(W_n\big(\{P_n^1(k; \cdot)\}_{k=0}^n\big)\Big)\Big)$$
$$= \int_{\mathbb{R}} (2\pi T)^{-1/2} \tilde{g}\left(\exp\left(\underline{\sigma} x + \left(r - \frac{\underline{\sigma}^2}{2}\right)T\right)\right)\exp\left(-\frac{x^2}{2T}\right) dx.$$

Here $\underline{\sigma} = \{(\sigma(\sigma - \gamma)) \vee 0\}^{1/2}$.

REMARK. In the case of a European call option with the exercise price α, we have $G(w) = 0 \vee (w(T) - \alpha)$. So we can apply Corollary 1. This result naturally coincides with Lemma 1 in Boyle and Vorst (1992).

2. Preconsistent price systems. In this section, we introduce the notion of preconsistent price systems following the ideas in Harrison and Kreps (1979). Then we show that the replication cost $\pi^*(Y)$ of the portfolio Y is the supremum of preconsistent prices of Y.

PROPOSITION 2.1.

(i) $$f(\alpha z) = \alpha f(z),\ \alpha > 0,\ z \in \mathbb{R}.$$

(ii) $$f(z) + f(z') \geq f(z + z'),\ z, z' \in \mathbb{R}.$$

(iii) $$f(z + z') = f(z) + f(z') \text{ if } z, z' \geq 0 \text{ or } z, z' \leq 0.$$

(iv) $$X^0(k; x, I) + X^0(k; x', I') \leq X^0(k; x + x', I + I')$$

and

$$X^1(k; x, I) + X^1(k; x', I') = X^1(k; x + x', I + I')$$

for any $k = 0, 1, \dots, K$, $x, x' \in \mathbb{R}^2$ *and* $I, I' \in \mathcal{I}$.

(v) $$X(k; \alpha x, \alpha I) = \alpha X(k; x, I)$$

for any $\alpha > 0$, $k = 0, 1, \dots, K$, $x \in \mathbb{R}^2$ *and* $I \in \mathcal{I}$.

PROOF. Assertions (i), (ii) and (iii) are obvious. Assertion (iv) follows from assertion (ii) and assertion (v) follows from assertion (i). □

DEFINITION 2.2. A *price system* is a linear map π from $L^\infty(\Omega; \mathbb{R}^2, \mathcal{F}_K, dP)$ into $\mathbb{R}$ such that

$$\pi(X) > 0$$

for any $X = (X^0, X^1) \in L^\infty(\Omega; \mathbb{R}^2, \mathcal{F}_K, dP)$ such that

$$X^0 P^0(K) - f(-X^1) P^1(K) \geq 0, \quad P\text{-a.s.} \tag{2.3}$$

and

$$P(X^0 P^0(K) - f(-X^1) P^1(K) > 0) > 0. \tag{2.4}$$

PROPOSITION 2.5. (i) *For any price system* π, *there is a unique* $(\rho^0, \rho^1) \in L^1(\Omega; \mathbb{R}^2, \mathcal{F}_K, dP)$ *satisfying*

$$\rho^i > 0, \quad P\text{-a.s.}, i = 0, 1, \tag{2.6}$$

and

$$(1 - c_0)\tilde{P}(K)\rho^0 \leq \rho^1 \leq (1 + c_1)\tilde{P}(K)\rho^0, \quad P\text{-a.s.} \tag{2.7}$$

(ii) *Conversely, if* $(\rho^0, \rho^1) \in L^1(\Omega; \mathbb{R}^2, \mathscr{F}_K, dP)$ *satisfies* (2.6) *and* (2.7), *then the linear map* $\pi(\ ; \rho^0, \rho^1)$: $L^\infty(\Omega; \mathbb{R}^2, \mathscr{F}_K, dP) \to \mathbb{R}$ *given by*

$$(2.8) \quad \pi(X; \rho^0, \rho^1) = E^P[X^0\rho^0 + X^1\rho^1], \qquad X \in L^\infty(\Omega; \mathbb{R}^2, \mathscr{F}_K, dP),$$

is a price system.

PROOF. (i) Since $\#(\mathscr{F}_K) < \infty$, there is a $(\rho^0, \rho^1) \in L^1(\Omega; \mathbb{R}^2, \mathscr{F}_K, dP)$ such that $\pi(X) = E^P[X^0\rho^0 + X^1\rho^1]$, $X = (X^0, X^1) \in L^\infty(\Omega; \mathbb{R}^2, \mathscr{F}_K, dP)$. Let $\xi \in L^\infty(\Omega, \mathscr{F}_K, dP)$ such that $\xi \geq 0$ P-a.s. and $P(\xi > 0) > 0$. Then both $(\xi, 0)$, $(0, \xi) \in L^\infty(\Omega; \mathbb{R}^2, \mathscr{F}_K, dP)$ satisfy (2.3) and (2.4), and so we have $\pi((\xi, 0)) = E^P[\xi\rho^0] > 0$ and $\pi((0, \xi)) = E^P[\xi\rho^1] > 0$. This implies (2.6).

Let $\xi \in L^\infty(\Omega, \mathscr{F}_K, dP)$ with $\xi \geq 0$ P-a.s. and let $m \geq 1$. Let $X = (X^0, X^1) = (1/m - (1 - c_0)\tilde{P}(K)\xi, \ \xi) \in L^\infty(\Omega; \mathbb{R}^2, \mathscr{F}_K, dP)$. It is obvious that $X^0 - f(-X^1)\tilde{P}(K) \geq 1/m$. So we have $\pi(X) > 0$, which implies that

$$E^P\big[\xi(1 - c_0)\tilde{P}(K)\rho^0\big] \leq E^P[\xi\rho^1] + \frac{1}{m}E^P[\rho^0].$$

Letting $m \to \infty$, we have

$$E^P\big[\xi(1 - c_0)\tilde{P}(K)\rho^0\big] \leq E^P[\xi\rho^1].$$

This implies the first inequality in (2.7).

Also, let $X = (X^0, X^1) = (1/m + (1 + c_1)\tilde{P}(K)\xi, \ -\xi)$. Again, we have $X^0 - f(-X^1)\tilde{P}(K) \geq 1/m$ P-a.s. This implies

$$E^P[\xi\rho^1] \leq E^P\big[\xi(1 + c_1)\tilde{P}(K)\rho^0\big] + \frac{1}{m}E^P[\rho^0].$$

Letting $m \to \infty$, we have the second inequality in (2.7).

(ii) Assume that (2.6) and (2.7) are satisfied. Suppose that $X \in L^\infty(\Omega; \mathbb{R}^2, \mathscr{F}_K, dP)$ satisfies (2.3) and (2.4). By (2.3), we have $X^0\rho^0 - f(-X^1)\tilde{P}(K)\rho^0 \geq 0$. So we see that $X^0\rho^0 + X^1\rho^1 \geq 0$ and $P(X^0\rho^0 + X^1\rho^1 > 0) > 0$. Therefore, by (2.8) we have

$$\pi(X; \rho^0, \rho^1) = E^P[X^0\rho^0 + X^1\rho^1] > 0.$$

This implies our assertion and completes the proof. □

From now on, we denote by $\pi(\ ; \rho^0, \rho^1)$ the price system given by (2.8) for $(\rho^0, \rho^1) \in L^1(\Omega; \mathbb{R}^2, \mathscr{F}_K, dP)$ satisfying (2.6) and (2.7).

DEFINITION 2.9. We say that a price system π is *preconsistent* if

$$(2.10) \qquad \pi((1, 0)) = P^0(0),$$

$$(2.11) \qquad (1 - c_0)P^1(0) \leq \pi((0, 1)) \leq (1 + c_1)P^1(0)$$

and

$$(2.12) \qquad \pi(X(K; 0, I)) \leq 0, \qquad I \in \mathscr{I}.$$

For each $(c_0, c_1) \in [0, 1) \times [0, \infty)$. Let $\mathscr{P}(c_0, c_1)$ be the set of all preconsistent price systems for which transaction costs are given by (c_0, c_1).

The following proposition is obvious.

PROPOSITION 2.13. (i) *$\mathscr{P}(c_0, c_1)$ is a convex set for any $(c_0, c_1) \in [0, 1) \times [0, \infty)$.*

(ii) *$\mathscr{P}(c_0, c_1) \subset \mathscr{P}(c'_0, c'_1)$ for any $(c_0, c_1), (c'_0, c'_1) \in [0, 1) \times [0, \infty)$ such that $c_0 \le c'_0$ and $c_1 \le c'_1$.*

PROPOSITION 2.14. *Let $(\rho^0, \rho^1) \in L^1(\Omega; \mathbb{R}^2, \mathscr{F}_K, dP)$, $\rho^0 > 0$, $\rho^1 > 0$, P-a.s. Then $\pi(\ ;\rho^0, \rho^1)$ is a preconsistent price system if and only if the following are satisfied:*

$$E^P[\rho^0] = P^0(0); \tag{2.15}$$

$$\begin{aligned}(1 - c_0)\tilde{P}(k)E^P[\rho^0 \mid \mathscr{F}_k] &\le E^P[\rho^1 \mid \mathscr{F}_k] \\ &\le (1 + c_1)\tilde{P}(k)E^P[\rho^0 \mid \mathscr{F}_k], \quad P\text{-a.s.},\end{aligned} \tag{2.16}$$

for any $k = 0, 1, \dots, K$.

PROOF ("Only if" part). (2.15) is obvious from (2.10), so we prove (2.16). Let $k \in \{0, \dots, K\}$ and ξ be an $\mathscr{F}_k$-measurable function, and let $I(l) = 0$, $l \ne k$, and $I(k) = \xi$. Then we see that $X(K; 0, I) = (-f(\xi)\tilde{P}(k), \xi)$. So we have

$$\begin{aligned}0 \ge \pi(X(K; 0, I); \rho^0, \rho^1) &= E^P[-f(\xi)\tilde{P}(k)\rho^0 + \xi\rho^1] \\ &= E^P\big[\xi E^P[\rho^1 \mid \mathscr{F}_k] - f(\xi)\tilde{P}(k)E^P[\rho^0 \mid \mathscr{F}_k]\big].\end{aligned}$$

This implies that for any $\mathscr{F}_k$-measurable nonnegative function ξ,

$$E^P\big[\xi\{E^P[\rho^1 \mid \mathscr{F}_k] - (1 + c_1)\tilde{P}(k)E^P[\rho^0 \mid \mathscr{F}_k]\}\big] \le 0$$

and

$$E^P\big[\xi\{E^P[\rho^1 \mid \mathscr{F}_k] - (1 - c_0)\tilde{P}(k)E^P[\rho^0 \mid \mathscr{F}_k]\}\big] \ge 0.$$

These imply (2.16).

("If" part). Suppose that (2.15) and (2.16) are satisfied. Letting $k = 0$ or K in (2.16), we have

$$(1 - c_0)\tilde{P}(0)E^P[\rho^0] \le E^P[\rho^1] \le (1 + c_1)\tilde{P}(0)E^P[\rho^0]$$

and

$$(1 - c_0)\tilde{P}(K)\rho^0 \le \rho^1 \le (1 + c_1)\tilde{P}(K)\rho^0.$$

These and (2.15) imply (2.7), (2.10) and (2.11). So we only have to check (2.12). However, by (2.16) we see that for any $I \in \mathscr{I}$ and $k = 0, \dots, K$, we have

$$-f(I(k))\tilde{P}(k)E^P[\rho^0 \mid \mathscr{F}_k] + I(k)E^P[\rho^1 \mid \mathscr{F}_k] \le 0.$$

This implies that

$$E^P[X^0(K;0,I)\rho^0 + X^1(K;0,I)\rho^1] \le 0, \qquad I \in \mathscr{I}.$$

So $\pi(\cdot;\ \rho^0,\ \rho^1)$ is a preconsistent price system. This completes the proof. □

PROPOSITION 2.17. *Suppose that $\mathscr{P}(c_0,\ c_1) \ne \phi$. Then for any $Y \in L^\infty(\Omega;\ \mathbb{R}^2, \mathscr{F}_K,\ dP)$,*

$$\pi^*(Y; c_0, c_1) = \sup\{\pi(Y); \pi \in \mathscr{P}(c_0, c_1)\}.$$

PROOF. Let $Y \in L^\infty(\Omega;\ \mathbb{R}^2,\ \mathscr{F}_K,\ dP)$. Let $x^0 \in \mathbb{R}$ and $I \in \mathscr{I}$ and suppose that $Y^0 \le X^0\ (K;(x^0,\ 0),\ I)$ and $Y^1 \le X^1(K;(x^0,\ 0),\ I)$, P-a.s. Then we see that for any preconsistent price system π,

$$\begin{aligned}\pi(Y) &\le \pi\big(X(K;(x^0,0),I)\big)\\ &= \pi\big((x^0,0)\big) + \pi\big(X(K;0,I)\big) \le x^0P^0(0).\end{aligned}$$

This implies that $\pi(Y) \le \pi^*(Y)$. Thus we have

$$\pi^*(Y; c_0, c_1) \ge \sup\{\pi(Y); \pi \in \mathscr{P}(c_0, c_1)\}.$$

Let $A_0 = \{Z \in L^\infty(\Omega;\ \mathbb{R}^2, \mathscr{F}_K,\ dP)$; there is an $I \in \mathscr{I}$ such that $Z^0 \le X^0(K;\ 0,\ I)$ and $Z^1 \le X^1(K;\ 0,\ I)$ P-a.s.$\}$. Then, by Proposition 2.1, we see that A_0 is a convex set containing 0. Now suppose that $b < \pi^*(Y)$. Let $A_1 = \{Z \in L^\infty(\Omega;\ \mathbb{R}^2, \mathscr{F}_K,\ dP);\ Y^0 - bP^0(0)^{-1} \le Z^0,\ Y^1 \le Z^1$ P-a.s.$\}$. Then A_1 is a convex set. Also, by the definition of the replication cost, we see that $A_0 \cap A_1 = \phi$. Since $\#(\mathscr{F}_K) < \infty$, there is a $(\rho^0,\ \rho^1) \in L^1(\Omega;\ \mathbb{R}^2,\ \mathscr{F}_K,\ dP) \setminus \{0\}$ such that

$$E^P[Z^0\rho^0 + Z^1\rho^1] \le 0, \qquad Z = (Z^0, Z^1) \in A_0 \tag{2.18}$$

and

$$E^P[Z^0\rho^0 + Z^1\rho^1] \ge 0, \qquad Z = (Z^0, Z^1) \in A_1. \tag{2.19}$$

By (2.19), we see that $\rho^0 \ge 0$, $\rho^1 \ge 0$, P-a.s. Then a similar argument in the proof of the only if part of Proposition 2.14 shows that $(\rho^0,\ \rho^1)$ satisfies (2.16). So we see that $E^P[\rho^0] > 0$ and $E^P[\rho^1] > 0$. So we may assume that $E^P[\rho^0] = P^0(0)$. Let π': $L^\infty(\Omega;\ \mathbb{R}^2,\ \mathscr{F}_K,\ dP) \to \mathbb{R}$ be given by

$$\pi'(Z) = E^P[Z^0\rho^0 + Z^1\rho^1], \qquad Z = (Z^0, Z^1) \in L^\infty(\Omega; \mathbb{R}^2, \mathscr{F}_K, dP).$$

Then we see that

$$b = \pi'\big((bP^0(0)^{-1}, 0)\big) = \pi'(Y) - \pi'\big(Y - (bP^0(0)^{-1}, 0)\big) \le \pi'(Y).$$

Now let us take a $\pi'' \in \mathscr{P}(c_0,\ c_1)$. Then for any $\varepsilon \in (0,\ 1)$, $(1 - \varepsilon)\pi' + \varepsilon\pi'' \in \mathscr{P}(c_0,\ c_1)$. This shows that $b \le \sup\{\pi(Y);\ \pi \in \mathscr{P}(c_0,\ c_1)\}$ and so we have

$$\pi^*(Y; c_0, c_1) \le \sup\{\pi(Y); \pi \in \mathscr{P}(c_0, c_1)\}.$$

This completes the proof. □

3. Proof of Theorem 1. Let G: $\mathbb{R}^2 \times (0, \infty) \to \mathbb{R}$ be given by $G(x; \tilde{p}) = -x^0 - f(x^1)\tilde{p}$, $x = (x^0, x^1) \in \mathbb{R}^2$, $\tilde{p} \in (0, \infty)$. Then we have the following proposition.

PROPOSITION 3.1. (i) *If* $G(x; \tilde{p}) \geq 0$ *and* $G(y; \tilde{p}) \geq 0$, *then* $G(x + y; \tilde{p}) \geq 0$, $x, y \in \mathbb{R}^2$, $\tilde{p} \in (0, \infty)$.

(ii) *Let* $x = (x^0, x^1)$, $y = (y^0, y^1) \in \mathbb{R}^2$ *and* $\tilde{p} \in (0, \infty)$.

(a) *If* $x^1 \geq y^1 \geq 0$ *and if* $G(x; \tilde{p}) \geq 0$ *and* $G(y; \tilde{p}) = 0$, *then* $G(x - y; \tilde{p}) \geq 0$.

(b) *If* $x^1 \leq y^1 \leq 0$ *and if* $G(x; \tilde{p}) \geq 0$ *and* $G(y; \tilde{p}) = 0$, *then* $G(x - y; \tilde{p}) \geq 0$.

(iii) *If* $x = (x^0, x^1) \in \mathbb{R}^2$, $x^1 > 0$, $0 < \tilde{p}' \leq \tilde{p} < \infty$ *and* $G(x; \tilde{p}) \geq 0$, *then* $G(x; \tilde{p}') \geq 0$. *If* $x = (x^0, x^1) \in \mathbb{R}^2$, $x^1 > 0$, $0 < \tilde{p}' \leq \tilde{p} < \infty$ *and* $G(x; \tilde{p}) \geq 0$, *then* $G(x; \tilde{p}') \geq 0$. *If* $x = (x^0, x^1) \in \mathbb{R}^2$, $x^1 < 0$, $0 < \tilde{p} \leq \tilde{p}' < \infty$ *and* $G(x; \tilde{p}) \geq 0$, *then* $G(x; \tilde{p}') \geq 0$.

PROOF. By Proposition 2.1, we have

$$\begin{aligned} G(x + y; \tilde{p}) &= -(x^0 + y^0) - f(x^1 + y^1)\tilde{p} \\ &\geq -(x^0 + y^0) - \big(f(x^1) + f(y^1)\big)\tilde{p}. \end{aligned}$$

So we have assertion (i). Also, if $x^1 \geq y^1 \geq 0$, we have

$$\begin{aligned} G(x - y; \tilde{p}) &= -(x^0 - y^0) - f(x^1 - y^1)\tilde{p} \\ &= -(x^0 - y^0) - \big(f(x^1) - f(y^1)\big)\tilde{p} \\ &= G(x; \tilde{p}) - G(y; \tilde{p}). \end{aligned}$$

So we have assertion (ii)(a). The proof of assertion (ii)(b) is similar. Assertion (iii) is obvious. This completes the proof. □

PROPOSITION 3.2. *Let* $x \in \mathbb{R}^2$ *and* $I \in \mathcal{I}$. *Then*

$$G\big(X(k + 1; x, I) - X(k; x, I); \tilde{P}(k + 1)\big) = 0, \quad P\text{-a.s.}, k = 0, 1, \dots, K - 1.$$

PROOF. This is obvious from the fact that

$$X(k + 1; x, I) - X(k; x, I) = \big(-f(I(k + 1))\tilde{P}(k + 1), I(k + 1)\big). \quad \square$$

PROPOSITION 3.3. *Let* $x, x' \in \mathbb{R}^2$ *and* $I, I' \in \mathcal{I}$. *Assume that the strategy* I *is efficient and* $G(X(K; x, I) - X(k; x', I'); \tilde{P}(K)) \geq 0$, *P-a.s. Then for any* $k = 0, 1, \dots, K$, $G(X(k; x, I) - X(k; x', I'); \tilde{P}(k)) \geq 0$, *P-a.s.*

PROOF. We prove our assertion by induction on $K - k$. First by the assumption, our assertion is true for $k = K$. Suppose that our assertion is true for $k + 1$. So we have

$$G\big(X(k + 1; x, I) - X(k + 1; x', I'); \tilde{P}(k + 1)\big) \geq 0, \quad P\text{-a.s.} \tag{3.4}$$

By Propositions 3.1(i) and 3.2, we have

$$G\big(X(k+1; x, I) - X(k; x', I'); \tilde{P}(k+1)\big) \geq 0, \quad P\text{-a.s.} \tag{3.5}$$

Let A be a nontrivial atom of $\mathscr{F}_k$, that is, $A \in \mathscr{F}_k$, $P(A) > 0$, and if $A' \in \mathscr{F}_k$ and if $A' \subset A$, then $P(A') = 0$ or $P(A \setminus A') = 0$. Then by the definition of efficiency, we see that there are nontrivial atoms A_1, A_2 of $\mathscr{F}_{k+1}$ such that $A_1 \subset A$, $A_2 \subset A$ and

$$I(k+1, \omega) \geq 0, \qquad \tilde{P}(k+1, \omega) \geq \tilde{P}(k, \omega), \qquad \omega \in A_1$$

and

$$I(k+1, \omega) \leq 0, \qquad \tilde{P}(k+1, \omega) \leq \tilde{P}(k, \omega), \qquad \omega \in A_2.$$

There are two cases as follows:

Case 1. $X^1(k; x, I) \geq X^1(k; x', I')$ on A.
Case 2. $X^1(k; x, I) \leq X^1(k; x', I')$ on A.

In Case 1, we see that $X^1(k+1; x, I) \geq X^1(k; x, I) \geq X^1(k; x', I')$ on A_1. So by Propositions 3.1(ii)(a), 3.2 and 3.5, we have

$$G\big(X(k; x, I) - X(k; x', I'); \tilde{P}(k+1)\big) \geq 0 \quad \text{on } A_1.$$

So by Proposition 3.1(iii), we have

$$G\big(X(k; x, I) - X(k; x', I'); \tilde{P}(k)\big) \geq 0 \quad \text{on } A_1.$$

Since A is a nontrivial atom, we have

$$G\big(X(k; x, I) - X(k; x', I'); \tilde{P}(k)\big) \geq 0 \quad \text{on } A. \tag{3.6}$$

A similar argument works also in Case 2 and we have (3.6) again. So we see that our assertion is true for k. This completes the induction. □

Proof of Theorem 1. Suppose that $y^0 \in \mathbb{R}$, $I' \in \mathscr{I}$ and $X^i(K; (x^0, 0), I) \leq X^i(K; (y^0, 0), I')$, P-a.s., $i = 0, 1$. Then we have $G(X(K; (x^0, 0), I) - X(K; (y^0, 0), I'); \tilde{P}(K)) \geq 0$, P-a.s. So by Propositions 2.1 and 3.3, we have

$$\begin{aligned}
0 &\leq G\big(X(0; (x^0, 0), I) - X(0; (y^0, 0), I'); \tilde{P}(0)\big) \\
&= -\big\{\big(x^0 - f(I(0))\tilde{P}(0)\big) - \big(y^0 - f(I'(0))\tilde{P}(0)\big)\big\} \\
&\quad - f(I(0) - I'(0))\tilde{P}(0) \\
&= (y^0 - x^0) - \{f(I(0) - I'(0)) + f(I'(0)) - f(I(0))\}\tilde{P}(0) \\
&\leq y^0 - x^0.
\end{aligned}$$

So we see that $x^0 P^0(0) \leq \pi^*(X(K; (x^0, 0), I))$. It is obvious that $x^0 P^0(0) \geq \pi^*(X(K; (x^0, 0), I)$. So we have our assertion. □

4. Preparations for the proof of Theorem 2. Henceforth, we place ourselves in the setting of the last part of Section 1. Let $\mathscr{P}_n(c_{0,n}, c_{1,n})$, $n = 1, 2, \dots$, be the set of preconsistent price systems. First, we have the following proposition.

PROPOSITION 4.1. $\mathscr{P}_n(c_{0,n}, c_{1,n}) \neq \phi$, $n \geq 1$.

PROOF. By (1.3), it is easy to see that there is a probability measure Q_n equivalent to P such that $\{\tilde{P}_n(k; \omega); k = 0, 1, \dots, n\}$ is a martingale under Q_n. So we see that $\mathscr{P}_n(0, 0) \neq \phi$. Therefore, by Proposition 2.13(ii), we have our assertion. □

The following lemma is the main result in this section.

LEMMA 4.2. *Let* $\{\pi_n; n \in \mathbb{N}\} \in \prod_{n=1}^{\infty} \mathscr{P}_n(c_{0,n}, c_{1,n})$. *Then we have the following*:

(i)
$$\sup_{n \geq 1} \pi_n\Big(\max\big\{P_n^1(k; \omega); k = 0, 1, \dots, n\big\}^p (1, 1)\Big) < \infty$$

and

$$\sup_{n \geq 1} \pi_n\Big(\max\big\{|\log P_n^1(k; \omega)|; k = 0, 1, \dots, n\big\}^p (1, 1)\Big) < \infty.$$

(ii) *For any increasing sequence* $\{n_l\}_{l=1}^{\infty}$ *of integers, there is a subsequence* $\{n'_l\}_{l=1}^{\infty}$ *of* $\{n_l\}_{l=1}^{\infty}$ *and some* $Q \in \mathscr{P}_M(\sigma, \gamma, r)$ *such that*

$$\lim_{l \to \infty} \pi_{n_l}\Big(F\Big(W_{n_l}\Big(\big\{P_{n_l}^1(k; \omega)\big\}_{k=0}^{n_l}\Big)\Big)\Big) = E^Q\big[F_0(w) + e^{-rT} w(T) F_1(w(t))\big]$$

for any bounded continuous function $F = (F_0, F_1)$: $C([0, T]; \mathbb{R}) \to \mathbb{R}^2$.

In order to prove this lemma, we make some preparations. Let $\tilde{P}_n(k; \omega) = \exp(\sigma_n \sum_{l=1}^{k} Z(l) + (\mu_n - r_n)k)$, $k = 0, \dots, n$, $\omega \in \Omega$, and let $(\rho_n^0, \rho_n^1) \in L^1(\Omega; \mathbb{R}^2, \mathscr{F}_n, dP)$, $n \in \mathbb{N}$, be such that $\pi_n = \pi(\cdot; \rho_n^0, \rho_n^1)$. By Propositions 2.5 and 2.14, we have the following:

$$\rho_n^i > 0, \quad P\text{-a.s.}, i = 0, 1. \tag{4.3}$$

$$E^P\big[\rho_n^0\big] = 1, \tag{4.4}$$

$$\begin{aligned}(1 - c_{0,n})\tilde{P}_n(k) &\leq E^P\big[\rho_n^0 \mid \mathscr{F}_k\big]^{-1} E^P\big[\rho_n^1 \mid \mathscr{F}_k\big] \\ &\leq (1 + c_{1,n})\tilde{P}_n(k), \quad P\text{-a.s.},\end{aligned} \tag{4.5}$$

for any $k = 0, 1, \dots, n$. Let Q_n be the probability measure on Ω given by $dQ_n = \rho_n^0 \, dP$. Then it is well known that

$$E_n[g \mid \mathscr{F}_k] = E^P\big[\rho_n^0 \mid \mathscr{F}_k\big]^{-1} E^P\big[g \rho_n^0 \mid \mathscr{F}_k\big], \qquad k = 0, 1, 2, \dots, n,$$

for any bounded measurable function g: $\Omega \to \mathbb{R}$. Here we denote $E^{Q_n}[\cdot]$ by $E_n[\cdot]$. Let $M_n(k) = E_n[(\rho_n^0)^{-1}\rho_n^1 \mid \mathscr{F}_k]$, $k = 0, 1, \dots, n$. Then we see that

$\{M_n(k)\}_{k=0}^n$ is a positive martingale under the probability measure Q_n and that

$$(4.6)\quad \begin{aligned}(1 - c_{0,n})\tilde{P}_n(k) &\le M_n(k)\\ &\le (1 + c_{1,n})\tilde{P}_n(k), \qquad Q_n\text{-a.s., } k = 0, 1, 2, \dots, n.\end{aligned}$$

Now let $\gamma_n = (1 - c_{0,n})^{-1}(1 + c_{1,n}) - 1$. Then it is easy to see that

$$(4.7)\qquad (T/n)^{-1/2}\log(1 + \gamma_n) \to \gamma = c_0 + c_1, \qquad n \to \infty.$$

PROPOSITION 4.8. (i) *For any* $n \ge 1$, $k = 0, 1, \dots, n - 1$,

$$|\log M_n(k + 1) - \log M_n(k)| \le \sigma_n + \log(1 + \gamma_n) + |\mu_n - r_n|$$

and

$$|M_n(k)^{-1}(M_n(k + 1) - M_n(k))| \le a_{0,n},$$

where $a_{0,n} = \exp(\sigma_n + \log(1 + \gamma_n) + |\mu_n - r_n|) - 1$.

(ii) *There are sequences* $\{a_{1,n}\}_{n=1}^\infty$ *and* $\{a_{2,n}\}_{n=1}^\infty$ *such that*

$$(4.9)\qquad \lim_{n\to\infty} (T/n)^{-1} a_{1,n} = \tfrac{1}{2}\sigma(\sigma - \gamma),$$

$$(4.10)\qquad \lim_{n\to\infty} (T/n)^{-1} a_{2,n} = \tfrac{1}{2}\sigma(\sigma + \gamma),$$

$$(4.11)\quad E_n[\log M_n(k + l) - \log M_n(k) \mid \mathscr{F}_k] \le -l a_{1,n} + (\log(1 + \gamma_n))^2$$

and

$$(4.12)\quad E_n[\log M_n(k + l) - \log M_n(k) \mid \mathscr{F}_k] \ge -l a_{2,n} - (\log(1 + \gamma_n))^2$$

for any $n \ge 1$, $k = 0, 1, \dots, n$ *and* $l \ge 1$ *with* $k + l \le n$.

$$\text{(iii)}\qquad \sup_{n\ge 1} E_n\Big[\max\{M_n(k);\ k = 0, 1, \dots, n\}^{2m}\Big] < \infty$$

and

$$\sup_{n\ge 1} E_n\Big[\max\{|\log M_n(k)|;\ k = 0, 1, \dots, n\}^{2m}\Big] < \infty$$

for any $m \ge 1$.

(iv) *There is a sequence* $\{a_{3,n}\}_{n=1}^\infty$ *of positive numbers such that* $\lim_{n\to\infty} n a_{3,n} < \infty$ *and that*

$$E_n\Big[(M_n(k + l) - M_n(k))^4\Big] \le (a_{3,n} l)^2$$

for any $n \ge 1$, $k = 0, 1, \dots, n$ *and* $l \ge 1$ *with* $k + l \le n$.

PROOF. Let $Y_n(k) = \log M_n(k) - \log \tilde{P}_n(k) - \frac{1}{2}\log((1 - c_{0,n})(1 + c_{1,n}))$, $k = 0, 1, \dots, n$. Then by (4.6), we have

$$(4.13)\qquad |Y_n(k)| \le \tfrac{1}{2}\log(1 + \gamma_n), \qquad k = 0, 1, \dots, n.$$

Note that

$$\begin{aligned}\log M_n(k+1) - \log M_n(k) \\ = Y_n(k+1) - Y_n(k) + \{\sigma_n Z(k+1) + (\mu_n - r_n)\}.\end{aligned} \tag{4.14}$$

So (4.13) and (4.14) imply our assertion (i).

Let $\varphi_1(z) = \log(1+z) - z$ and $\varphi_2(z) = \varphi_1(z) + z^2/2$, $z > -1$. Let $d_{1,n} = \sup\{|\varphi_1(z)|;\ -a_{0,n} \le z \le a_{0,n}\} = O(n^{-1})$ and $d_{2,n} = \sup\{|\varphi_2(z)|;\ -a_{0,n} \le z \le a_{0,n}\} = O(n^{-3/2})$. Note that

$$\begin{aligned}\log M_n(k+1) - \log M_n(k) \\ = M_n(k)^{-1}(M_n(k+1) - M_n(k)) \\ + \varphi_1\big(M_n(k)^{-1}(M_n(k+1) - M_n(k))\big).\end{aligned} \tag{4.15}$$

Therefore, we have

$$|E_n[\log M_n(k+1) \mid \mathscr{F}_k] - \log M_n(k)| \le d_{1,n} \tag{4.16}$$

and

$$\begin{aligned}|\log M_n(k+1) - E_n[\log M_n(k+1) \mid \mathscr{F}_k] \\ - M_n(k)^{-1}(M_n(k+1) - M_n(k))| \le 2d_{1,n}.\end{aligned} \tag{4.17}$$

So we have

$$\begin{aligned}\Big|2E_n[\log M_n(k+1) - \log M_n(k) \mid \mathscr{F}_k] \\ +E_n\Big[\{\log M_n(k+1) - E_n[\log M_n(k+1) \mid \mathscr{F}_k]\}^2 \mid \mathscr{F}_k\Big]\Big| \\ \le 2d_{2,n} + 4d_{1,n}^2 + 4d_{1,n}a_{0,n} = O(n^{-3/2}) \quad \text{as } n \to \infty.\end{aligned} \tag{4.18}$$

By (4.13) and (4.14), we see that

$$\begin{aligned}\Big|E_n\Big[(\log M_n(k+1) - \log M_n(k) + Y_n(k))^2 \mid \mathscr{F}_k\Big] \\ -\Big\{E_n\Big[Y_n(k+1)^2 \mid \mathscr{F}_k\Big] + \sigma_n^2 \\ +2\sigma_n E_n[Y_n(k+1)Z(k+1) \mid \mathscr{F}_k]\Big\}\Big| \le d_{3,n}.\end{aligned} \tag{4.19}$$

Here $d_{3,n} = |\mu_n - r_n|(2\sigma_n + \log(1+\gamma_n)) + (\mu_n - r_n)^2 = O(n^{-3/2})$. Combining (4.13), (4.16), (4.18) and (4.19), we see that there are positive numbers $d_{4,n}$, $n \ge 1$, such that $\lim_{n\to\infty} nd_{4,n} = 0$ and that

$$\begin{aligned}\Big|2E_n[\log M_n(k+1) - \log M_n(k) \mid \mathscr{F}_k] \\ +\Big(E_n\Big[Y_n(k+1)^2 \mid \mathscr{F}_k\Big] - Y_n(k)^2\Big) \\ +\sigma_n^2 + 2\sigma_n E_n[Y_n(k+1)Z(k+1) \mid \mathscr{F}_k]\Big| \le d_{4,n}.\end{aligned} \tag{4.20}$$

Note that by (4.13), we have

$$-\sigma_n \log(1+\gamma_n) \le 2\sigma_n E_n[Y_n(k+1)Z(k+1) \mid \mathcal{F}_k] \le \sigma_n \log(1+\gamma_n).$$

So we have

$$\begin{aligned}(4.21)\qquad & 2E_n[\log M_n(k+l) - \log M_n(k) \mid \mathcal{F}_k] \\ &\le -E_n\left[Y_n(k+l)^2 \mid \mathcal{F}_k\right] + Y_n(k)^2 \\ &\quad - l\sigma_n(\sigma_n - \log(1+\gamma_n)) + ld_{4,n} \\ &\le -l\sigma_n(\sigma_n - \log(1+\gamma_n)) + ld_{4,n} + (\log(1+\gamma_n)/2)^2\end{aligned}$$

and

$$\begin{aligned}& 2E_n[\log M_n(k+l) - \log M_n(k) \mid \mathcal{F}_k] \\ (4.22)\qquad & \ge -E_n\left[Y_n(k+l)^2 \mid \mathcal{F}_k\right] + Y_n(k)^2 - l\sigma_n(\sigma_n + \log(1+\gamma_n)) - ld_{4,n} \\ & \ge -l\sigma_n(\sigma_n + \log(1+\gamma_n)) - ld_{4,n} - (\log(1+\gamma_n)/2)^2.\end{aligned}$$

These imply our assertion (ii).

By assertion (i), we have

$$\begin{aligned}(4.23)\qquad & E_n\left[M_n(k+1)^{2m}\right] \\ &= \sum_{j=0}^{2m} \binom{2m}{j} E_n\Big[M_n(k)^{2m} E_n \\ &\qquad \times \left[\left\{M_n(k)^{-1}(M_n(k+1) - M_n(k))\right\}^j \mid \mathcal{F}_k\right]\Big] \\ &\le \left\{1 + \sum_{j=2}^{2m} \binom{2m}{j} a_{0,n}^j\right\} E_n\left[M_n(k)^{2m}\right].\end{aligned}$$

Since $a_{0,n} = O(n^{-1/2})$ as $n \to \infty$, we see that $\sup_n E_n[M_n(n)^{2m}] < \infty$. This and Doob's inequality imply the first part of assertion (iii).

Since we have

$$E_n\left[(M_n(k+1) - M_n(k))^4\right] \le a_{0,n}^4 E_n\left[\max\{M_n(j); j = 0, \dots, n\}^4\right],$$

we have assertion (iv) by Burkholder's inequality.

Let $N_n(k) = \log M_n(k) - E_n[\log M_n(k) \mid \mathcal{F}_{k-1}]$, $k = 1, \dots, n$. Then we have by (4.14) and (4.16),

$$(4.24)\qquad |\log M_n(k+1) - \log M_n(k) - N_n(k+1)| \le 2d_{1,n} = O(n^{-1})$$

and

$$(4.25)\qquad |N_n(k+1)| \le (\sigma_n + \log(1+\gamma_n) + |\mu_n - r_n|) + d_{1,n} = O(n^{-1/2}).$$

Again by Burkholder's inequality, we see that there is a constant C_m depending only on m such that

$$(4.26)\quad E_n\left[\max\left\{\left|\sum_{j=1}^{k} N_n(j)\right|; k=1,\ldots,n\right\}^{2m}\right] \le C_m E_n\left[\left\{\sum_{k=1}^{n} N_n(k)^2\right\}^{m}\right].$$

Combining (4.24), (4.25) and (4.26), we have the latter part of assertion (iii). This completes the proof. □

As a consequence of Proposition 4.8, we have the following proposition.

PROPOSITION 4.27. *Let $\tilde{Q}_n$, $n \ge 1$, be the probability law on $C([0, T]; \mathbb{R})$ of $\{e^{rt}W_n(\{M_n(k; \omega)\}_{k=0}^n)(t); t \in [0, T]\}$ under Q_n. Then the sequence $\{\tilde{Q}_n; n \in \mathbb{N}\}$ is tight. Moreover, any cluster point of $\{\tilde{Q}_n; n \in \mathbb{N}\}$ belongs to $\mathscr{P}_M(\sigma, \gamma, r)$.*

PROOF. By Proposition 4.8(iii) and (iv), we see that there is a constant $C_0 \in (0, \infty)$ such that

$$E_n\left[|W_n(\{M_n(k)\}_{k=0}^n)(t) - W_n(\{M_n(k)\}_{k=0}^n)(s)|^4\right] \le C_0|t-s|^2$$

for any t, $s \in [0, T]$ and $n \ge 1$. This implies that $\{\tilde{Q}_n; n \in \mathbb{N}\}$ is tight.

Let Q be a cluster point of $\{\tilde{Q}_n; n \in \mathbb{N}\}$. Then it is obvious that $Q(w(0) = 1) = 1$ and that $\{e^{-rt}w(t); t \in [0, T]\}$ is a martingale under $Q(dw)$. Moreover, by Proposition 4.8(iii) we see that

$$\sup_n E^{\tilde{Q}_n}\left[\max\{|w(t)|; t \in [0, T]\}^p\right] < \infty, \qquad p \in (1, \infty)$$

and

$$\sup_n E^{\tilde{Q}_n}\left[\max\{|\log w(t)|; t \in [0, T]\}^p\right] < \infty, \qquad p \in (1, \infty).$$

Let g: $[0, T] \times C([0, T]; \mathbb{R}) \to [0, \infty)$ be an arbitrary adapted bounded continuous function. Here we say that the function g: $[0, T] \times C([0, T]; \mathbb{R}) \to [0, \infty)$ is adapted if $g(t, \cdot)$: $C([0, T]; \mathbb{R}) \to [0, \infty)$ is $\sigma\{w(s); s \in [0, t]\}$-measurable for all $t \in [0, T]$. Then by Proposition 4.8(ii), we see that

$$\limsup_{n\to\infty} E^{\tilde{Q}_n}\Big[g(s, w)\Big\{\big(\log(e^{-rt}w(t)) - \log(e^{-rs}w(s))\big) + \tfrac{1}{2}\sigma(\sigma-\gamma)(t-s)\Big\}\Big] \le 0$$

and

$$\liminf_{n\to\infty} E^{\tilde{Q}_n}\Big[g(s, w)\Big\{\big(\log(e^{-rt}w(t)) - \log(e^{-rs}w(s))\big) + \tfrac{1}{2}\sigma(\sigma+\gamma)(t-s)\Big\}\Big] \ge 0$$

for any s, $t \in [0, T]$ with $s < t$. This implies that

$$E^{Q}\Big[g(s, w)\Big\{\big(\log(e^{-rt}w(t)) - \log(e^{-rs}w(s))\big) + \tfrac{1}{2}\sigma(\sigma-\gamma)(t-s)\Big\}\Big] \le 0$$

and

$$E^Q\left[g(s,w)\left\{\left(\log(e^{-rt}w(t)) - \log(e^{-rs}w(s))\right) + \tfrac{1}{2}\sigma(\sigma+\gamma)(t-s)\right\}\right] \geq 0$$

for any s, $t \in [0, T]$ with $s < t$. Since $\{e^{-rt}w(t);\ t \in [0, T]\}$ under $Q(dw)$ is a martingale, $\langle \log w \rangle_t$ is the bounded variation part of the semimartingale $-2\log(e^{-rt}w(t))$. So we have

$$\sigma(\sigma-\gamma)E^Q\left[\int_0^T g(t,w)\,dt\right] \leq E^Q\left[\int_0^T g(t,w)\,d\langle \log w\rangle_t\right]$$
$$\leq \sigma(\sigma+\gamma)E^Q\left[\int_0^T g(t,w)\,dt\right]$$

for any adapted bounded measurable function g: $[0, T] \times C([0, T]; \mathbb{R}) \to [0, \infty)$. This implies that $\sigma(\sigma-\gamma)\,dt \leq d\langle \log w\rangle_t \leq \sigma(\sigma+\gamma)\,dt$, Q-a.s. w. So we see that $Q \in \mathscr{P}_M(\sigma, \gamma, r)$. This completes the proof. □

PROOF OF LEMMA 4.2. First note that

$$\pi_n(X) = E^P\left[\rho_n^0 X_0 + \rho_n^1 X_1\right] = E_n\left[X_0 + M_n(n)X_1\right] \tag{4.28}$$

for any $X = (X^0, X^1) \in L^\infty(\Omega; \mathbb{R}^2, \mathscr{F}_n, dP)$. Also, we have

$$\max\{P_n^1(k;\omega);\ k = 0, 1, \dots, n\}$$
$$\leq \exp(nr_n + \log(1+\gamma_n))\max\{M_n(k;\omega);\ k = 0, 1, \dots, n\}$$

and

$$\max\{|\log P_n^1(k;\omega)|;\ k = 0, 1, \dots, n\}$$
$$\leq nr_n + \log(1+\gamma_n) + \max\{|\log M_n(k;\omega)|;\ k = 0, 1, \dots, n\}.$$

These and Proposition 4.8(iii) imply assertion (i).

Now note that

$$\max_{t\in[0,T]} |W_n(\{P_n^1(k);\ k = 0, \dots, n\})(t) - e^{rt}W_n(\{M_n(k);\ k = 0, \dots, n\})(t)|$$
$$\leq (\gamma_n + |\exp(rT) - \exp(nr_n)| + r_n\exp(nr_n))$$
$$\times \max_{t\in[0,T]} |W_n(\{M_n(k);\ k = 0, \dots, n\})(t)|.$$

So for any bounded continuous function $F = (F_0, F_1)$: $C([0, T]; \mathbb{R}) \to \mathbb{R}^2$, we have, by (4.28),

$$\limsup_{n\to\infty}\left|\pi_n\left(F\left(W_n\left(\{P_n^1(k;\omega)\}_{k=0}^n\right)\right)\right) - E^{\tilde{Q}_n}\left[F_0(w) + e^{-rT}w(T)F_1(w)\right]\right| = 0.$$

Thus by Proposition 4.27, we have assertion (ii), and this completes the proof of Lemma 4.2. □

5. Limit set of preconsistent price systems. Let $\tilde{\mathscr{P}}$ be the set of probability measures Q on $C([0, T]; \mathbb{R})$ such that there is a sequence $\{\pi_n\}_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathscr{P}_n(c_{0,n}, c_{1,n})$ such that

$$\lim_{n \to \infty} \pi_n\Big(F\Big(W_n\Big(\{P_n^1(k)\}_{k=0}^n\Big)\Big)\Big) = E^Q\big[F_0(w) + e^{-rT}w(T)F_1(w)\big]$$

for any bounded continuous function $F = (F_0, F_1)$: $C([0, T]; \mathbb{R}) \to \mathbb{R}^2$.
Then we have the following proposition.

PROPOSITION 5.1. *$\tilde{\mathscr{P}}$ is a closed convex subset in the space of probability measures on $C([0, T]; \mathbb{R})$.*

The main purpose of this section is to prove the following lemma.

LEMMA 5.2. $\tilde{\mathscr{P}} = \mathscr{P}_M(\sigma, \gamma, r)$.

Let μ be the standard Wiener measure in $C([0, T]; \mathbb{R})$. For any $w \in C([0, T]; \mathbb{R})$, let $\|w\|_{C([0,T];\mathbb{R})}$ denote $\max\{|w(t)|;\ t \in [0, T]\}$. Then $\|\cdot\|_{C([0,T];\mathbb{R})}$ is a norm in $C([0, T]; \mathbb{R})$. Let $\mathscr{B}_t = \sigma\{w(s);\ s \in [0, t]\}$, $t \in [0, T]$. Then we have the following proposition.

PROPOSITION 5.3. *Let g: $[0, T] \times C([0, T]; \mathbb{R}) \to [0, \infty)$ be a bounded continuous function satisfying the following statements.*

(i) *$g(t, \cdot)$: $C([0, T]; \mathbb{R}) \to \mathbb{R}$ is $\mathscr{B}_t$-measurable, $t \in [0, T]$.*
(ii) *There is a $\delta > 0$ such that $g(t, w)^2 \in [(0 \vee (\sigma(\sigma - \gamma)) + \delta, \sigma(\sigma + \gamma) - \delta]$, $t \in [0, T]$, $w \in C([0, T]; \mathbb{R})$.*
(iii) *There is a constant $C > 0$ such that*

$$|g(t, w) - g(t', w')| \le C\big\{|t - t'| + \|w - w'\|_{C([0,T];\mathbb{R})}\big\}$$

for any t, $t' \in [0, T]$ and w, $w' \in C([0, T]; \mathbb{R})$. Let $\{X(t);\ t \in [0, T]\}$ be a stochastic process defined on the probability space $(C([0, T]; \mathbb{R}), \mathscr{B}_T, \mu)$ by

$$X(t, w) = \exp\left(\int_0^T g(s, w)\, dw(s) + rt - \tfrac{1}{2}\int_0^t g(s, w)^2\, ds\right), \qquad t \in [0, T].$$

Then the probability law of $\{X(t);\ t \in [0, T]\}$ under μ belongs to $\tilde{\mathscr{P}}$.

PROOF. For each $n \ge 1$, let $a_n(k)$: $\Omega \to \mathbb{R}$ and $B_n(k)$: $\Omega \to \mathbb{R}$, $k = 0, 1, \dots, n$, be inductively given by

$$a_n(0) = 0, \qquad B_n(0) = 0,$$

$$\begin{aligned} B_n(k) = B_n(k-1) &+ \sigma^{-1}(1 + 2a_n(k))^{-1/2} \\ &\times\{\exp((1 + a_n(k))\sigma_n Z(k) - a_n(k-1)\sigma_n Z(k-1) \\ &\qquad + (\mu_n - r_n)) - 1\}, \end{aligned}$$

$$a_n(k) = \tfrac{1}{2}\Big(\sigma^{-2} g\Big((k-1)T/n, W_n\Big(\{B_n(l \wedge (k-1))\}_{l=0}^n\Big)\Big)^2 - 1\Big)$$

for $k = 1, \ldots, n$. Then we see inductively that $B_n(k)$ is $\mathscr{F}_k$-measurable, $k = 0, 1, \ldots, n$, and that $a_n(k)$ is $\mathscr{F}_{k-1}$-measurable, $k = 1, \ldots, n$. Also, we see that there are an $n_1 \geq 1$ and $C_1 > 0$ such that

$$(5.4)\quad |a_n(k)| \leq (2\sigma_n)^{-1} \log(1 + \gamma_n), \quad P\text{-a.s.}, k = 0, 1, \ldots, n, n \geq n_1,$$

$$(5.5)\quad -\tfrac{1}{2} + (2\sigma^2)^{-1} \delta \leq a_n(k), \quad P\text{-a.s.}, k = 0, \ldots, n, n \geq n_1,$$

$$(5.6)\quad |B_n(k) - B_n(k-1)| \leq C_1 n^{-1/2}, \qquad k = 1, \ldots, n, n \geq n_1$$

and

$$(5.7)\quad |a_n(k) - a_n(k-1)| \leq C_1 n^{-1/2}, \qquad k = 1, \ldots, n, n \geq n_1.$$

Let $q_n(k)$, $k = 1, \ldots, n$, $n \geq 1$, be given by

$$(5.8)\quad q_n(k) = \frac{\exp(a_n(k-1)\sigma_n Z(k-1) - (\mu_n - r_n)) - \exp(-(1 + a_n(k))\sigma_n)}{\exp((1 + a_n(k))\sigma_n) - \exp(-(1 + a_n(k))\sigma_n)}.$$

Then by (5.5), we see that there is an $n_2 \geq 1$ such that $q_n(k) \in (0, 1)$, P-a.s., $k = 1, \ldots, n$, $n \geq n_2$. It is obvious that $q_n(k)$ is $\mathscr{F}_{k-1}$-measurable.

Let

$$\tilde{q}_n(k, \omega) = \begin{cases} q_n(k, \omega), & \text{if } Z(k, \omega) = 1, \\ 1 - q_n(k, \omega), & \text{if } Z(k, \omega) = -1, \end{cases} \qquad k = 1, \ldots, n,$$

and let $\rho_n^0 \colon \Omega \to (0, \infty)$, $n \geq n_2$, be given by

$$\rho_n^0(\omega) = 2^n \prod_{k=1}^{n} \tilde{q}_n(k; \omega).$$

Then we see that $E^P[\rho_n^0] = 1$ and

$$E^P\left[\rho_n^0 \mid \mathscr{F}_k\right] = 2^k \prod_{l=1}^{k} \tilde{q}_n(l), \qquad k = 1, \ldots, n.$$

Also, it is easy to see that $E^P[(B_n(k) - B_n(k-1))\tilde{q}_n(k) \mid \mathscr{F}_{k-1}] = 0$, $k = 1, 2, \ldots, n$. This implies that

$$(5.9)\quad E^P\left[(B_n(k) - B_n(k-1))\rho_n^0 \mid \mathscr{F}_{k-1}\right] = 0, \qquad k = 1, 2, \ldots, n.$$

Let $Q^{(n)}$, $n \geq n_2$, be a probability measure on Ω given by $Q^{(n)}(d\omega) = \rho_n^0(\omega)P(d\omega)$.

Let $M_n(k) \colon \Omega \to (0, \infty)$, $k = 0, 1, \ldots, n$, $n \geq n_2$, be given by

$$M_n(k) = (1 - c_{0,n})^{1/2}(1 + c_{1,n})^{1/2} \times \exp\left(\sigma_n\left\{\sum_{l=1}^{k-1} Z(l) + (1 + a_n(k))Z(k)\right\} + k(\mu_n - r_n)\right).$$

Then by (5.4) we have

$$(5.10)\quad \begin{aligned} (1 - c_{0,n})P_n^1(k) &\leq M_n(k)\exp(kr_n) \\ &\leq (1 + c_{1,n})P_n^1(k), \qquad k = 0, 1, \ldots, n, \end{aligned}$$

$n \le n_2$. Note that

$$M_n(k)^{-1}M_n(k+1) = \exp((1+a_n(k+1))\sigma_n Z(k+1) \\ -a_n(k)\sigma_n Z(k) + (\mu_n - r_n)),$$

So we have

$$\begin{aligned} &M_n(k+1) \\ (5.11) \quad &= M_n(k)\left\{1 + \sigma(1+2a_n(k+1))^{1/2}(B_n(k+1) - B_n(k))\right\}. \end{aligned}$$

This and (5.9) imply that

$$(5.12) \quad E^P[M_n(k+1)\rho_n^0 \mid \mathscr{F}_k] = E^P[\rho_n^0 \mid \mathscr{F}_k]M_n(k), \\ k = 0, 1, \dots, n,\ n \ge n_2.$$

So we see that $\{M_n(k);\ k = 0, 1, \dots, n\}$ and $\{B_n(k);\ k = 0, 1, \dots, n\}$ are $\mathscr{F}_k$-martingales under the probability measure $Q^{(n)}$, $n \ge n_2$. Moreover, by (5.10) and Proposition 2.14, we see that $\pi_n = \pi(\cdot;\ \rho_n^0, M_n(n)\rho_n^0)$ is a preconsistent price system.

By (5.8), we see that there is a constant $C_2 > 0$ such that

$$\left| q_n(k) - \frac{a_n(k-1)Z(k-1) + (1+a_n(k))}{2(1+a_n(k))} \right| \\ \le C_2 n^{-1/2}, \qquad k = 1, \dots, n,\ n \ge n_2.$$

This implies that

$$(5.13) \quad \left| (2q_n(k) - 1)Z(k-1) - \frac{a_n(k-1)}{1+a_n(k)} \right| \\ \le 2C_2 n^{-1/2}, \qquad k = 1, \dots, n,\ n \ge n_2.$$

Also, by (5.11), we see that there is a constant $C_3 > 0$ such that

$$\begin{aligned} &|M_n(k)^{-1}M_n(k+1) - 1 \\ (5.14) \quad &-\{(1+a_n(k+1))\sigma_n Z(k+1) - a_n(k)\sigma_n Z(k))\}| \le C_3 n^{-1}, \\ &\qquad k = 1, \dots, n,\ n \ge n_2. \end{aligned}$$

Note that

$$\begin{aligned} E^{Q^{(n)}}&\left[((1+a_n(k+1))\sigma_n Z(k+1) - a_n(k)\sigma_n Z(k))^2 \mid \mathscr{F}_k\right] \\ &= \sigma_n^2\left\{(1+a_n(k+1))^2 + a_n(k)^2 \right. \\ &\qquad \left. -2(1+a_n(k+1))a_n(k)(2q_n(k+1) - 1)Z(k)\right\}. \end{aligned}$$

So combining this with (5.11), (5.13) and (5.14), we see that there is constant $C_4 > 0$ such that

$$\left| E^{Q^{(n)}}\left[\left(M_n(k)^{-1} M_n(k+1) - 1 \right)^2 \mid \mathscr{F}_k \right] - \sigma_n^2 (1 + 2a_n(k+1)) \right| \\ \leq C_4 n^{-3/2}, \qquad k = 0, 1, \dots, n-1,\ n \geq n_2,$$

and

$$\left| E^{Q^{(n)}}\left[(B_n(k+1) - B_n(k))^2 \mid \mathscr{F}_k \right] - \sigma^{-2}\sigma_n^2 \right| \\ \leq C_4 n^{-3/2}, \qquad k = 0, 1, \dots, n-1,\ n \geq n_2.$$

So by (5.10) and Burkholder's inequality, we see that the sequence of the probability laws $(W_n(\{P_n^1(l)\}_{l=0}^n), W_n(\{M_n(l)\}_{l=0}^n), W_n(\{B_n(l)\}_{l=0}^n))$ under $Q^{(n)}$, $n = n_2, n_2 + 1, \dots,$ is tight in the space of probability measures on $C([0, T]; \mathbb{R}^3)$. Let $\nu(dP, dM, dB)$ be a cluster point of this sequence. Then by (5.9), (5.10) and (5.11), we see that $P(t) = e^{rt}M(t)$, $t \in [0, T]$, ν-a.s. (P, M, B), $\{(M(t), B(t));\ t \in [0, T]\}$ is a martingale under ν, $\{B(t);\ t \in [0, T]\}$ under ν is a Brownian motion and that the following Itô-type SDE is satisfied under ν:

$$\text{(5.15)} \qquad \begin{aligned} dB(t) &= dB(t), \\ dM(t) &= M(t) g(t, B(\cdot))\, dB(t), \end{aligned} \qquad t \in [0, T].$$

From these facts, we see that

$$\text{(5.16)} \qquad \begin{aligned} &\lim_{n\to\infty} \pi_n\left(F\left(W_n\left(\{P_n^1(k)\}_{k=0}^n \right) \right) \right) \\ &\quad = E^{\nu}[F_0(\{P(t); t \in [0, T]\}) \\ &\qquad + M(T) F_1(\{P(t); t \in [0, T]\})] \end{aligned}$$

for any bounded continuous function $F = (F_0, F_1)$: $C([0, T]; \mathbb{R}) \to \mathbb{R}^2$. Since the coefficients of the SDE (5.15) are Lipschitz continuous, the uniqueness theorem [e.g., Ikeda and Watanabe (1989), Chapter 4, Theorem 2.2] implies that

$$M(t) = \exp\left(\int_0^t g(s, B(\cdot))\, dB(s) - \tfrac{1}{2} \int_0^t g(s, B(\cdot))^2\, ds \right), \qquad t \in [0, T].$$

So the probability law of $\{(P(t), M(t));\ t \in [0, T]\}$ under ν and the probability law of $\{(X(t), e^{-rt}X(t));\ t \in [0, T]\}$ under μ are the same. Thus by (5.16), we have our assertion, and this completes the proof. □

COROLLARY 5.17. *Let g: $[0, T] \times C([0, T]; \mathbb{R}) \to [0, \infty)$ be an adapted bounded measurable function satisfying*

$$(0 \vee (\sigma(\sigma - \gamma))) \leq g(t, w)^2 \leq \sigma(\sigma + \gamma),\ t \in [0, T], \quad w \in C([0, T]; \mathbb{R}).$$

Let $\{X(t);\ t \in [0, T]\}$ be the stochastic process defined in the probability space $(C([0, T]; \mathbb{R}), \mathscr{B}_T, \mu)$ by

$$X(t, w) = \exp\left(\int_0^t g(s, w)\, dw(s) + rt - \tfrac{1}{2}\int_0^t g(s, w)^2\, ds\right), \quad t \in [0, T].$$

Then the probability law of $\{X(t);\ t \in [0, T]\}$ under μ belongs to $\tilde{\mathscr{P}}$.

Proof. It is easy to see that there are bounded continuous functions g_m: $[0, T] \times C([0, T]; \mathbb{R}) \to [0, \infty)$, $m = 1, 2, \dots$, satisfying the following statements:

(i) $g_m(t, \cdot)$: $C([0, T]; \mathbb{R}) \to \mathbb{R}$ is $\mathscr{B}_t$-measurable, $t \in [0, T]$, $m \geq 1$.

(ii) $g_m(t, w)^2 \in [(0 \vee (\sigma(\sigma - \gamma))) + m^{-1}\sigma(\sigma \wedge \gamma),\ \sigma(\sigma + \gamma) - m^{-1}\sigma\gamma]$, $t \in [0, T]$, $w \in C([0, T]; \mathbb{R})$, $m \geq 1$.

(iii) $|g_m(t, w) - g_m(t', w')| \leq m\{|t - t'| + \|w - w'\|_{C([0,T];\mathbb{R})}\}$, $t, t' \in [0, T]$, $w, w' \in C([0, T]; \mathbb{R})$, $m \geq 1$.

(iv) $E^{\mu}[\int_0^T |g(t, w) - g_m(t, w)|^2\, dt] \to 0$, $m \to \infty$.

Let $\{X_m(t);\ t \in [0, T]\}$ be the stochastic process given by

$$X_m(t, w) = \exp\left(\int_0^t g_m(s, w)\, dw(s) + rt - \tfrac{1}{2}\int_0^t g_m(s, w)^2\, ds\right), \qquad t \in [0, T].$$

Then by Lemma 5.3, the probability law of $\{X_m(t);\ t \in [0, T]\}$ under μ belongs to $\tilde{\mathscr{P}}$. Since $E^{\mu}[\max\{|X(t) - X_m(t)|;\ t \in [0, T]\}] \to 0$, $m \to \infty$, we have our assertion by Proposition 5.1. □

Proof of Lemma 5.2. Let $\{M(t);\ t \in [0, T]\}$ be an arbitrary positive continuous martingale such that $M(0) = 1$, P-a.s., and that $\sigma(\sigma - \gamma)\, dt \leq d\langle \log M \rangle_t \leq \sigma(\sigma + \gamma)\, dt$, P-a.s. Let $N(t)$ be the martingale part of $\log M(t)$. Then we see that $\log M(t) = N(t) - \frac{1}{2}\langle N \rangle_t$. Let $\{Z(t);\ t \in [0, T]\}$ be a standard Brownian motion independent of $\{N(t);\ t \in [0, T]\}$. Let $N_m(t) = (1 - m^{-1})N(t) + m^{-1}\sigma Z(t)$, $t \in [0, T]$, $m \geq 1$, and let $M_m(t) = \exp(N_m(t) - \frac{1}{2}\langle N_m \rangle_t)$, $t \in [0, T]$, $m \geq 1$. Then it is obvious that $M_m(0) = 1$, P-a.s., and that $(m^{-2}\sigma^2 \vee \sigma(\sigma - \gamma))\, dt \leq d\langle \log M_m \rangle_t \leq \sigma(\sigma + \gamma)\, dt$, P-a.s. Moreover, we see that $E^P[\sup_{t \in [0,T]} |M(t) - M_m(t)|^p] \to 0$, $m \to \infty$, for any $p \in (1, \infty)$.

Let $g_m(t) = d/dt\langle N_m \rangle_t$, a.e. $t \in [0, T]$. Then we see that $g_m(t) \geq m^{-2}\sigma^2$. Let $B_m(t) = \int_0^t g_m(s)^{-1/2}\, dN_m(s)$, $t \in [0, T]$, $k_{m,n}(t) = \sigma$, $t \in [0, n^{-1}T)$ and $k_{m,n}(t) = (n/T)\int_{([nt/T]-1)T/n}^{[nt/T]T/n} g_m(s)^{1/2}\, ds$, $t \in [n^{-1}T, T]$. Then we see that

$$E^P\left[\sup_{t \in [0,T]}\left|M_m(t) - \exp\left(\int_0^t k_{m,n}(s)\, dB_m(s) - \tfrac{1}{2}\int_0^t k_{m,n}(s)^2\, ds\right)\right|^p\right] \to 0,$$

and $n \to \infty$ for any $p \in (1, \infty)$ and $m \geq 1$.

Fix $m, n \geq 1$ for a while. Let $\mathscr{B}_m(t) = \sigma\{B_m(s);\ s \in [0, t]\}$, $t \in [0, T]$. For any probability measure ρ on $\mathbb{R}$, let $F(\cdot; \rho)$: $\mathbb{R} \to \mathbb{R}$ be given by $F(x; \rho) = \inf\{y \in \mathbb{R};\ x \leq \rho((-\infty, y])\}$, $x \in [0, 1]$. Then we see that the probability law of $F(\xi; \rho)$ under Lebesgue measure $d\xi$ on $(0, 1)$ is ρ. Let $\rho_1(dy; w) =$

$P[k_{m,n}(0) \in dy \mid B_m(\cdot) = w]$ and

$$\rho_k(dy; y_1, \dots, y_{k-1}, w)$$
$$= P\Big[k_{m,n}\big((k-1)n^{-1}T\big) \in dy \mid B_m(\cdot) = w,$$
$$k_{m,n}\big((j-1)n^{-1}T\big) = y_j, j = 1, \dots, k-1\Big],$$

$k = 2, \dots, n$. We define a map f_k: $(0,1)^k \times C([0,T]; \mathbb{R}) \to \mathbb{R}$ to be given by

$$f_1(\underline{\xi}, w) = F(\xi_1; \rho_1(\cdot; w))$$

and

$$f_k(\underline{\xi}, w) = F\Big(\xi_\kappa; \rho_{k-1}\big(\cdot; f_1(\underline{\xi}, w), \dots, f_{k-1}(\underline{\xi}, w), w\big)\Big), \qquad k = 2, \dots, n.$$

Then we see that the probability law of $(w, f_1(\underline{\xi}, w), \dots, f_n(\underline{\xi}, w))$ under $d\xi \otimes d\mu$ is the same as the probability law of $(B_m(\cdot), k_{m,n}(0), k_{m,n}(n^{-1}T), \dots, k_{m,n}((n-1)n^{-1}T))$ under dP. Since $\sigma\{B_m(t) - B_m(kn^{-1}T);$ $t \in [kn^{-1}T, T]\}$ is independent of $k_{m,n}((j-1)n^{-1}T)$, $j = 1, \dots, k$, we see that $\rho_k(dy; y_1, \dots, y_{k-1}, w)$ is $\mathscr{B}_{kT/n}$-measurable and so is $f_k(\underline{\xi}, w)$.

Let $\tilde{f}(t, w; \underline{\xi}) = f_{[nt/T]}(\underline{\xi}, w)$, $t \in [0, T]$, and let

$$X_{m,n}(t) = \exp\bigg(\int_0^t k_{m,n}(s)\, dB_m(s) + rt - \tfrac{1}{2}\int_0^t k_{m,n}(s)^2\, ds\bigg), \qquad t \in [0, T].$$

Then the probability law of

$$\bigg\{\exp\bigg(\int_0^t \tilde{f}(s, w; \xi)\, dw(s) + rt - \tfrac{1}{2}\int_0^t \tilde{f}(s, w, \xi)^2\, ds\bigg); t \in [0, T]\bigg\}$$

under $d\xi \otimes d\mu$ is the same as $\{X_{m,n}(t); t \in [0, T]\}$ under dP. So by Proposition 5.1 and Corollary 5.16, we see that the probability law of $\{X_{m,n}(t);$ $t \in [0, T]\}$ belongs to $\tilde{\mathscr{P}}$. Since

$$\lim_{m\to\infty} \lim_{n\to\infty} E^P\big[\max\{|M(t) - X_{m,n}(t)|; t \in [0, T]\}\big] = 0,$$

again by Proposition 5.1, we see that the probability law of $\{e^{rt}M(t);$ $t \in [0, T]\}$ under P belongs to $\tilde{\mathscr{P}}$. This shows that $\mathscr{P}_M(\sigma, \gamma, r) \subset \tilde{\mathscr{P}}$. By Lemma 4.2, we have $\tilde{\mathscr{P}} \subset \mathscr{P}_M(\sigma, \gamma, r)$. So this completes the proof of Lemma 5.2. □

6. Proof of Theorem 2 and Corollary 1. Let us prove Theorem 2. Let F: $C([0, T]; \mathbb{R}) \to \mathbb{R}^2$ be a continuous function such that there are $C \in (0, \infty)$ and $p \in (1, \infty)$ such that

$$|F(w)| \le C\Big(1 + \max_{t\in[0,T]} |w(t)|\Big)^p.$$

By Proposition 2.1, we see that there are subsequences $\{n_l\}_{l=1}^\infty$ and $\{\pi_{n_l}\}_{l=1}^\infty$ $\in \prod_{l=1}^\infty \mathscr{P}_{n_l}(c_{0,n_l}, c_{1,n_l})$ such that

$$\limsup_{n\to\infty} \pi_n^*\Big(F\Big(W_n\Big(\big\{P_n^1(k;\cdot)\big\}_{k=0}^n\Big)\Big)\Big) = \lim_{l\to\infty} \pi_{n_l}\Big(F\Big(W_{n_l}\Big(\big\{P_{n_l}^1(k;\cdot)\big\}_{k=0}^n\Big)\Big)\Big).$$

By Lemma 4.2, we see that there is a subsequence $\{n'_l\}_{l=1}^{\infty}$ of the sequence $\{n_l\}_{l=1}^{\infty}$ and some $Q \in \mathscr{P}_M(\sigma, \gamma, r)$ such that

$$\lim_{l\to\infty} \pi_{n_l}\Big(F\Big(W_{n_l}\big(\{P^1_{n_l}(k;\cdot)\}_{k=0}^{n}\big)\Big)\Big) = E^Q\big[F_0(w) + e^{-rT}w(T)F_1(w)\big].$$

So we see that

$$\text{(6.1)}\quad \begin{aligned}&\limsup_{n\to\infty} \pi_n^*\Big(F\Big(W_n\big(\{P^1_n(k;\cdot)\}_{k=0}^{n}\big)\Big)\Big)\\ &\qquad \le \sup\big\{E^Q\big[F_0(w) + e^{-rT}w(T)F_1(w)\big]; Q\in\mathscr{P}_M(\sigma,\gamma)\big\}.\end{aligned}$$

On the other hand, by Lemmas 5.2 and 4.2(i), we see that for any $Q \in \mathscr{P}_M(\sigma, \gamma, r)$, there is a $\{\pi_n\}_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathscr{P}_n(c_{0,n}, c_{1,n})$ such that

$$\lim_{n\to\infty} \pi_n\Big(F\Big(W_n\big(\{P^1_n(k;\cdot)\}_{k=0}^{n}\big)\Big)\Big) = E^Q\big[F_0(w) + e^{-rT}w(T)F_1(w)\big].$$

This and Proposition 2.17 imply that

$$\text{(6.2)}\quad \begin{aligned}&\liminf_{n\to\infty} \pi_n^*\Big(F\Big(W_n\big(\{P_1(k;\cdot)\}_{k=0}^{n}\big)\Big)\Big)\\ &\qquad \ge E^Q\big[F_0(w) + e^{-rT}w(T)F_1(w)\big], \qquad Q\in\mathscr{P}_M(\sigma,\gamma,r).\end{aligned}$$

(6.1) and (6.2) imply Theorem 2, and this completes the proof of Theorem 2. □

Now let us prove Corollary 1. First we prove assertion (i). It is sufficient to prove

$$\sup\big\{E^Q\big[G(\{e^{rt}w(t); t\in[0,T]\})\big]; Q\in\mathscr{P}_M(\sigma,\gamma,r)\big\} = E^{\overline{Q}}[G(w)].$$

Let $\{M(t); t \in [0, T]\}$ be a continuous martingale such that $P(M(0) = 1) = 1$ and $(0 \vee \sigma(\sigma - \gamma))\,dt \le d\langle \log M\rangle_t \le \sigma(\sigma+\gamma)\,dt$, P-a.s. Let $g(t) = (d/dt)\langle \log M\rangle_t$ a.e. $t \in [0, T]$, P-a.s. Let us take a standard Brownian motion $\{Z(t); t \in [0, T]\}$, that is independent of $\tilde{\mathscr{B}} = \sigma\{M(t); t \in [0, T]\}$. Let $X_0(t) = e^{rt}M(t)$ and

$$\begin{aligned}X_1(t) &= e^{rt}M(t)\exp\bigg(\int_0^T\big(\sigma(\sigma+\gamma) - g(s)\big)^{1/2}\,dZ(s)\\ &\qquad -\tfrac{1}{2}\int_0^t(\sigma(\sigma+\gamma) - g(s))\,ds, \qquad t\in[0,T].\end{aligned}$$

Then it is obvious that $X_0(t) = E^P[X_1(t) \mid \tilde{\mathscr{B}}]$, $t \in [0, T]$, P-a.s. Therefore, Jensen's inequality implies that

$$E^P[G(X_0(\cdot))] = E^P\Big[G\big(E^P[X_1(\cdot)\mid\tilde{\mathscr{B}}]\big)\Big] \le E^P[G(X_1(\cdot))].$$

It is obvious that $\langle \log X_1\rangle_t = \sigma(\sigma+\gamma)t$, and so by Doob's theorem, we see that the probability law of $\{X_1(T); t \in [0, T]\}$ is $\overline{Q}$.

The above observation implies that

$$E^{\overline{Q}}[G(w)] \ge E^Q[G(w)], \qquad Q\in\mathscr{P}_M(\sigma,\gamma,r).$$

So we have our assertion (i).

Now let us prove assertion (ii). Let $Q \in \mathscr{P}_M(\sigma, \gamma, r)$. Then we see that $\langle \log w \rangle_T \geq \underline{\sigma}^2 T$, Q-a.s. w. Let $\tau(w) = \inf\{t \geq 0; \langle \log w \rangle_t \geq \underline{\sigma}^2 T\}$. Then by the representation theorem of martingales [e.g., Ikeda and Watanabe (1989), Theorem 7.2], we see that $\log w(\tau) - r\tau + \underline{\sigma}^2 T/2$ is normally distributed with mean 0 and variance $\underline{\sigma}^2 T$. Then by Jensen's inequality we have

$$E^Q[\tilde{g}(w(T))] \leq E^Q\left[\tilde{g}\left(e^{rT}E\left[e^{-rT}w(T) \mid \mathscr{F}_\tau\right]\right)\right]$$
$$= \int_{\mathbb{R}} (2\pi T)^{-1/2} \tilde{g}\left(\exp\left(\underline{\sigma} x + \left(rT - \frac{\underline{\sigma}^2 T}{2}\right)\right)\exp\left(-\frac{x^2}{2T}\right)\right) dx.$$

If we take as Q a probability law of $\{\exp(\underline{\sigma} B(t) + (rt - \underline{\sigma}^2 t/2)); t \in [0, T]\}$, the above inequality becomes an equality. So we have assertion (ii), and this completes the proof of Corollary 1. □

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