The Annals of Applied Probability
1995, Vol. 5, No. 1, 121–127

ON THE WEAK CONVERGENCE OF DEPARTURES FROM AN INFINITE SERIES OF $\cdot/M/1$ QUEUES[1]

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In this note we observe that the recent argument of Ekhaus and Gray combined with the approach of Liggett and Shiga shows that the limit from passing a stationary ergodic arrival process of rate $\alpha < 1$ through a sequence of independent, rate one, exponential server queues is a Poisson process of rate α. This builds on work of Liggett and Shiga and Anantharam.

1. Some background. Given a stationary point process $\mathbf{A}$ on $(\Omega, \mathscr{F}, P)$ indexed by two-sided time, we define

$$(1) \qquad A(t) = \begin{cases} 0, & \text{if } t = 0, \\ \text{number of points of } \mathbf{A} \text{ in } (0, t], & \text{if } t > 0, \\ -\text{number of points of } \mathbf{A} \text{ in } (t, 0], & \text{if } t < 0. \end{cases}$$

By stationarity,

$$\lim_{t \to \infty} \frac{A(t)}{t} = \lim_{t \to -\infty} \frac{A(t)}{t} = V(\omega) \quad \text{exists } P \text{ a.s.}$$

We will be exclusively concerned with the case where this limit is P a.s. constant, equalling $\alpha < 1$. In this case we will construct a stationary output process $\mathbf{D}$ obtained from passing $\mathbf{A}$ through an independent single-server queue whose service times are i.i.d. exponentials of rate 1. As in [1], we view a $\cdot/M/1$ node (the symbol "$\cdot/M/1$" stands for a single-server queue with i.i.d. exponential service times and an arbitrary arrival process that is independent of the service process) as specified by its "virtual departure process," which is a Poisson process of rate 1. The construction of $\mathbf{D}$ from $\mathbf{A}$ takes place in two steps:

STEP 1. Given $\mathbf{A}$ and an independent rate 1 Poisson process $\mathbf{N}$ (which we think of as the *service process*), we construct for $T > 0$, the departure process $\mathbf{D}^T$ by ignoring all points of $\mathbf{A}$ that are less than $-T$. Thus, $\mathbf{D}^T$ is the departure process from the queue if it were started empty at time $-T$ and is processing arrivals since that time. The corresponding queue-size process X^T

Received August 1993; revised June 1994.
[1] Research supported in part by grants from the NSF, Sloan Foundation and the Rosenbaum Foundation.
AMS 1991 *subject classifications*. 60K25, 60K35.
Key words and phrases. Coupling, Poisson limit.

is a nonnegative integer-valued process with almost surely right continuous paths that are constant outside points in **A** and **N** and varying at points of **A** and **N** as follows:

$$(2)\qquad X^T(t) \doteq \begin{cases} 0, & \text{if } t < -T, \\ X^T(t^-) + 1, & \text{for } t \geq -T,\ t \in \mathbf{A}, \\ X^T(t^-) - 1, & \text{for } t \geq -T,\ t \in \mathbf{N} \text{ and } X^T(t^-) > 0. \end{cases}$$

Of course, by the independence of **A** and **N** we need not consider points in both **A** and **N**. The departure process $\mathbf{D}^T$ can then be defined as the points $\{t \geq -T: t \in \mathbf{N} \text{ and } X^T(t^-) > 0\}$.

STEP 2. It is easy to see that as T increases, the processes $X^T(t)$ and D^T increase. What is not so easy to see but is true by virtue of Loynes' construction (Loynes [7]) and the fact that the arrival rate is strictly less than the service rate (i.e., $\alpha < 1$) is that both $\mathbf{D} = \lim_{T\to\infty} \mathbf{D}^T$ and $X(t) = \lim_{T\to\infty} X^T(t)$ exist as finite increasing limits. As Prabhakar and Bambos [8] note, this is a nonstandard application of Loynes' construction, which is possible because of the memoryless property of the Poisson process.

If we set

$$(3)\qquad N(t) = \begin{cases} 0, & \text{if } t = 0, \\ \text{number of points of } \mathbf{N} \text{ in } (0, t], & \text{if } t > 0, \\ -\text{number of points of } \mathbf{N} \text{ in } (t, 0], & \text{if } t < 0 \end{cases}$$

and similarly define $D(t)$, we at once see that

$$D(t) = \int_0^t 1_{\{X(s^-) > 0\}}\, dN(s).$$

In fact, this is how a $G/M/1$ queue is defined in [4]. Also if **A** is stationary and ergodic of rate $\alpha < 1$, then so is **D**. To see this, notice that if **A** and **N** are shifted by t, then by construction, so is the output process **D**. Thus **D** is stationary if **A** and **N** are. By the independence of **A** and **N**, the joint process (**A**, **N**) is ergodic. Because any shift invariant function of **D** is a shift invariant function of (**A**, **N**), ergodicity of **D** follows from that of (**A**, **N**). See [1], Theorem 2, for more details.

Given the concluding remark of Step 2, we may rename **D** as $\mathbf{A}^1 = \mathscr{H}(\mathbf{N}, \mathbf{A})$, where $\mathscr{H}(\mathbf{N}, \cdot)$ is an exponential server node with service process **N** mapping arrival processes to departure processes. Now $\mathbf{D} = \mathbf{A}^1$ can be thought of as the arrival process to a second independent queue. And, as before, we may "feed" it through a Poisson process of service times $\mathbf{N}^1$, independent of **A**, **N** to obtain a departure process $\mathbf{A}^2 = \mathscr{H}(\mathbf{N}^1, \mathbf{A}^1)$, which we can think of as an arrival process, and so on, obtaining a sequence of arrival processes $\mathbf{A}^n = \mathscr{H}(\mathbf{N}^{n-1}, \mathbf{A}^{n-1})$. We may now state the result we shall prove.

THEOREM 1. *If* $\mathbf{A}$ *is a stationary, ergodic point process of rate* $\alpha < 1$, *then* $\mathbf{A}^n$ *converges in distribution to a Poisson process of rate* α.

A result crucial to our argument is the classical Burke's theorem, which states that when a Poisson process of rate α (< 1) is fed into a rate 1 exponential server queue, the output is also a rate α Poisson process. Liggett and Shiga [6] and Anantharam [1] show that this is the unique (ergodic) fixed distribution of rate α. Our result confirms a conjecture made by Reiman and Simon [9] in 1981. This conjecture was motivated by a desire to replace a complex flow in a queueing system by a Poisson process for the purpose of approximation.

A conclusion similar to Theorem 1 is obtained in the classical result of Vere-Jones [10], which states that if a stationary ergodic mixing point process is passed through a sequence of independent $\cdot/GI/\infty$ queues, then the limiting output is a Poisson process of the same rate as the original arrival process.

2. Coupling. As in [6], we rely on coupling arguments. We fix our (independent) sequence of Poisson processes of rate 1, $\mathbf{N}^n$ and consider a Poisson process $\mathbf{P}$ of rate α, independent of these Poisson processes and $\mathbf{A}$. As with the $\mathbf{A}^n$, we obtain the sequence of point processes $\mathbf{P}^n$ by $\mathbf{P}^1 = \mathscr{H}(\mathbf{N}, \mathbf{P})$; $\mathbf{P}^{n+1} = \mathscr{H}(\mathbf{N}^n, \mathbf{P}^n)$. Given that Poisson processes of rate strictly less than 1 are fixed by transformations $\mathscr{H}$, all the point processes $\mathbf{P}^n$ are Poisson of rate α. We will adopt the colouring scheme of Ananthram [1]: points of $\mathbf{A}^n \cup \mathbf{P}^n$ are coloured yellow, blue or red according to the rules:

1. Points in $\mathbf{A}^n \cap \mathbf{P}^n$ are coloured yellow.
2. Points in $\mathbf{A}^n$ but not in $\mathbf{P}^n$ are coloured blue.
3. Points in $\mathbf{P}^n$ but not in $\mathbf{A}^n$ are coloured red.

Of course no points (or customers) in $\mathbf{A}$ or $\mathbf{P}$ are coloured yellow, by the independence of $\mathbf{A}$ and $\mathbf{P}$. For each n, define $\mathbf{Y}^n$, $\mathbf{B}^n$ and $\mathbf{R}^n$ to be the process of yellow, blue and red customers, respectively. It is not hard to see that for every n, the joint process $(\mathbf{Y}^n, \mathbf{B}^n, \mathbf{R}^n)$ is stationary and ergodic. The problem is that a limiting distribution of $(\mathbf{Y}^n, \mathbf{B}^n, \mathbf{R}^n)$ need not be so.

We now give our labelling procedure for customers in $\mathbf{A}$ and $\mathbf{P}$ that will enable us to speak of a customer's path. It would be natural to adopt a "first in, first out" policy for queues. The disadvantage is that such a policy would have as a consequence the possibility that a customer of $\mathbf{A}$ could be yellow in $\mathbf{A}^n$ but blue in $\mathbf{A}^{n+1}$. It is vital for us that once a customer (of $\mathbf{A}$ or $\mathbf{P}$) becomes yellow, they remain yellow forever. Thus at a queue we adopt the following rules for all n:

(a) Yellow customers in $\mathbf{A}^n$ or $\mathbf{P}^n$ observe a "first in, first out" rule.

(b) Yellow customers in $\mathbf{A}^n$ or $\mathbf{P}^n$ take priority over any blue or red customers.

(c) If a blue customer in $\mathbf{A}^n$ arrives at a queue at which there are red customers, then she immediately "coalesces" with the red customer who arrived first and has not yet coalesced. Both the "coalesced" customers will be coloured yellow in $\mathbf{A}^{n+1}$ and $\mathbf{P}^{n+1}$. Likewise if a red customer of $\mathbf{P}^n$ arrives at a queue at which blue customers are present, then she "coalesces" with the blue customer who arrived first and has not yet coalesced.

Note that because a yellow customer remains yellow forever, the (nonrandom) density of yellow customers is nondecreasing in n. In fact, as Liggett and Shiga [6] show, for an exponential server queue the density of yellow points in $\mathbf{A}^n$ strictly increases with n (unless almost every path has either yellow and blue points or yellow and red points, but not both blue and red points simultaneously). With the symbol $\mathscr{D}(\cdot)$ denoting density, we formally record the preceding information.

FACT 1 (Liggett and Shiga [6]). For every n, $\mathscr{D}(\mathbf{Y}^{n+1}) \geq \mathscr{D}(\mathbf{Y}^n)$, and if $\mathscr{D}(\mathbf{Y}^{n+1}) = \mathscr{D}(\mathbf{Y}^n)$, then either $\mathbf{A}^n \cap \mathbf{P}^n = \mathbf{A}^n$ or $\mathbf{A}^n \cap \mathbf{P}^n = \mathbf{P}^n$ pathwise.

As a consequence of Fact 1 and the fact that mixtures of Poisson processes of rate $\alpha < 1$ are invariant for the $\cdot/M/1$ node, Liggett and Shiga [6] deduced that the limit from passing an arbitrary ergodic stationary arrival process $\mathbf{A}$ through an infinite tandem of $\cdot/M/1$ queues is a mixture of Poisson processes. In this note, we rule out the nonergodic mixtures as possible limits. Our method of proof will be to try and prove that as n tends to infinity, the density of red and blue points tends to zero and that, therefore, the limiting distribution of $\mathbf{A}^n$ is that of $\mathbf{P}^n$, that is to say, Poisson of rate α. In Section 3 we will assume that the density of yellow customers does not increase up to α and argue by contradiction. Specifically, this assumption implies that there exist customers in the initial arrival processes $\mathbf{A}$ and $\mathbf{P}$ that never coalesce and hence never become yellow. We call these customers "everblues" and "everreds," respectively.

NOTATION. Given a customer V (in either $\mathbf{A}$ or $\mathbf{P}$), we write their departure times from the nth queue as $V(n)$.

From our construction of $\mathbf{A}^{n+1}$ and $\mathbf{P}^{n+1}$ from $\mathbf{A}^n$, $\mathbf{P}^n$, $\mathbf{N}^n$ and the foregoing labelling scheme we readily obtain the following lemma.

LEMMA 2.1. *Let V and U be two customers* (*in* $\mathbf{A}$ *or* $\mathbf{P}$, *not necessarily belonging to the same initial point process*) *such that* $V(n) > U(n)$. *If* $U(n+1) > V(n+1)$, *then customer V must be coloured yellow after* $n+1$ *queues.*

The importance of Lemma 2.1 is that among customers that never become (and therefore remain) yellow, order is preserved: if an "everblue" in $\mathbf{A}$ arrives before an "everred" in $\mathbf{P}$, then it will arrive before the "everred" after passing through any number of queues. That is, the only way a customer can get ahead is by being yellow.

3. Proof of Theorem 1. In this section we will argue by contradiction and hence assume that the density of customers in **A** that never coalesce is strictly positive with strictly positive probability. Consider the distribution of points in **A** that never become yellow. Given the stationarity of **A** and the translation invariant nature of the transformation $\mathscr{H}$, these customers give a stationary point process. It should be noted that while **A** is an ergodic point process, it is not a priori true that the process of everblue customers in **A** is so. Accordingly, a priori the limiting density of everblues of **A** (and therefore of all the $\mathbf{A}^n$'s) must be considered as a random quantity I, which satisfies $E[I] = c > 0$.

However, the following fact follows from ergodicity of $\mathbf{A}^n$, $\mathbf{P}^n$. The (nonrandom) density $c(n)$ of blue customers in $\mathbf{A}^n$ must decrease to c. Similarly for red customers in $\mathbf{P}^n$. Thus I is nonrandom.

Therefore, initially there coexist, with probability 1, everblue and everred customers, both of density $c > 0$. It is possible to derive a contradiction from this by considering couplings of **A** with Poisson processes of order different from α as in [6]; however, we prefer to follow Ekhaus and Gray's [5] method. Because we have a.s. (according to our assumption) customers in **A** that are everblue and customers in **P** that are everred, we may consider the points of $\mathbf{A} \cup \mathbf{P}$ that represent customers that never become yellow as an alternation of intervals of everblue customers with intervals of everred customers. By Lemma 2.1, these intervals will have their "orderings" preserved. Thus if a customer is the left endpoint of an interval of everblues in **A**, then she will be the left endpoint of an interval of everblues in $\mathbf{A}^n$ for every n (and similarly with the everreds). Let us call these customers "left everblues". The process of left everblues is stationary (for the same reason as with the everblues) and thus possesses a (possibly random) density. Additionally, because the density of everblues $c > 0$, the density of left everblues is a strictly positive random variable, which we will denote by L (following Ekhaus and Gray [5] slavishly). Therefore, there exists $\varepsilon > 0$ such that $P(L > \varepsilon) > \varepsilon$. Now $L = L_l + L_g$, where L_l is the density of left everblues V such that there is another left everblue $U \in (V, V + 2/\varepsilon]$ and L_g is the density of the remaining left everblues for whom the next left everblue is at a distance greater than $2/\varepsilon$. Because by definition, $L_g \le \varepsilon/2$ P a.s., this means $L_l \ge \varepsilon/2$ whenever $L > \varepsilon$. By similar reasoning, for any n, the density of left everblues V such that there exists another left everblue U such that $U(n) \in (V(n), V(n) + 2/\varepsilon]$ must be at least $\varepsilon/2$ with probability greater than ε. A vital observation of Ekhaus and Gray [5] is that between two left everblues there must be an everred. We deduce from this that the density of everreds W in $\mathbf{P}^n$ such that there is an everblue V with $V(n) \in (W(n), W(n) + 2/\varepsilon]$ must be at least $\varepsilon/2$ with probability ε. Now everblues and everreds are distinguished customers, identified by looking into the future; however, they must be, respectively, blue and red in $\mathbf{A}^n$ and $\mathbf{P}^n$ for any n. For any n consider the event S:

$$\{\varepsilon/2 \le \text{the density of red customers } W \text{ in } \mathbf{P}^n \text{ s.t. } \exists \text{ a blue customer of } \mathbf{A}^n \text{ in } (W, W + 2/\varepsilon]\}.$$

Clearly, this event is translation invariant and is contained in the jointly stationary and ergodic process ($\mathbf{B}^n$, $\mathbf{R}^n$). Moreover, by the preceding arguments, $P(S) > \varepsilon$. Thus $P(S) = 1$ and we obtain the following lemma.

LEMMA 3.1. *If the density of everblues is strictly positive, then there exists an* ε, *not depending on* n, *such that the* (*nonrandom*) *density in* $\mathbf{P}^n$ *of red customers* W *satisfying*

$$\text{there exist blue customers of } \mathbf{A}^n \text{ in } (W(n), W(n) + 2/\varepsilon]$$

must be at least $\varepsilon/2$.

Let $X^n(\cdot)$ denote the queue size at the $n+1$st queue that has $\mathbf{P}^n$ as the arrival process and $\mathbf{N}^n$ as the service process. Thus $X^n(t)$ denotes the total number of red and yellow customers present in queue $n+1$ at time t. Now for any n, $\mathbf{P}^n$ is a Poisson process of rate $\alpha < 1$, independent of Poisson process $\mathbf{N}^n$. Accordingly, for any $t_p \in \mathbf{P}^n$,

$$P\big(X^n(t_p^-) = 0|\mathbf{P}^n\big) = P\big(X^n(t_p^-) = 0|\mathbf{Y}^n, \mathbf{R}^n, \mathbf{B}^n\big) > 0,$$

where the equality follows from the fact that $X^n(\cdot)$ is the queue-size process corresponding to yellow and red customers only and so is independent of the process $\mathbf{B}^n$. Given this, we easily see the following lemma.

LEMMA 3.2. *Under the assumptions of Lemma* 3.1, *there exist strictly positive* ε *and* δ *such that for every* n, *red points* t_p *in* $\mathbf{P}^n$ *with the properties* (a) *there exists a blue point* (*or customer*) *of* $\mathbf{A}^n$ *in* $(t_p, t_p + 2/\varepsilon]$ *and* (b) $P(X^n(t_p^-) = 0|\mathbf{P}^n) > \delta$ *have density at least* $\varepsilon/3$.

Suppose that a red customer R satisfies properties (a) and (b) of Lemma 3.2. Then because of property (b), the chance that R finds the $n+1$st queue empty of yellow and red customers upon entering it is at least δ. By the Markov property of the Poisson process $\mathbf{N}^n$, given this, the chance that there are no services in the time interval $(t_p, t_p + 2/\varepsilon]$ is exactly $e^{-2/\varepsilon}$. However, if there are no services in this time, then because by property (a) a blue customer must arrive during this interval, it must be the case that R is yellow in $\mathbf{P}^{n+1}$. Therefore, under the assumptions of Lemma 3.1 we have shown that for all n, the density of red (or blue) customers in $\mathbf{P}^n$ ($\mathbf{A}^n$) minus the corresponding density in $\mathbf{P}^{n+1}$ ($\mathbf{A}^{n+1}$) is at least $\varepsilon\delta e^{-2/\varepsilon}/3$. This contradiction proves Theorem 1.

Conclusion. An old conjecture in queueing theory has been considered and proved to be true. The general case is still undecided. That is, does every $\cdot/G/1$ queue have an invariant distribution (an issue originally discussed in Bambos and Walrand [3])? If yes, will an arbitrary ergodic stationary arrival process converge weakly to the invariant distribution as it traverses through an infinite tandem of $\cdot/G/1$ queues? As Anantharam [1] has remarked, the methods used by him and those used here do not translate readily to the general case. This is due to the fact that the memoryless property of the service Poisson processes $\mathbf{N}^n$, which facilitated couplings, is typical only of $\cdot/M/1$ (and $\cdot/M/k$) queues. Thus new methods need to be devised.

Acknowledgment. The authors would like to thank Professor Tom Liggett and Professor Nicholas Bambos for many fruitful discussions.

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