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LARGE DEVIATION RATES FOR BRANCHING PROCESSES. II. THE MULTITYPE CASE

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Let $\{Z_n: n \geq 0\}$ be a p-type ($p \geq 2$) supercritical branching process with mean matrix M. It is known that for any l in R^p,

$$\left(\frac{l \cdot Z_n}{1 \cdot Z_n} - \frac{l \cdot (Z_n M)}{1 \cdot Z_n}\right) \quad \text{and} \quad \left(\frac{l \cdot Z_n}{1 \cdot Z_n} - \frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}}\right)$$

converge to 0 with probability 1 on the set of nonextinction, where $v^{(1)}$ is the left eigenvector of M corresponding to its maximal eigenvalue ρ and 1 is the vector with all components equal to one. In this paper we study the large deviation aspects of this convergence. It is shown that the large deviation probabilities for these two sequences decay geometrically and under appropriate conditioning supergeometrically.

1. Introduction. Let $\{Z_n: n \geq 0\}$ be a supercritical p-type Galton–Watson branching process (see [2] for a definition) with offspring generating functions $f^{(i)}(s), i = 1, 2, \ldots, p$, and mean matrix M. Let ρ be the maximal eigenvalue of M (necessarily greater than 1) with the corresponding left and right eigenvectors $v^{(1)}$ and $u^{(1)}$, respectively. It is known that for any vector l (see [2])

$$\left(\frac{l \cdot Z_{n+1}}{1 \cdot Z_n} - \frac{l \cdot (Z_n M)}{1 \cdot Z_n}\right) \quad \text{and} \quad \left(\frac{l \cdot Z_n}{1 \cdot Z_n} - \frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}}\right)$$

converge to 0 with probability 1 (wp1) on the set of nonextinction and that $\{W_n \equiv (u^{(1)} \cdot Z_n)/\rho^n: n \geq 0\}$ is a nonnegative martingale sequence and hence converges to a nonnegative random variable W with probability 1.

The questions addressed in this paper concern the large deviation aspects of the above convergence. It turns out that, under certain moment conditions, the rate of decay of the probabilities of large deviations is geometric, while conditionally on $W \geq a$ ($a > 0$), the rate is supergeometric. The corresponding results for single-type branching process are available in [1] and [3]. In [8] large deviation aspects of $P(W \leq x)$ and $P(W \geq x)$ a $x \to 0$ and $x \to \infty$, respectively, are studied.

As in those papers, we reduce the problem (using the moment conditions on the offspring distributions) to a study of decay rates of iterates of generating functions $f^{(i)}$.

The paper is organized as follows: Section 2 contains notations, definitions and assumptions, Section 3 gives statements of the results and Section 4 con-

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tains some of the proofs. For ease of exposition, we assume $p = 2$ throughout the rest of the paper.

2. Notations, definitions and assumptions.

1. $\mathscr{C}_2 = [0,1]\times[0,1]$ is the unit square in R^2, the two-dimensional Euclidean space.
2. $\mathscr{A}_2 = \{(i_1, i_2)\colon i, \in \mathbb{Z}_+,\ i_2 \in \mathbb{Z}_+\}$, where $\mathbb{Z}_+$ is the set of all nonnegative integers.
3. For $s \in \mathscr{C}_2$ and $j \in \mathscr{A}_2$, $s^j = s_1^{j_1} s_2^{j_2}$
4. $\mathbf{1} = (1,1)$, $e_1 = (1,0)$, $e_2 = (0,1)$ and $\mathbf{0} = (0,0)$.
5. $Z_n = (Z_n^{(1)}, Z_n^{(2)})$ is the population vector of the nth generation.
6. $P_i(\cdot) = P(\cdot|Z_0 = e_i)$ is the probability measure for the process with $Z_0 = e_i$ and $E_i(\cdot) = (E(\cdot|Z_0 = e_i))$ is the corresponding expectation for $i = 1, 2$.
7. $P_i(j_1, j_2) = P(Z_1 = (j_1, j_2)|Z_0 = e_i)$.
8. For $s \in \mathscr{C}_2$, $f_i^{(n)}(s) = E(s^{Z_n}|Z_0 = e_i)$, $i = 1, 2$. If $n = 1$, we shall write $f_1(s)$ and $f_2(s)$ for $f_1^{(1)}(s)$ and $f_2^{(1)}(s)$.
9. For $n \geq 0$ and $s \in \mathscr{C}_2$, $f^{(n)}(s) = (f_1^{(n)}(s), f_2^{(n)}(s))$, where for $n = 0$, $f_1^{(0)}(s) \triangleq s$ and $f(s) \equiv f^{(1)}(s)$. It is known that (see [2]) for all $n \geq 1$,

$$f^{(n)}(s) = f(f^{(n-1)}(s)).$$

10. For $s \in \mathscr{C}_2$, $||s|| = \max(s_1, s_2)$ and $||E(\cdot)|| = \max(|E_1(\cdot)|, |E_2(\cdot)|)$.
11. $R_+^2 = \{(x_1, x_2)\colon x_1 \geq 0,\ x_2 \geq 0\}$.
12. For $i,\ j = 1,\ 2$, $D_{ij}(s) = (\partial f_i(s))/\partial s_j$, $a_{ij} = D_{ij}(0)$, $m_{ij} = D_{ij}(1-)$, $A = ((q_{ij}))$ and $M = ((m_{ij}))$.
13. For any matrix E its transpose will be denoted by E^t.

ASSUMPTIONS (not all valid at all times).

A1. $f(0,0) = 0$.
A2. M is positively regular with maximum eigenvalue ρ and the associated right and left eigenvectors $u^{(1)}$ and $v^{(1)}$, respectively.
A3. $\rho > 1$
A4. There exists $0 < \gamma < 1$ such that $A^n\gamma^{-n}$ converges to a matrix P_0 that is nonzero and has finite entries.
A5. $||E(\exp(\theta_0(\mathbf{1}\cdot Z_1))|| < \infty$ for some $\theta_0 > 0$.
A6. $||E(\mathbf{1}\cdot Z_1)^{2r_0}|| < \infty$, where r_0 is such that $\rho^{r_0}\gamma > 1$.
A7. $P_i(Z_i^{(1)} \leq 1) = 0$ and $P_i(Z_i^{(1)} = 2) > 0$ for $i = 1, 2$.

3. Statements of results. It is known (see [2]) that $f^{(n)}(s) \to 0$ (for s in $\mathscr{C}_2 - \mathbf{1}$) as $n \to \infty$. Our first theorem gives the corresponding rate of convergence under A4 and is the key to the main result of the paper contained in Theorem 2.

THEOREM 1. *Under* A1 *and* A4, *there exists a map* $Q\colon \mathscr{C}_2 \to R_+^2$ *such that*

(1) $$\frac{f_n(s)}{\gamma^n} \to Q(s) \quad \text{as } n \to \infty$$

and $Q(\cdot)$ *is the unique solution of the vector functional equation*

(2) $$Q(f(s)) = \gamma Q(s)$$

subject to

(3) $$Q(0) = 0, \qquad Q'(0) = P_0 \quad \text{and} \quad 0 < Q(s) < \infty \quad \text{for } 0 < s < 1$$

(*see* A4 *for definition of* P_0).

REMARK 1. Assumption A1 can be removed by considering $f_n(s) - q$, where q is the extinction probability vector, that is, $q_i = P_i(Z_n = 0 \text{ for some } n \geq 1)$ and $A = (a_{ij})$, where $a_{ij} = D_{ij}(q)$.

The next theorem is a large deviation result for functionals of the process under a moment hypothesis on the offspring distribution function.

THEOREM 2. *Assume that* A1–A4 *and* A6 *hold. Let* $l = (l_1, l_2)$ *be a nonzero vector with* $l_1 \neq l_2$. *Then, for every* $\varepsilon > 0$ *and* $i = 1, 2$,

(4) $$\lim_{n\to\infty} \gamma^{-n} P_i\left(\left|\frac{l \cdot Z_{n+1}}{1 \cdot Z_n} - \frac{l \cdot (Z_n M)}{1 \cdot Z_n}\right| > \varepsilon\right),$$

(5) $$\lim_{n\to\infty} \gamma^{-n} P_i\left(\left|\frac{l \cdot Z_n}{1 \cdot Z_n} - \frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}}\right| > \varepsilon\right)$$

exist and are positive and finite.

The proofs of Theorem 1 and (5) of Theorem 2 are presented in the next section. We also have a number of results that are related to these and generalize the corresponding one-type case results proved in [1]. These are stated below for the sake of completeness. Their proofs as well as that of (4) of Theorem 2, though similar in spirit to those of the one-type case, are not entirely straightforward. We do not provide the proofs for considerations of space. The reader is referred to [4] and [9] for complete proofs.

The next theorem gives a rate of decay for the generating functions when $A = 0$ and every particle produces at least two particles of its kind.

THEOREM 3. *Under* A7,

$$\lim_{n\to\infty} \frac{\log f_i^{(n)}(s)}{2^n} \equiv R_i(s) \quad \text{for } i = 1, 2$$

exists and satisfies the vector functional equations

$$R_i(f(s)) = 2R_i(s)$$

and

$$\lim_{s \downarrow 0} R_i(s) = -\infty.$$

Our next theorem considers the case when A4 does not hold but A7 does. In this case the rate of decay of probabilities of large deviations is supergeometric.

THEOREM 4. *Assume* A1–A3, A5 *and* A7 *hold. Then for* $l \neq 0$ *and* $\varepsilon > 0$, *there exist constants* $0 < C_1(\varepsilon)$, $C_2(\varepsilon) < \infty$ *and* $0 < \lambda_1(\varepsilon)$, $\lambda_2(\varepsilon) < 1$ *such that*

$$P_i\left(\left|\frac{l \cdot Z_{n+1}}{1 \cdot Z_n} - \frac{l \cdot (Z_n M)}{1 \cdot Z_n}\right| \geq \varepsilon\right) \leq C_1 \cdot \lambda_1^{(2^n)}$$

and

$$P_i\left(\left|\frac{l \cdot Z_n}{1 \cdot Z_n} - \frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}}\right| \geq \varepsilon\right) \leq C_2 \lambda_2^{(2^n)}$$

for $i = 1, 2$.

The next result is of independent interest and in also needed in the proof of Theorem 6 below (see [4]).

THEOREM 5. *Under* A5 *there exists* $\theta_0 > 0$ *such that*

$$\sup_{n \geq 1} ||E(\exp(\theta_0 W_n))|| < \infty. \tag{6}$$

Theorem 6 asserts that the decay rate of $P(|W - W_n| \geq \varepsilon)$ is always supergeometric.

THEOREM 6. *Assume* A1–A3 *and* A5 *hold. Then there exist constants* $0 < C_3 < \infty$ *and* $0 < \lambda_3 < \infty$ *such that for* $\varepsilon > 0$,

$$P(|W - W_n| \geq \varepsilon) \leq C_3 \exp(-\lambda_3 \varepsilon^{2/3} (\rho^{1/3})^n).$$

The next theorem shows that conditioned on $W \geq a$, $a > 0$, large deviation probabilities in Theorem 2 decay supergeometrically.

THEOREM 7. *Assume* A1–A3 *and* A5 *hold. Then there exist constants* $0 < C_4, C_5, C_6, C_7 < \infty$ *and* $\lambda_4, \lambda_5 > 0$ *such that for every* $\varepsilon > 0$ *and* $a > 0$ *there exists* $0 < I(\varepsilon) < \infty$ *such that*

$$P_i\left(\left|\frac{l \cdot Z_{n+1}}{1 \cdot Z_n} - \frac{l \cdot (Z_n M)}{1 \cdot Z_n}\right| > \varepsilon | W \geq a\right)$$
$$\leq C_4 \exp(-a I(\varepsilon) \xi \rho^n) + C_5 \exp(-\lambda_4 (a(1 - \xi))^{2/3} (\rho^{2/3})^n)$$

and

$$P_i\left(\left|\frac{l\cdot Z_n}{1\cdot Z_n}-\frac{l\cdot v^{(1)}}{1\cdot v^{(1)}}\right|>\varepsilon|W\geq a\right)$$
$$\leq C_6\exp(-aI(\varepsilon)\xi\rho^n)+C_7\exp(-\lambda_5(a(1-\xi))^{2/3}(\rho^{2/3})^n)$$

for every $0<\xi<1$.

4. Proofs. As mentioned earlier we supply the proofs of only Theorem 1 and (5) of Theorem 2.

PROOF OF THEOREM 1. For $s\in C_2$ write f as

$$f(s)=sB+g(s)\quad\text{where } B=A^t.$$

Iterating the above equation and dividing by γ^n yields

$$\gamma^{-n}f^{(n)}(s)=\gamma^{-n}sB^n+\sum_{k=0}^{n-1}\gamma^{-n}g(f^{(n-1-k)}(s))B^k. \tag{7}$$

The first term on the right converges to sP_0^t (by A4) as $n\to\infty$.

The second term is the same as

$$\gamma^{-1}\left[\sum_{k=0}^{n-1}\gamma^{-k}g(f^{(k)}(s))(\gamma^{-1}B)^{n-1-k}\right].$$

We shall show that

$$\sum_{k\geq 0}\frac{g(f^{(k)}(s))}{\gamma^k}<\infty. \tag{8}$$

From this it will follow (by the dominated convergence theorem applied to counting measure space $\mathscr{A}_2$) (see [1] and [7]) that

$$\lim_{n\to\infty}\frac{f^{(n)}(s)}{\gamma^n}=sP_0^t+\left(\sum_{k\geq 0}\frac{g(f^{(k)}(s))}{\gamma^{k+1}}\right)P_0^t\triangleq Q(s). \tag{9}$$

Note that for $s\in\mathscr{C}_2$,

$$\|f(s)\|\leq\|s\|\quad\text{and}\quad\|g(s)\|\leq\|s\|^2. \tag{10}$$

Using this it is easy to see that, to establish (8), it is enough to establish that for $s\in\mathscr{C}_2-1$,

$$\left\{\frac{f_n(s)}{\gamma^n}\colon n\geq 0\right\}\text{ is a bounded sequence.}$$

We do this via the following two lemmas.

LEMMA 1. *Let $\{a_n: n \geq 0\}$ be a sequence of positive numbers satisfying*

$$a_n \leq C_1 + C_2 \sum_{k=0}^{n-1} \eta^k a_k, \qquad n \geq 1, \tag{11}$$

where $0 < \eta < 1$ and C_1 and C_2 are finite positive constants. Then

$$\sup_{n \geq 0} a_n < \infty.$$

The proof straightforward and therefore omitted (see [4]).

LEMMA 2. *For each $s \in \mathscr{C}_2 - 1$, there exist positive constants C and δ (depending on s) with $0 < \delta < 1$, such that for every $n \geq 1$,*

$$||f^n(s)|| \leq C\delta^n. \tag{12}$$

PROOF. Assume for the moment that

$$0 < \sum_{j=1}^{2} a_{ij} < 1 \quad \text{for all } 1 \leq i \leq 2. \tag{13}$$

We see by the continuity of $\sum_{j=1}^{2} D_{ij}(s)$ that there exist a positive number η and $0 < \delta < 1$ such that for $||s|| < \eta$,

$$\sum_{j=1}^{2} D_{ij}(s) \leq \delta \quad \text{for all } i = 1, 2.$$

For $n \geq 1$,

$$\begin{aligned} ||f^{(n+N_0)}(s)|| &= ||f^{(n+N_0)}(s) - f^{(n+N_0)}(0)|| \\ &= \max_{1 \leq i \leq 2} (f_i(f^{(n+N_0-1)}(s)) - f_i(f^{(n+N_0-1)}(0))). \end{aligned}$$

For $s \in \mathscr{C}_2 - 1$, since $f_n(s) \to 0$, given $\eta > 0$, $\exists\, N_0(\eta, s)$ such that for $n \geq N_0$,

$$||f_n(s)|| < \eta.$$

However, by mean value theorem, for each i,

$$\begin{aligned} & f_i(f^{(n+N_0-1)}(s)) - f_i(f^{(n+N_0-1)}(0)) \\ &= \sum_{j=1}^{2} f_j^{(n+N_0-1)}(s) \frac{\partial f_i}{\partial s_j}(s^*) \qquad [\text{for some } s^* \in (0, f^{(n+N_0-1)}(s))] \\ &\leq ||f^{(n+N_0-1)}(s)|| \sum_{j=1}^{2} D_{ij}(s^*) \\ &\leq ||f^{(n+N_0-1)}(s)|| \delta. \end{aligned}$$

Iterating, we have

$$||f^{(n+N_0)}(s)|| \leq ||f^{(N_0)}(s)|| \delta^n \leq C\delta^{n+N_0} \quad \text{for some } 0 < C < \infty,$$

completing the proof of the lemma under (13).

Since for each i, $1 \cdot Z_n \to \infty$ with probability 1, under P_i it follows that there exists N such that

$$\sum_{j=1}^{2} a_{ij}^{(N)} = P_i(1 \cdot Z_N = 1) < 1 \quad \text{for all } i.$$

Applying the above argument to the sequence $\{f^{nN}(s)\colon 0 \le n < \infty\}$ we see that $\|f^{nN}(s)\| < C\delta^n$. Now use (10).

To complete the proof of (11) [and hence (1)], observe that by (7), (10), Lemma 2 and (A4), $a_n \equiv \|f^{(n)}(s)/\gamma^n\|$ satisfies the hypothesis of Lemma 1 for an appropriate choice of C_1, C_2 and η.

Finally, the facts that Q is nontrivial and satisfies the functional equation (2) subject to (3) follow from (9). The proof of uniqueness is standard and omitted. □

REMARK 2. Even for the single-type case one can construct a proof based on the above method. However, in this case, finiteness of $\sum_{n\ge 1}(g(f^{(n)}(s)))/\gamma^n$ [see (8)], where $\gamma = f'(0)$, can be seen by an application of ratio test. The advantage of the above method is that it gives an explicit formula for the limit $Q(\cdot)$.

REMARK 3. Using Theorem 1 one can show that there does not exist a large deviation principle (see [6]) for the convergence of averages in a multitype branching process. The details are similar to the single-type case (see [3]).

PROOF OF THEOREM 3. We prove only assertion (5). Without loss of generality assume l is not a multiple of the vector 1. Let k_0 be fixed (to be chosen later). For $n > k_0$,

$$\begin{aligned} P_i&\left(\left|\frac{l \cdot Z_n}{1 \cdot Z_n} - \frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}}\right| > \varepsilon\right) \\ &= E_i\left(P\left(\left|\frac{l \cdot Z_n}{1 \cdot Z_n} - \frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}}\right| > \varepsilon \,\middle|\, Z_{n-k_0}\right)\right) \\ &= \sum_{j \in \mathscr{A}_2} P\left(\left|\frac{l \cdot Z_n}{1 \cdot Z_n} - \frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}}\right| > \varepsilon \,\middle|\, Z_{n-k_0} = j\right) P(Z_{n-k_0} = j). \end{aligned}$$

Consider the event

$$\left\{\left(\frac{l \cdot Z_n}{1 \cdot Z_n} - \frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}}\right) > \varepsilon\right\}$$

conditioned on $Z_{n-k_0} = j$. By the branching property,

$$Z_n = \sum_{r=1}^{j_1} Z_{k_0,r}^{(1)} + \sum_{r=1}^{j_2} Z_{k_0,r}^{(2)}, \tag{14}$$

where for fixed i, i.i.d., $\{Z_{k_0,r}^{(i)}\}_{r=1}^{\infty}$ are $\mathscr{A}_2$-valued random variables distributed as the population at time k_0 initiated by a particle of type i at time 0. Now

$$\frac{l \cdot Z_n}{1 \cdot Z_n} > \left(\frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}} + \varepsilon\right)$$

if and only if

$$\begin{aligned} l \cdot Z_n - l \cdot (jM^{k_0}) > {} & \left(\frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}} + \varepsilon\right)(1 \cdot Z_n - 1 \cdot (jM^{k_0})) \\ & + \left(\frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}} + \varepsilon\right)(1 \cdot (jM^{k_0})) - l \cdot (jM^{k_0}) \\ & \Leftrightarrow \left(l - \left(\frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}} + \varepsilon\right)1\right) \cdot \frac{(Z_n - (jM^{k_0}))}{(u^{(1)} \cdot j)\rho^{k_0}} \\ & > \frac{((C+\varepsilon)1 - l) \cdot jM^{k_0}}{(u^{(1)} \cdot j)\rho^{k_0}}, \end{aligned}$$

where $C = (l \cdot v^{(1)})/(1 \cdot v^{(1)})$.

From the Frobenius theorem (see [2], Lemma 1, page 194) it is known that if $F = \{x = (x_1, x_2) | x_i > 0,\ x \cdot u^{(1)} = 1\}$, then

$$\lim_{n\to\infty} \sup_{x \in F} ||xM^n \rho^{-n} - v^{(1)}|| = 0.$$

Consequently, for each $\eta > 0$ there is $k_0 < \infty$ such that

$$\sup_{x \in F} ||xM^{k_0}\rho^{-k_0} - v||(2||l|| + \varepsilon) < \frac{\varepsilon(1 \cdot v)}{2}.$$

Thus, for $j \neq 0$,

$$\frac{l \cdot Z_n}{1 \cdot Z_n} > \frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}} + \varepsilon \quad \text{and} \quad Z_{n-k} = j$$

imply

$$\left(l - \frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}} + \varepsilon\right)1 \cdot \frac{(Z_n - jM^{k_0})}{(u^{(1)} \cdot j)\rho^{k_0}} > ((C+\varepsilon)1 - l) \cdot v^{(1)} - \frac{\varepsilon(1 \cdot v^{(1)})}{2}.$$

The left side above is a sum of two random walks with mean 0 and hence, by Markov's inequality,

$$P\left(\frac{l \cdot Z_n}{1 \cdot Z_n} > \frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}} + \varepsilon \mid Z_{n-k_0} = j\right) = O\left(\frac{1}{(1 \cdot j)^{r_0}}\right) \tag{15}$$

due to the following lemma.

LEMMA 3. *Let $\{X_i\}_1^\infty$ be i.i.d. with $EX_1 = 0$, $EX_1^{2r} < \infty$ for some $r \geq 1$. Then*

$$P(|\bar{X}_n| > \varepsilon) = O\left(\frac{1}{n^r}\right).$$

PROOF.

$$P(|\bar{X}_n| > \varepsilon) = P(\sqrt{n}|\bar{X}_n| > \varepsilon\sqrt{n}) \leq \frac{E(\sqrt{n}|\bar{X}_n|)^{2r}}{\varepsilon^r n^r}.$$

By a result of Brown (see [5]), under the given hypothesis $\sup_n E(\sqrt{n}|\bar{X}_n|)^{2r} < \infty$ and the lemma follows.

Thus

$$E\left(P\left(\left|\frac{l \cdot Z_n}{1 \cdot Z_n} - \frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}}\right| > \varepsilon | Z_n\right)\right) \leq C\left(\frac{1}{1 \cdot Z_{n-k_0}}\right)^r. \tag{16}$$

For any positive random variable X and $0 < r < \infty$,

$$\Gamma(r)E(X^{-r}) = \int_0^\infty E(e^{-tX})t^{r-1}\,dt, \tag{17}$$

where

$$\Gamma(r) = \int_0^\infty e^{-x}x^{r-1}\,dx.$$

Applying (17) to $E(1 \cdot Z_n)^{-r}$, we have

$$\Gamma(r)E_i(1 \cdot Z_n)^{-r} = \int_0^\infty f_i^{(n)}(e^{-t}, e^{-t})t^{r-1}\,dt.$$

By Theorem 1, for each $0 < t < \infty$,

$$h_{n,i}(t) \equiv \frac{f_i^{(n)}(e^{-t}, e^{-t})}{\gamma^n}$$

converges to $Q_i(e^{-t}, e^{-t})$. Also from (7) and the boundedness of $\{\gamma^{-j}B^j\}$ (due to A4) there exists $0 < C < \infty$ such that

$$h_{n,i}(t) \leq CQ_i(e^{-t}, e^{-t}) \quad \text{for } 0 < t < \infty.$$

If we now show that for each i,

$$\int_0^\infty Q_i(e^{-t}, e^{-t})t^{r-1}dt < \infty, \tag{18}$$

then by LDCT it would follow that

$$\lim_n \int_0^\infty h_{n,i}(t)t^{r-1}\,dt = \int_0^\infty Q_i(e^t, e^{-t})t^{r-1}\,dt < \infty.$$

This in turn would imply by a generalization of LDCT (see [1]) that for each i,

$$\frac{1}{\gamma^n}P_i\left(\left|\frac{l \cdot Z_n}{1 \cdot Z_n} - \frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}}\right| > \varepsilon\right) = \sum_j P_i(|(\cdot)| > \varepsilon | Z_{n-k_0} = j)\frac{P_i(Z_{n-k_0} = j)}{\gamma^n}$$

converges to

$$\sum_j \frac{\phi(j, k_0, \varepsilon)}{\gamma k_0} q_{i,j} < \infty, \tag{19}$$

where

$$\phi(j, k_0, \varepsilon) = P\left(\left|\frac{l \cdot Z_{k_o}}{1 \cdot Z_{k_0}} - \frac{l \cdot v^{(1)}}{1 \cdot v^{(1)}}\right| > \varepsilon | Z_0 = j\right).$$

For l not a multiple of 1, the infinite series (19) is nonzero since $\phi(j, k_0, \varepsilon) > 0$ for each $j \neq 0$. We now establish (18). Since $||Q(s)|| = O(||s||)$ in $\{s\colon ||s|| \leq \lambda < 1\}$, $\int_1^\infty Q_i(e^{-t}, e^{-t})t^{r-1}\, dt < \infty$ and so it suffices to show that $\sum_1^\infty I_n < \infty$, where

$$I_n = \int_{\rho^{-n}}^{\rho^{-n+1}} Q_i(e^{-t}, e^{-t})t^{r-1}\, dt.$$

Setting $t = x\rho^{-n}$ and using (2),

$$I_n = (\gamma\rho^r)^{-n} \int_1^\rho Q_i(f_n^{(n)}(e^{-\rho^{-n}}1))x^{r-1}\, dx.$$

Note that $f_j^{(n)}(\exp(-x\rho^n)1) = E_j(\exp(-x(1 \cdot Z_n)\rho^{-n}))$ and it converges uniformly to $E_j(\exp(x(1 \cdot v^{(1)})W))$, where W is the limit of the martingale $W_n \equiv u^{(1)} \cdot Z_n\rho^{-n}$. Since $r \geq 1$, $E_i(1 \cdot Z_1)^{2r} < \infty$ for all i, the $X \log X$ condition is satisfied and so W is nontrivial and hence

$$\sup_{\substack{n \geq 1 \\ 1 \leq x \leq \rho \\ 1 \leq j \leq 2}} Q_i(f_j^{(n)}(\exp(-x\rho^{-x})1)) < \infty.$$

Now since $\gamma\rho^r > 1$, we have that $\sum_{n \geq 1} I_n < \infty$. We are grateful to the referee for this elegant modification of our earlier argument. □

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