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THE DISTRIBUTION OF THE QUANTILE OF A BROWNIAN MOTION WITH DRIFT AND THE PRICING OF RELATED PATH-DEPENDENT OPTIONS

BY ANGELOS DASSIOS

London School of Economics

The study of the quantile of a Brownian motion with a drift is undertaken. An explicit formula for its density, as well as a representation of its distribution as the sum of the maximum and the minimum of two rescaled independent Brownian motions with drift, is given. The result is used in the pricing of a financial path-dependent option due to Miura.

1. Introduction and statement of results. Let $(B_t, t \geq 0)$ be a one-dimensional Brownian motion starting from 0. Let $\sigma \in \mathbb{R}^+$, $\mu \in \mathbb{R}$ and define $X_t = \sigma B_t + \mu t$ (a Brownian motion with drift). The study of the distribution of

$$\int_0^t \exp(X_s)\, ds$$

for some fixed time t is closely related to the so-called (financial) Asian options. Recent results on this topic can be found in Geman and Yor (1993) and Yor (1992).

Asian options can be considered as a special case of a more general class of "look-back" (path-dependent) options and other contingent claims. If the stochastic process $Y_t = Y_0 \exp(X_t)$ represents the price of a stock at time t, we are seeking the "no-arbitrage" price [see Harrison and Pliska (1981)] of the contract $h(V_T)$, where h is a known real function, T is a fixed time and V_t is an $\mathscr{F}_t$-measurable process, where $\mathscr{F}_t = \sigma\{B_s: 0 \leq s \leq t\}$. In particular, one could consider $h(v) = [v - c]^+$ and take V_t to be the median or more generally an α-quantile $(0 < \alpha < 1)$ of the process Y_t. This option, called "α-percentile option" was first introduced by Miura (1992). The study of the pricing of this option has also been undertaken by Akahori (1994), using similar mathematical tools, where a generalized arc-sine law is obtained. However, the result in Theorem 2 is new and is the main result of this paper.

The distribution of the α-quantile of Y_t can of course immediately be obtained from the distribution of the α-quantile of X_t, the Brownian motion

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with drift. This will be the object of study of this paper. We define the α-quantile of X_s $(0 \le s \le t)$ as

$$M(\alpha, t) = \inf\left\{x: \int_0^t 1_{(X_s \le x)}\, ds > \alpha t\right\}. \tag{1.1}$$

The aim of this paper is to obtain a representation of the distribution of $M(\alpha,t)$ in terms of the distributions of the maximum $(\sup_{0 \le s \le t} X_s)$ and the minimum $(\inf_{0 \le s \le t} X_s)$ of X_t and thus an explicit formula for its density. At this stage it should be remarked that $\lim_{\alpha \to 0} M(\alpha, t) = \inf_{0 \le s \le t} X_s$ and $\lim_{\alpha \to 1} M(\alpha, t) = \sup_{0 \le s \le t} X_s$.

The result we will attempt to prove is the following theorem.

THEOREM 1. *Let* $0 < \alpha < 1$. *Then* $M(\alpha, t)$ *is a continuous random variable. Furthermore, define* $g(x; \alpha, t)$ *by*

$$\Pr(M(\alpha, t) \in dx) = g(x; \alpha, t)\, dx. \tag{1.2}$$

Then

$$g(x; \alpha, t) = \int_{-\infty}^{\infty} g_1(x - y; \alpha t) g_2(y; (1 - \alpha)t)\, dy, \tag{1.3}$$

where

$$g_1(x;t) = \begin{cases} \dfrac{1}{\sigma}\left(\dfrac{2}{\pi t}\right)^{1/2} \exp\left\{-\dfrac{(x-\mu t)^2}{2\sigma^2 t}\right\} \\ \quad - \dfrac{2\mu}{\sigma^2} \exp\left(\dfrac{2\mu x}{\sigma^2}\right)\left[1 - \Phi\left(\dfrac{x + \mu t}{\sigma\sqrt{t}}\right)\right], & x > 0, \\ 0, & x \le 0, \end{cases} \tag{1.4}$$

$$g_2(x;t) = \begin{cases} 0, & x \ge 0, \\ \dfrac{1}{\sigma}\left(\dfrac{2}{\pi t}\right)^{1/2} \exp\left\{-\dfrac{(x-\mu t)^2}{2\sigma^2 t}\right\} \\ \quad + \dfrac{2\mu}{\sigma^2} \exp\left(\dfrac{2\mu x}{\sigma^2}\right)\Phi\left(\dfrac{x + \mu t}{\sigma\sqrt{t}}\right), & x < 0, \end{cases} \tag{1.5}$$

and $\Phi(x) = \int_{-\infty}^{x}(2\pi)^{-1/2}\exp(-y^2/2)\, dy$ *is the cumulative distribution function of a standard normal random variable.*

This theorem leads to the following representation:

THEOREM 2. *Let* $0 < \alpha < 1$ *and suppose* Y_1 *and* Y_2 *are two* independent *random variables such that*

$$Y_1 = \sup_{0 \le s \le \alpha t} X_s \quad \textit{in law } [\textit{the maximum of } X_s \textit{ up to a time } \alpha t], \tag{1.6}$$

$$Y_2 = \inf_{0 \le s \le (1-\alpha)t} X_s \quad \textit{in law } [\textit{the minimum of } X_s \textit{ up to time } (1 - \alpha)t]. \tag{1.7}$$

Then

$$M(\alpha, t) = Y_1 + Y_2 \quad \textit{in law}. \tag{1.8}$$

We will organize the paper as follows. In Section 2, we state and prove two results about extremes of X_t. They are not mathematically new, but they are nowhere to be found in the literature in explicit form. In Section 3, we prove the main results of the paper. Finally, in Section 4, we make some comments about the results as well as demonstrate how they can be used in the pricing of an "α-quantile" option.

2. Distributions of extremes.

PROPOSITION 3. *Define $g_1(x;t)$ by*

$$\Pr\Big(\sup_{0 \le s \le t} X_s \in dx\Big) = g_1(x,t)\,dx. \tag{2.1}$$

Then, for all $\gamma > 0$, $x > 0$,

$$\begin{aligned} \hat{g}_1(x;\gamma) &= \int_0^\infty e^{-\gamma t} g_1(x;t)\,dt \\ &= \frac{\sqrt{\mu^2 + 2\sigma^2\gamma} - \mu}{\sigma^2\gamma} \exp\left\{-\frac{\sqrt{\mu^2 + 2\sigma^2\gamma} - \mu}{\sigma^2} x\right\} \end{aligned} \tag{2.2}$$

and

$$g_1(x;t) = \begin{cases} \dfrac{1}{\sigma}\left(\dfrac{2}{\pi t}\right)^{1/2} \exp\left\{-\dfrac{(x-\mu t)^2}{2\sigma^2 t}\right\} \\ \quad -\dfrac{2\mu}{\sigma^2}\exp\left(\dfrac{2\mu x}{\sigma^2}\right)\left[1 - \Phi\left(\dfrac{x+\mu t}{\sigma\sqrt{t}}\right)\right], & x > 0, \\ 0, & x \le 0. \end{cases} \tag{2.3}$$

PROOF. Let $x > 0$ and define $T_x = \inf_{t>0}\{t\colon X_t = x\}$. It is a well-known result that the Laplace transform of the density of T_x is given by

$$\exp\left\{-\frac{\sqrt{\mu^2 + 2\sigma^2\gamma} - \mu}{\sigma^2} x\right\}, \qquad \gamma > 0$$

[see, e.g., Karlin and Taylor (1975), page 362]. This density is defective for $\mu < 0$, since in this case, $\Pr(T_x = \alpha) = \exp\{-2|\mu|x/\sigma^2\}$. We therefore have that

$$\int_0^\infty e^{-\gamma t}\Pr(T_x < t)\,dt = \frac{1}{\gamma}\exp\left\{-\frac{\sqrt{\mu^2 + 2\sigma^2\gamma} - \mu}{\sigma^2} x\right\}, \qquad \gamma > 0. \tag{2.4}$$

However, the events $\{T_x < t\}$ and $\{\sup_{0 \le s \le t} x_s > x\}$ are identical, so

$$\int_0^\infty e^{-\gamma t} \Pr\Big(\sup_{0 \le s \le t} X_s > x\Big)\, dt = \frac{1}{\gamma} \exp\left\{ - \frac{\sqrt{\mu^2 + 2\sigma^2\gamma} - \mu}{\sigma^2} x \right\}, \quad \gamma > 0. \tag{2.5}$$

By differentiating both sides of (2.5) with respect to x, we obtain (2.2). The density of T_x is given by

$$\frac{x}{\sigma t^{3/2} (2\pi)^{1/2}} \exp\left\{ - \frac{(x - \mu t)^2}{2\sigma^2 t} \right\}$$

[see Karlin and Taylor (1975), page 363]. This implies that

$$\begin{aligned} \Pr\Big(\sup_{0 \le s \le t} X_s > x\Big) &= \Pr(T_x < t) \\ &= \int_0^t \frac{x}{\sigma s^{3/2} (2\pi)^{1/2}} \exp\left\{ - \frac{(x - \mu s)^2}{2\sigma^2 s} \right\} ds \qquad (2.6) \\ &= \exp\left(\frac{2\mu x}{\sigma^2} \right) \left[1 - \Phi\left(\frac{x + \mu t}{\sigma\sqrt{t}} \right) \right] + 1 - \Phi\left(\frac{x - \mu t}{\sigma\sqrt{t}} \right). \end{aligned}$$

This can be verified by direct differentiation of the r.h.s. of (2.6) with respect to t.

So, differentiating (2.6) with respect to x, we obtain (2.3). Alternatively, we could obtain (2.3) by inverting the Laplace transform (2.2). □

PROPOSITION 4. *Define $g_2(x;t)$ by*

$$\Pr\Big(\inf_{0 \le s \le t} X_s \in dx\Big) = g_2(x, t)\, dx. \tag{2.7}$$

Then, for all $\gamma > 0$, $x > 0$,

$$\begin{aligned} \hat{g}_2(x;\gamma) &= \int_0^\infty e^{-\gamma t} g_2(x;t)\, dt \\ &= \frac{\sqrt{\mu^2 + 2\sigma^2\gamma} + \mu}{\sigma^2\gamma} \exp\left\{ \frac{\sqrt{\mu^2 + 2\sigma^2\gamma} + \mu}{\sigma^2} x \right\} \end{aligned} \tag{2.8}$$

and

$$g_2(x;t) = \begin{cases} 0, & x \ge 0, \\ \dfrac{1}{\sigma} \left(\dfrac{2}{\pi t} \right)^{1/2} \exp\left\{ - \dfrac{(x - \mu t)^2}{2\sigma^2 t} \right\} \\ \quad + \dfrac{2\mu}{\sigma^2} \exp\left(\dfrac{2\mu x}{\sigma^2} \right) \Phi\left(\dfrac{x + \mu t}{\sigma\sqrt{t}} \right), & x < 0. \end{cases} \tag{2.9}$$

PROOF. Define $X_t^* = -X_t$. Then note that $\inf_{0 \le s \le t} X_s = -\sup_{0 \le s \le t} X_s^*$ and X_t^* is a Brownian motion with drift $-\mu$. Replacing X and μ with $-X$ and $-\mu$ in (2.2) and (2.3), and noting that $1 - \Phi(-y) = \Phi(y)$ yields (2.8) and (2.9). □

3. Proofs of Theorems 1 and 2. Consider the occupation time $L(a,t) = \int_0^t 1_{(X_s \le a)}\, ds$. Define $\beta > 0$, $\gamma > -\beta$ and suppose $f(x)$ is the bounded continuously differentiable solution of the equation

$$1 - (\beta + \gamma 1_{(x \le a)}) f(x) + \mu f'(x) + (\sigma^2/2) f''(x) = 0. \tag{3.1}$$

Then the Feynman–Kac formula implies that [see Kac (1951), Itô and McKean (1965) and Karlin and Taylor (1981), and for a similar example for the driftless case, Karatzas and Shreve (1988), Proposition 4.11]

$$f(x) = E\left(\int_0^\infty \exp(-\beta t) \exp(-\gamma L(a,t))\, dt \mid X_0 = x \right) \tag{3.2}$$

and, since in our case $X_0 = 0$,

$$f(0) = E\left(\int_0^\infty \exp(-\beta t) \exp(-\gamma L(a,t))\, dt \right). \tag{3.3}$$

Solving (3.1) we obtain

$$f(x) = \begin{cases} \dfrac{1}{\beta+\gamma} + \dfrac{\gamma}{\beta(\beta+\gamma)} \dfrac{\sqrt{\mu^2 + 2\sigma^2\beta} + \mu}{\sqrt{\mu^2 + 2\sigma^2\beta} + \sqrt{\mu^2 + 2\sigma^2(\beta+\gamma)}} \\ \quad \times \exp\left\{ \dfrac{\sqrt{\mu^2 + 2\sigma^2(\beta+\gamma)} - \mu}{\sigma^2} (x - a) \right\}, & x \le a, \\ \dfrac{1}{\beta} - \dfrac{\gamma}{\beta(\beta+\gamma)} \dfrac{\sqrt{\mu^2 + 2\sigma^2\beta} - \mu}{\sqrt{\mu^2 + 2\sigma^2\beta} + \sqrt{\mu^2 + 2\sigma^2(\beta+\gamma)}} \\ \quad \times \exp\left\{ -\dfrac{\sqrt{\mu^2 + 2\sigma^2\beta} + \mu}{\sigma^2} (x - a) \right\}, & x \ge a. \end{cases} \tag{3.4}$$

Setting $x = 0$ in (3.4) we obtain the double Laplace transform

$$\begin{aligned} f(0) &= E\left(\int_0^\infty \exp(-\beta t) \exp(-\gamma L(a,t))\, dt \right) \\ &= \begin{cases} \dfrac{1}{\beta+\gamma} + \dfrac{\gamma}{\beta(\beta+\gamma)} \dfrac{\sqrt{\mu^2 - 2\sigma^2\beta} + \mu}{\sqrt{\mu^2 + 2\sigma^2\beta} + \sqrt{\mu^2 + 2\sigma^2(\beta+\gamma)}} \\ \quad \times \exp\left\{ -\dfrac{\sqrt{\mu^2 + 2\sigma^2(\beta+\gamma)} - \mu}{\sigma^2} a \right\}, & a \ge 0, \\ \dfrac{1}{\beta} - \dfrac{\gamma}{\beta(\beta+\gamma)} \dfrac{\sqrt{\mu^2 + 2\sigma^2\beta} - \mu}{\sqrt{\mu^2 + 2\sigma^2\beta} + \sqrt{\mu^2 + 2\sigma^2(\beta+\gamma)}} \\ \quad \times \exp\left\{ \dfrac{\sqrt{\mu^2 - 2\sigma^2\beta} + \mu}{\sigma^2} a \right\}, & a \le 0. \end{cases} \end{aligned} \tag{3.5}$$

Now, note that

$$\gamma \int_0^\infty \int_0^\infty e^{-\beta t} e^{-\gamma v} \Pr(L(a,t) < v)\, dv\, dt = E\left(\int_0^\infty e^{-\beta t} e^{-\gamma L(a,t)}\, dt \right). \tag{3.6}$$

Since $\Pr(L(a,t) < v) = 1$ for $v > t$, we rewrite the left-hand side of (3.6) as

$$\gamma\int_0^\infty e^{-\beta t}\int_0^t e^{-\gamma v}\Pr(L(a,t) < v)\,dv\,dt + \gamma\int_0^\infty e^{-\beta t}\int_t^\infty e^{-\gamma v}\,dv\,dt$$
$$= \gamma\int_0^\infty e^{-\beta t}\int_0^t e^{-\gamma v}\Pr(L(a,t) < v)\,dv\,dt + \frac{1}{\beta+\gamma}.$$

This implies that

$$(3.7)\qquad \begin{aligned}&\gamma\int_0^\infty e^{-\beta t}\int_0^t e^{-\gamma v}\Pr(L(a,t) < v)\,dv\,dt\\ &\quad = E\left(\int_0^\infty e^{-\beta t}e^{-\gamma L(a,t)}\,dt\right) - \frac{1}{\beta+\gamma}.\end{aligned}$$

Observing that the events $\{L(a,t) < v\}$ and $\{M(v/t,t) > a\}$ are identical and combining (3.5), (3.6) and (3.7), we obtain

$$(3.8)\qquad \begin{aligned}&\int_0^\infty e^{-\beta t}\int_0^t e^{-\gamma v}\Pr\left(M\left(\frac{v}{t},t\right) > a\right)dv\,dt\\ &= \begin{cases}\dfrac{1}{\beta(\beta+\gamma)}\dfrac{\sqrt{\mu^2+2\sigma^2\beta}+\mu}{\sqrt{\mu^2+2\sigma^2\beta}+\sqrt{\mu^2+2\sigma^2(\beta+\gamma)}}\\ \qquad\times\exp\left\{-\dfrac{\sqrt{\mu^2+2\sigma^2(\beta+\gamma)}-\mu}{\sigma^2}a\right\}, & a \ge 0,\\ \dfrac{1}{\beta(\beta+\gamma)}\left[1-\dfrac{\sqrt{\mu^2+2\sigma^2\beta}-\mu}{\sqrt{\mu^2+2\sigma^2\beta}+\sqrt{\mu^2+2\sigma^2(\beta+\gamma)}}\right.\\ \qquad\left.\times\exp\left\{\dfrac{\sqrt{\mu^2+2\sigma^2\beta}+\mu}{\sigma^2}a\right\}\right], & a \le 0.\end{cases}\end{aligned}$$

Setting $\lambda = \beta + \gamma$ ($\lambda > 0$, since $\gamma > -\beta$) and $t = v + s$ in the double integral, we get

$$(3.9)\qquad \begin{aligned}&\int_0^\infty\int_0^\infty e^{-\beta s}e^{-\lambda v}\Pr\left(M\left(\frac{v}{v+s},v+s\right) > a\right)dv\,ds\\ &= \begin{cases}\dfrac{1}{\beta\lambda}\dfrac{\sqrt{\mu^2+2\sigma^2\beta}+\mu}{\sqrt{\mu^2+2\sigma^2\beta}+\sqrt{\mu^2+2\sigma^2\lambda}}\\ \qquad\times\exp\left\{-\dfrac{\sqrt{\mu^2+2\sigma^2\lambda}-\mu}{\sigma^2}a\right\}, & a \ge 0,\\ \dfrac{1}{\beta\lambda}\left[1-\dfrac{\sqrt{\mu^2+2\sigma^2\beta}-\mu}{\sqrt{\mu^2+2\sigma^2\beta}+\sqrt{\mu+2\sigma^2\lambda}}\right.\\ \qquad\left.\times\exp\left\{\dfrac{\sqrt{\mu^2+2\sigma^2\beta}+\mu}{\sigma^2}a\right\}\right], & a \le 0.\end{cases}\end{aligned}$$

We define the measure $G(\cdot\,; v/(v+s), v+s)$ by $\Pr(M(v/(v+s) \le a) = G(a; v/(v+s), v+s)$ as in (1.2). Then (3.9) can be rewritten as

$$(3.10)\qquad \int_0^\infty \int_0^\infty e^{-\beta s} e^{-\lambda v} \int_{(a,\infty)} dG\left(a; \frac{v}{v+s}, v+s\right) dv\, ds = \int_a^\infty \hat{g}(x; \lambda, \beta)\, dx,$$

where $\hat{g}(\cdot; \lambda, \beta)$ is defined as the derivative of the right-hand side of (3.9) with respect to a; that is,

$$(3.11)\qquad \hat{g}(x;\lambda,\beta) = \begin{cases} \dfrac{1}{\sigma^2\beta\lambda} \dfrac{\left(\sqrt{\mu^2+2\sigma^2\beta}+\mu\right)\left(\sqrt{\mu^2+2\sigma^2\lambda}-\mu\right)}{\sqrt{\mu^2+2\sigma^2\beta}+\sqrt{\mu^2+2\sigma^2\lambda}} \\ \qquad \times \exp\left\{-\dfrac{\sqrt{\mu^2+2\sigma^2\lambda}-\mu}{\sigma^2}x\right\}, & x \ge 0, \\ \dfrac{1}{\sigma^2\beta\lambda} \dfrac{\left(\sqrt{\mu^2+2\sigma^2\beta}+\mu\right)\left(\sqrt{\mu^2+2\sigma^2\lambda}-\mu\right)}{\sqrt{\mu^2+2\sigma^2\beta}+\sqrt{\mu^2+2\sigma^2\lambda}} \\ \qquad \times \exp\left\{\dfrac{\sqrt{\mu^2+2\sigma^2\beta}+\mu}{\sigma^2}x\right\}, & x \le 0. \end{cases}$$

We than observe that

$$(3.12)\qquad \hat{g}(x;\lambda,\beta) = \int_{-\infty}^{\infty} \hat{g}_1(y;\lambda)\hat{g}_2(x-y;\beta)\, dy,$$

where

$$(3.13)\qquad \hat{g}_1(x;\gamma) = \frac{\sqrt{\mu^2+2\sigma^2\gamma}-\mu}{\sigma^2\gamma} \exp\left\{-\frac{\sqrt{\mu^2+2\sigma^2\gamma}-\mu}{\sigma^2}x\right\} 1_{(x\ge 0)},$$

$$(3.14)\qquad \hat{g}_2(x;\gamma) = \frac{\sqrt{\mu^2+2\sigma^2\gamma}+\mu}{\sigma^2\gamma} \exp\left\{\frac{\sqrt{\mu^2+2\sigma^2\gamma}+\mu}{\sigma^2}x\right\} 1_{(x\le 0)},$$

which are the same as (2.2) and (2.8), respectively. By the uniqueness of Laplace transforms, we deduce that

$$(3.15)\qquad \int_{(a,\infty)} dG\left(x; \frac{v}{v+s}, v+s\right) = \int_a^\infty \int_{-\infty}^\infty g_1(y;v) g_2(x-y;s)\, dy\, dx,$$

where $g_1(\cdot\,;\cdot)$ and $g_2(\cdot\,;\cdot)$ are as defined in Propositions 3 and 4, respectively. We then conclude that the measure $G(\cdot\,; v/(v+s), v+s)$ has a density $g(\cdot\,; v/(v+s)$ and

$$(3.16)\qquad g\left(x; \frac{v}{v+s}, v+s\right) = \int_{-\infty}^\infty g_1(y;v) g_2(x-y;s)\, dy\, dx \quad \text{a.e.}$$

Setting $v = \alpha t$ and $s = (1-\alpha)$ completes the proofs of both theorems. □

4. Comments: The price of a quantile option.

1. For $\mu = 0$, calculating the convolution (1.6), we obtain

$$g(x;\alpha,t) = \begin{cases} \dfrac{2}{\sigma}\left(\dfrac{2}{\pi t}\right)^{1/2} \exp\left\{-\dfrac{x^2}{2\sigma^2 t}\right\} \\ \quad \times\left\{1 - \Phi\left[\left(\dfrac{1-\alpha}{\alpha t}\right)^{1/2} \dfrac{x}{\sigma}\right]\right\}, & x \geq 0, \\ \dfrac{2}{\sigma}\left(\dfrac{2}{\pi t}\right)^{1/2} \exp\left\{-\dfrac{x^2}{2\sigma^2 t}\right\} \\ \quad \times\Phi\left[\left(\dfrac{\alpha}{(1-\alpha)t}\right)^{1/2} \dfrac{x}{\sigma}\right], & x \leq 0. \end{cases} \tag{4.1}$$

   This result has also been derived by Yor (1995), where other representations special to the driftless case are obtained.
2. It is an interesting problem to investigate for which class of processes the property in Theorem 2 is true. It is of course trivially true for all nondecreasing or nonincreasing processes.
3. Using Theorem 1, we can derive the price of an "α-quantile" call option. Suppose, as in Black and Scholes (1973), that the market consists of the stock $Y_t = Y_0 \exp(X_t)$, where the initial price Y_0 is known and a deterministic bond $R_t = R_0 e^{rt}$, where r is a constant. Then, applying results from Harrison and Pliska (1981), we see that the "no-arbitrage" price of $[Y_0 e^{M(\alpha,T)} - c]^+$ at any time t $(0 \leq t \leq T)$ is given by

$$e^{-r(T-t)} E^*\left[\left(Y_0 e^{M(\alpha,T)} - c\right)^+ | \mathscr{F}_t\right], \tag{4.2}$$

   where E^* is calculated under the equivalent martingale measure, under which the discounted price of the stock $Y_t e^{-rt}$ is a martingale. Note, that $Y_0 e^{M(\alpha,T)}$ is the "α-quantile" of Y_s, $0 \leq s \leq t$. Also, (4.2) can be expressed as

$$e^{-r(T-t)} \int_c^\infty \Pr{}^*\left[\left(Y_0 e^{M(\alpha,T)}\right) > z | \mathscr{F}_t\right] dz, \tag{4.3}$$

   where $\Pr^*$ is again calculated under the equivalent martingale measure. Now, denoting Y_t by y and $\int_0^t 1_{\{Y_s \leq z\}}\, ds$ by $l(z,t)$, we have

$$\begin{aligned} &\Pr{}^*\left[\left(Y_0 e^{M(\alpha,T)}\right) > z | \mathscr{F}_t\right] \\ &\quad = \Pr{}^*\left[M(\alpha,T) > \ln(z/Y_0) | \mathscr{F}_t\right] \\ &\quad = \Pr{}^*\left[\int_0^t 1_{\{X_s \leq \ln(z/Y_0)\}}\, ds + \int_t^T 1_{\{X_s \leq \ln(z/Y_0)\}}\, ds < \alpha T | \mathscr{F}_t\right] \\ &\quad = \Pr{}^*\left[\int_t^T 1_{\{X_s \leq \ln(z/Y_0)\}}\, ds < \alpha T - l(z,t) | \mathscr{F}_t\right], \end{aligned} \tag{4.4}$$

where $Y_t = y$, $X_t = ln(y/Y_0)$ and by the strong Markov property of X_t,

$$
\begin{aligned}
\Pr^*\left[\int_t^T 1_{\{X_s \le \ln(z/Y_0)\}}\,ds < \alpha T - l(z,t) | \mathscr{F}_t\right]
&= \Pr^*\left[\int_0^{T-t} 1_{\{X_s \le \ln(z/Y_0) - \ln(y/Y_0)\}}\,ds < \alpha T - l(z,t)\right] \\
&= \Pr^*\left[\int_0^{T-t} 1_{\{X_s \le \ln(z/y)\}}\,ds < \alpha T - l(z,t)\right] \\
&= \Pr^*\left[M\left(\frac{\alpha T - l(z,t)}{T-t}, T-t\right) > \ln\left(\frac{z}{y}\right)\right] 1_{\{t-(1-\alpha)T \le l(z,t) < \alpha T\}} \\
&\quad + 1_{\{l(z,t) < t-(1-\alpha)T\}}.
\end{aligned}
\tag{4.5}
$$

Under the equivalent martingale measure, X_t is a Brownian motion with drift $r - \sigma^2/2$ and variance coefficient σ^2. So

$$
\Pr^*[M(\alpha,t) > a] = \int_a^\infty g^*(x;\alpha,t)\,dx,
\tag{4.6}
$$

where $g^*(\cdot\,;\cdot,\cdot)$ is defined as $g(\cdot\,;\cdot,\cdot)$ with $\mu = r - \sigma^2/2$, and the price of the option is given by

$$
\begin{aligned}
&e^{-r(T-t)}\int_c^\infty \int_{\ln(z/y)}^\infty g^*\left(x;\frac{\alpha T - l(z,t)}{T-t}, T-t\right) \\
&\quad \times dx\, 1_{\{t-(1-\alpha)T \le l(z,t) < \alpha T\}}\,dz \\
&\quad + e^{-r(T-t)}\int_c^\infty 1\{l(z,t) < t-(1-\alpha)T\}\,dz.
\end{aligned}
\tag{4.7}
$$

For $t < \min\{\alpha T, (1-\alpha)T\}$ the formula simplifies somewhat to

$$
e^{-r(T-t)}\int_c^\infty \int_{\ln(z/y)}^\infty g^*\left(x;\frac{\alpha T - l(z,t)}{T-t}, T-t\right) dx\,dz.
\tag{4.8}
$$

It still presents computational difficulties since all occupation times $l(z,t)$ have to be recorded. For $t = 0$, however, it simplifies to

$$
\begin{aligned}
&e^{-rT}\int_c^\infty \int_{\ln(z/y)}^\infty g^*(x;\alpha,T)\,dx\,dz \\
&\quad = e^{-rT}\int_{\ln(c/y)}^\infty (ye^x - c)\,g^*(x;\alpha,T)\,dx.
\end{aligned}
\tag{4.9}
$$

4. The calculations for (4.2) are equivalent to the proof of Theorem 2.1 of Akahori (1994).

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DEPARTMENT OF STATISTICS
LONDON SCHOOL OF ECONOMICS
HOUGHTON STREET
LONDON WC2A 2AE
UNITED KINGDOM