

SPECTRAL THEORY AND LIMIT THEOREMS FOR GEOMETRICALLY ERGODIC MARKOV PROCESSES

BY I. KONTOYIANNIS¹ AND S. P. MEYN²

Brown University and University of Illinois

Consider the partial sums $\{S_t\}$ of a real-valued functional $F(\Phi(t))$ of a Markov chain $\{\Phi(t)\}$ with values in a general state space. Assuming only that the Markov chain is geometrically ergodic and that the functional F is bounded, the following conclusions are obtained:

Spectral theory. Well-behaved solutions \check{f} can be constructed for the “multiplicative Poisson equation” $(e^{\alpha F} P)\check{f} = \lambda\check{f}$, where P is the transition kernel of the Markov chain and $\alpha \in \mathbb{C}$ is a constant. The function \check{f} is an eigenfunction, with corresponding eigenvalue λ , for the kernel $(e^{\alpha F} P) = e^{\alpha F(x)} P(x, dy)$.

A “multiplicative” mean ergodic theorem. For all complex α in a neighborhood of the origin, the normalized mean of $\exp(\alpha S_t)$ (and not the logarithm of the mean) converges to \check{f} exponentially fast, where \check{f} is a solution of the multiplicative Poisson equation.

Edgeworth expansions. Rates are obtained for the convergence of the distribution function of the normalized partial sums S_t to the standard Gaussian distribution. The first term in this expansion is of order $(1/\sqrt{t})$ and it depends on the initial condition of the Markov chain through the solution \bar{F} of the associated Poisson equation (and not the solution \check{f} of the multiplicative Poisson equation).

Large deviations. The partial sums are shown to satisfy a large deviations principle in a neighborhood of the mean. This result, proved under geometric ergodicity alone, cannot in general be extended to the whole real line.

Exact large deviations asymptotics. Rates of convergence are obtained for the large deviations estimates above. The polynomial preexponent is of order $(1/\sqrt{t})$ and its coefficient depends on the initial condition of the Markov chain through the solution \check{f} of the multiplicative Poisson equation.

Extensions of these results to continuous-time Markov processes are also given.

1. Introduction. Consider a Markov process $\Phi = \{\Phi(t) : t \in \mathbb{T}\}$ taking values in a general state space X and with time being either continuous, $\mathbb{T} = [0, \infty)$, or discrete, $\mathbb{T} = \{0, 1, \dots\}$. Let $F : X \rightarrow \mathbb{R}$ be a given functional on the state space of Φ .

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Our interest lies in the long-term behavior of

$$(1) \quad S_t = \int_{[0,t)} F(\Phi(s)) ds, \quad t \in \mathbb{T},$$

where, in discrete time, the integral is a sum and S_t are simply the partial sums

$$(2) \quad S_n = \sum_{i=0}^{n-1} F(\Phi(i)), \quad n \geq 1.$$

1.1. *Multiplicative ergodic theory.* For simplicity we first discuss the case of a discrete-time Markov chain Φ with a countable state space X . If Φ is positive recurrent with invariant probability measure π , then for any F with finite mean $\pi(F) = \sum_x \pi(x)F(x)$,

$$(3) \quad \frac{1}{n} E_x[S_n] \rightarrow \pi(F), \quad n \rightarrow \infty,$$

where $x = \Phi(0)$ is the initial condition, S_n are the partial sums defined above, P_x is the law of Φ conditional on $\Phi(0) = x$ and E_x is the corresponding expectation.

Often we can quantify the rate of convergence in (3) by showing that the limit

$$(4) \quad \widehat{F}(x) = \lim_{n \rightarrow \infty} E_x[S_n - n\pi(F)]$$

exists where, in fact, the function \widehat{F} solves the *Poisson equation*

$$(5) \quad P\widehat{F} = \widehat{F} - F + \pi(F).$$

Here P denotes the transition kernel of Φ , $P(x, y) := \Pr\{\Phi(1) = y \mid \Phi(0) = x\}$ and P acts on functions $f : X \rightarrow \mathbb{R}$ via $Pf(x) = \sum_y P(x, y)f(y)$. Results of this kind hold for a wide class of Markov chains on a general state space, as shown in [37] in discrete time and in [36, 38] in continuous time.

In this paper we seek multiplicative versions of the ergodic results in (3)–(5). Let $\alpha \in \mathbb{C}$ and consider the product

$$\prod_{i=0}^{n-1} \exp(\alpha F(\Phi(i))) = \exp(\alpha S_n).$$

For countable state space chains in discrete time, multiplicative results corresponding to the ergodic theorems (3)–(5) were established in [2] when α is a real number. The mean ergodic theorem (3) corresponds to the multiplicative limit

$$(6) \quad \frac{1}{n} \log E_x[\exp(\alpha S_n)] \rightarrow \Lambda(\alpha), \quad n \rightarrow \infty,$$

for some analytic function $\Lambda(\alpha) \in \mathbb{R}$, and the stronger limit theorem (4) has the multiplicative counterpart

$$(7) \quad \check{f}_\alpha(x) = \lim_{n \rightarrow \infty} E_x[\exp(\alpha S_n - n\Lambda(\alpha))],$$

where \check{f}_α solves the natural analog of (5), the *multiplicative Poisson equation*

$$(8) \quad P\check{f}_\alpha = \exp(-\alpha F + \Lambda(\alpha))\check{f}_\alpha.$$

Our first aim is to provide natural conditions under which the multiplicative ergodic results (6)–(8) hold. As we indicate in several instances, our conditions (and the results obtained under them) are often optimal or near optimal (see Proposition 6.2 and the examples in Section 7). Equipped with these results, we go on to prove precise expansions for some classical probabilistic limit theorems satisfied by the partial sums S_n . Specifically, the multiplicative mean ergodic theorem (7) leads to Edgeworth expansions for the central limit theorem and to exact large deviations asymptotics.

There are numerous approaches to multiplicative ergodic theory and its related spectral theory in the literature; a brief survey is given at the end of this Introduction. The conditions given in this paper considerably extend known criteria for the existence of solutions to the multiplicative Poisson equation and for the validity of the multiplicative mean ergodic theorem.

Most closely related to the approach taken here are the results of [2], developed for discrete-time Markov chains Φ on a discrete state space along the following lines. For any real α , define the new kernel \widehat{P}_α by

$$(9) \quad \widehat{P}_\alpha(x, y) = \exp(\alpha F(x))P(x, y), \quad x, y \in X,$$

where $P(x, y)$ is the transition kernel of the Markov chain Φ . In this notation, the multiplicative Poisson equation (8) can be rewritten as

$$(10) \quad \widehat{P}_\alpha\check{f}_\alpha = \lambda_\alpha\check{f}_\alpha$$

with $\lambda_\alpha = \exp(\Lambda(\alpha))$. That is, the solutions \check{f}_α of the multiplicative Poisson equation (8) are eigenfunctions for the new kernel \widehat{P}_α with associated eigenvalues λ_α . [Throughout this paper, we try to maintain the convention that lowercase letters denote quantities that are exponential versions of the corresponding uppercase letters; e.g., $\lambda = \exp(\Lambda)$.]

Under a monotonicity assumption on F , it is shown in [2] that well-behaved eigenfunctions for (10) exist for real α in a neighborhood of zero. Based on such an eigenfunction \check{f}_α with corresponding eigenvalue λ_α , the *twisted kernel* \check{P}_α is defined as

$$(11) \quad \check{P}_\alpha(x, y) = \lambda_\alpha^{-1}\check{f}_\alpha^{-1}(x)\widehat{P}_\alpha(x, y)\check{f}_\alpha(y)$$

and the convergence in (7) is deduced from the properties of \check{P}_α .

For bounded functionals F and assuming only that Φ is “geometrically ergodic,” results corresponding to (6)–(8) are obtained in Section 4 of the present paper for Markov processes Φ on a general state space, in continuous or discrete time and for complex α . For our purposes, a Markov chain Φ is *geometrically*

ergodic if it is ψ -irreducible, aperiodic and a Lyapunov function $V : X \rightarrow [1, \infty]$ exists such that the following condition holds:

(V4) For a “small” set $C \subset X$ and constants $\delta > 0$, $b < \infty$, $PV \leq (1 - \delta)V + b\mathbb{1}_C$.

Precise definitions and a more general version of condition (V4) for Markov processes in discrete or continuous time are given in Section 2.2.

Geometric ergodicity for Φ is our main assumption and it will remain in effect throughout the paper. Section 1.3 offers a discussion comparing (V4) to several of the standard assumptions in the relevant literature and in Section 7, geometric ergodicity is verified for several classes of important examples. Note also that what we call geometric ergodicity here is equivalent to the notion of geometric ergodicity used in [37], where it is stated slightly differently.

In the following section we briefly describe the probabilistic implications of the spectral theory outlined above. Along a different direction, in [26] we extended our present results to the case of products of random matrices. This extension leads to an interesting and nontrivial application of the present ideas to a stability question arising from systems theory.

1.2. *Probabilistic limit theorems.* The multiplicative mean ergodic theorems in (6) and (7) offer precise information about the asymptotic behavior, as $n \rightarrow \infty$, of

$$m_n(\alpha) := E_x[\exp(\alpha S_n)], \quad \alpha \in \mathbb{C}.$$

When $\alpha = i\omega$ is imaginary, $m_n(\alpha)$ is simply the characteristic function of the partial sums S_n , and it is well known that information about the convergence of the characteristic functions leads to Edgeworth expansions related to the central limit theorem [17, 24, 46]. Similarly, when α is real, $m_n(\alpha)$ is the moment-generating function of the partial sums S_n , and the precise convergence of the corresponding log-moment generating functions to a smooth limiting $\Lambda(\alpha)$ as in (6) leads to exact large deviations asymptotics; see [13, 7].

Suppose Φ is a geometrically ergodic Markov chain and let F be a bounded, nonlattice, real-valued functional on the state space of Φ . In Section 5, we obtain an Edgeworth expansion for the distribution function $G_n(y)$ of the normalized partial sums $[S_n - n\pi(F)]/\sigma\sqrt{n}$,

$$G_n(y) = P_x \left\{ \frac{S_n - n\pi(F)}{\sigma\sqrt{n}} \leq y \right\}, \quad y \in \mathbb{R},$$

where σ^2 is the asymptotic variance of S_n/\sqrt{n} . In Theorem 5.1 we show that, for all $x \in X$,

$$G_n(y) = \mathcal{G}(y) + \frac{\gamma(y)}{\sigma\sqrt{n}} \left[\frac{\rho_3}{6\sigma^2}(1 - y^2) - \widehat{F}(x) \right] + o(n^{-1/2}), \quad n \rightarrow \infty,$$

uniformly in $y \in \mathbb{R}$, where $\gamma(y)$ denotes the standard Normal density, $\mathcal{G}(y)$ is the corresponding distribution function, \widehat{F} is the solution to the Poisson equation (5) given in (4) and ρ_3 is a constant related to the third moment of S_n/\sqrt{n} .

A similar expansion is obtained in the case of lattice functionals F . These results generalize the Edgeworth expansions in [41, 30, 8], where they are derived under much more restrictive assumptions. In particular, in all these papers the conditions given are stronger than Doeblin recurrence, which is significantly stronger than the form of geometric ergodicity assumed in this paper; see the discussions in Sections 1.3 and 7.

In Section 6 we discuss moderate and large deviations for the partial sums S_n . Under geometric ergodicity, the multiplicative mean ergodic theorem (7) implies that a moderate deviations principle (MDP) holds for the partial sums S_n . Note that geometric ergodicity is essentially equivalent to the weakest conditions known to suffice for the MDP [10, 11] (although weaker assumptions can be used to obtain the MDP lower bound).

By standard large deviations techniques [13], the convergence of the log-moment generating functions in (6) to a smooth limiting $\Lambda(\alpha)$ can be used to prove large deviations estimates for the partial sums S_n : Suppose Φ is a Doeblin chain and let F be a bounded, real-valued functional on the state space of Φ . In Proposition 6.2 we show that under the stationary distribution π of Φ , the partial sums S_n satisfy a large deviations principle (LDP) in a neighborhood of the mean $\pi(F)$, that is, for any $c > \pi(F)$ close enough to the mean $\pi(F)$,

$$(12) \quad \frac{1}{n} \log \mathbb{P}_\pi \{S_n \geq nc\} \rightarrow -\Lambda^*(c), \quad n \rightarrow \infty,$$

where $\Lambda^*(c)$ is the Fenchel–Legendre transform of $\Lambda(\cdot)$. (A corresponding result holds for the lower tail.)

Note that this result *cannot* in general be extended to a full LDP on the whole real line. For example, Bryc and Dembo [5] have shown that the full LDP may even fail for the partial sums of a Doeblin chain with a countable state space.

Furthermore, the more precise convergence result (7) leads to exact large deviations expansions analogous to those obtained by Bahadur and Rao [1] for independent random variables: For geometrically ergodic chains and nonlattice functionals F , in Theorem 6.3 we obtain the following: For any $c > \pi(F)$ close enough to the mean $\pi(F)$ and all $x \in \mathbb{X}$,

$$(13) \quad \mathbb{P}_x \{S_n \geq nc\} \sim \frac{\check{f}_a(x)}{a\sqrt{2\pi n\sigma_a^2}} e^{-n\Lambda^*(c)}, \quad n \rightarrow \infty,$$

where $a \in \mathbb{R}$ is chosen such that $\Lambda'(a) = c$, $\check{f}_a(x)$ is the solution to the multiplicative Poisson equation (10), $\Lambda^*(\cdot)$ is as in (12) and $\sigma_a^2 = \Lambda''(a)$. A corresponding expansion is given for lattice functionals.

These results generalize those obtained by Miller [39] for finite-state chains and those in [30], proved under conditions stronger than Doeblin recurrence (in [30] a version of the domination assumption in (15) below is assumed, together with additional regularity conditions).

The problem of obtaining *exact* large deviations asymptotics [such as in (13) above] has been considered in [28, 42] using a “pinned” multiplicative mean ergodic theorem for a ψ -irreducible and aperiodic Markov chain. It is shown that for a “small” set $C \subset X$,

$$(14) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log E_x [\exp(\alpha S_n) \mathbb{I}(\Phi(n) \in C)] = \Lambda(\alpha),$$

and from this, under additional conditions [assuming a variant of the “uniform domination” condition (15) discussed in the following section], large deviations expansions are proved along the same lines as indicated above. The difference here is that, because of the additional constraint imposed by the small set C in (14), the resulting expansions are not for the probabilities $P_x\{S_n \geq nc\}$ as in (13), but for the “pinned” probabilities $P_x\{S_n \geq nc \text{ and } \Phi(n) \in C\}$.

Finally note that authors of much of the relevant literature often consider a Markov additive process model instead of simply the partial sums of a given Markov processes. For simplicity (and without loss of generality), we restrict our attention to the asymptotic behavior of the partial sums themselves.

1.3. *Related approaches.* In this paper we attempt to place within a single framework results from two previously disparate research areas: The theory of positive operators as developed in [45, 44], where $r_\alpha = (\lambda_\alpha)^{-1}$ is the *convergence parameter* for the semigroup generated by \widehat{P}_α , and from the theory of positive harmonic functions for diffusions, where $\Lambda(\alpha) = \log(\lambda_\alpha)$ is known as the *generalized principal eigenvalue* [47]. The reason that the constant λ_α is given two different names is that, so far, the discrete-time theory of ψ -irreducible Markov chains and the related continuous-time theory of positive harmonic functions have been developed independently. Looked at together, many of the results of the latter continuous-time theory can be replicated, improved or generalized by lifting results from the discrete-time setting.

These and some other relevant approaches in the existing literature are summarized below. Whereas this literature is very extensive, the following discussion is not intended to be a complete review.

A. *ψ -Irreducible operators.* The most general approach to understanding the eigenfunction equation (10) has been developed for discrete-time Markov chains, based on renewal theory and the theory of positive, ψ -irreducible operators; see Nummelin’s monograph [45]. In this framework $\Lambda(\alpha) = -\log(r_\alpha)$, where r_α is the *convergence parameter* for the semigroup generated by the kernel \widehat{P}_α defined in (9).

Although, in general, useful solutions to (8) cannot be constructed, if Φ is aperiodic and $r_\alpha > 0$, then from the definitions it can be shown directly that for any “small” set C ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log(\widehat{P}_\alpha^n(x, C)) = \Lambda(\alpha) = -\log(r_\alpha) \quad \text{a.e. } x \in X,$$

where \widehat{P}_α^n denotes the n -fold composition of the kernel \widehat{P}_α with itself. From this, the “pinned” multiplicative mean ergodic theorem (14) is easily obtained. The drawback to this approach is the restriction imposed by the small set C in (14). As we will see, this restriction is not necessary when Φ is geometrically ergodic. Nevertheless, in the case of first-order large deviations [as opposed to more precise estimates as in (13)], these methods provide what appear to be the most general large-deviations results to date [9, 12].

B. Lyapunov functions and compact sublevel sets. A well-behaved solution to the multiplicative Poisson equation (10) can be shown to exist under suitable bounds on the transition kernel P . For example, (8) will admit a *bounded* solution \check{f}_α under the “uniform domination” assumption of [51], Section 6: For some $\varepsilon > 0$ and all measurable $A \subset X$,

$$(15) \quad P(x, A) \geq \varepsilon P(y, A), \quad x, y \in X.$$

Condition (15), as well as its variants in [28, 16, 30, 13], are significantly stronger than geometric ergodicity and are rarely satisfied for noncompact state spaces. In particular, they imply that the process is Doeblin recurrent, a property that is equivalent to geometric ergodicity with a *bounded* Lyapunov function V ; see [37], Chapter 16.

Similar conditions are used in Donsker and Varadhan’s classic papers; see [52] for a general exposition. Variations on their assumptions are used throughout the large-deviations literature (including the recent work by Wu; see [56] and the references therein) and they all imply the validity of a condition stronger than geometric ergodicity, the *multiplicative regularity* condition (mV3), stated and discussed in Section 2.2. In particular, Varadhan [53] assumes directly that (mV3) holds.

C. Spectral gap. In all of the aforementioned works, only the case where $\alpha \in \mathbb{R}$ is considered. Specifically, the positivity of the semigroup generated by the kernel \widehat{P}_α in (9) is exploited in constructing solutions $(\lambda_\alpha, \check{f}_\alpha)$ to the eigenvalue problem (10). Nagaev [40] treated the special case of ergodic Markov chains that converge to the stationary distribution at a *uniform* geometric rate,

$$(16) \quad |P^t(x, A) - \pi(A)| \leq B_0 e^{-b_0 t} \quad \text{for all } x = \Phi(0), \text{ all measurable } A \subset X.$$

This condition is equivalent to Doeblin recurrence. A version of the multiplicative

mean ergodic theorem is proved, under (16), for purely imaginary $\alpha = i\omega$ in a neighborhood of zero. The gist of this approach is to formulate the problem in a vector-space setting similar to that considered here. Noting that the transition semigroup $\{P^n\}$ of the Markov chain converges in operator norm to the invariant probability measure (as $n \rightarrow \infty$, where P^n is viewed as a linear operator from $L_\infty \rightarrow L_\infty$), the continuity of the norm is exploited to obtain convergence of the semigroup $\{\widehat{P}_\alpha^n\}$.

Operator-theoretic approaches have been extensively used in the classical theory of Markov chains, and the assumption of *uniform* geometric ergodicity (16) is traditionally used to ensure a spectral gap, and hence convergence, as in [40]. Generalizations have typically involved an alternative vector-space setting, such as an L_p space for $p < \infty$; see [54, 29, 4] and also [22, 21]. In particular, under the assumption of *hypercontractivity*, Deuschel and Stroock [14] derived large deviations properties for Markov chains. Note that, as hypercontractivity implies L_2 -ergodicity at an exponential rate, it also implies that (V4) holds [37].

In a different vein, in [37, 38] the *weighted*- L_∞ space was considered,

$$L_\infty^V := \left\{ g : X \rightarrow \mathbb{C} : \sup_x \left[\frac{|g(x)|}{V(x)} \right] < \infty \right\},$$

with $V : X \rightarrow [1, \infty)$ being the Lyapunov function in condition (V4). The convergence of the semigroup $\{P^n\}$ in the induced operator norm on this space is equivalent to geometric ergodicity [37, 38] and, based on this equivalence, we show in this paper that (V4) leads to multiplicative mean ergodic theorems of the type (6)–(8) for complex α , and also to criteria for the existence of solutions to (10) under conditions far weaker than those used in, for example, [47, 52]. We also substantially strengthen the conclusions of both [47] and [42, 43] since we can apply the V -uniform ergodic theorem of [37] to obtain uniform geometric convergence in (7).

In earlier work related to the ergodic theory of Markov processes (as opposed to the multiplicative ergodicity and large deviations issues considered here), Kartashov [31, 32] considered weighted norms and a version of the V -uniform ergodic theorem for countable state space chains first appeared in [25].

D. Nonlinear semigroups. For a continuous-time Markov process Φ (typically a diffusion), Fleming [20] and Feng [18] considered a nonlinear operator \mathcal{H} defined as a modification of the generator \mathcal{A} of the process Φ :

$$\mathcal{H}(G) := \log((g^{-1})\mathcal{A}g) \quad \text{where } g = e^G.$$

For any function $F \in L_\infty$, the multiplicative Poisson equation is given in continuous time as $\mathcal{A}\check{f} = \exp(-F + \Lambda)\check{f}$, where $\Lambda = \Lambda(1)$ [recall the definition of $\Lambda(\cdot)$ in (6)]. If $g = \check{f}$ is a solution for a given F , then

$$\mathcal{H}(G) = \log[(g^{-1})(e^{-F+\Lambda}g)] = -F + \Lambda.$$

Define the functional \mathcal{G} on L_∞ as $\mathcal{G}(F) = \log(c\check{f})$, where \check{f} solves the multiplicative Poisson equation and $c = \pi(\check{f})^{-1}$ is a normalizing constant. The operator \mathcal{G} is an inverse of $-\mathcal{H}$ in the sense that $\mathcal{H} \circ \mathcal{G} = -I$ on some appropriately defined domain.

Under (V4), the results of the present paper imply that \mathcal{G} is a *bounded* nonlinear operator, whose domain contains an open ball in L_∞ centered at the origin. In particular, our results provide methods for verifying the structural assumptions of [18, 19]. A thorough investigation of this nonlinear structural theory and its intimate relationship to large deviations properties has been carried out in a subsequent work [33] for Markov processes that satisfy the stronger assumption of multiplicative regularity.

1.4. Organization. The rest of the paper is organized as follows. In Section 2 we collect the basic notation and definitions that will remain in effect throughout the paper. We present background results from the ergodic theory of Markov chains and processes, and briefly discuss several different conditions for ergodicity and the relationships between them.

In Section 3 we collect some results about the convergence parameter of a positive semigroup. Section 4 develops the spectral theory and multiplicative ergodic theory along the lines discussed above. Analogs of (6)–(8) are proved for geometrically ergodic Markov processes.

Sections 5 and 6 contain the probabilistic results outlined in Section 1.2. Finally in Section 7 we give numerous examples of Markov chains and processes that satisfy the assumption of geometric ergodicity.

2. Ergodicity. In this and the following section we review some necessary background results from certain parts of the ergodic theory of Markov chains and processes [37], and some results regarding the convergence parameter of a positive semigroup as defined in [45]. All of this concerns a ψ -irreducible and aperiodic chain or process Φ on a general state space X (see below for precise definitions). We assume that X is equipped with a σ -field \mathcal{B} and that \mathcal{B} is countably generated. The distribution of Φ is described by a transition semigroup $\{P^t : t \in \mathbb{T}\}$, where \mathbb{T} is taken to be either the nonnegative integers \mathbb{Z}_+ (in discrete time) or the nonnegative reals \mathbb{R}_+ (in continuous time), and where for each t , P^t is the transition kernel

$$P^t(x, A) := \Pr\{\Phi(t) \in A \mid \Phi(0) = x\}, \quad x \in X, A \in \mathcal{B}.$$

Recall that P^t acts on functions $f : X \rightarrow \mathbb{R}$ and signed measures ν on \mathcal{B} via

$$(17) \quad P^t f(\cdot) = \int_X P^t(\cdot, dy) f(y) \quad \text{and} \quad \nu P^t(\cdot) = \int_X \nu(dx) P^t(x, \cdot),$$

respectively.

2.1. ψ -Irreducibility. For any $\theta > 0$, we define the *resolvent kernel* R_θ by,

$$(18) \quad R_\theta := \begin{cases} \sum_{n=0}^{\infty} (1 - e^{-\theta}) e^{-\theta n} P^n, & \text{discrete time,} \\ \int_{[0, \infty)} \theta e^{-\theta t} P^t dt, & \text{continuous time,} \end{cases}$$

and we write R for R_1 .

If for some σ -finite measure ψ on \mathcal{B} , some $\theta > 0$, and all functions $s : X \rightarrow [0, \infty)$ with $\psi(s) = \int s(x) \psi(dx) > 0$, we have

$$R_\theta(x, s) := \int_X R_\theta(x, dy) s(y) > 0, \quad x \in X,$$

then the semigroup $\{P^t : t \in \mathbb{T}\}$ is called ψ -irreducible and ψ is called an *irreducibility measure*. If the transition semigroup $\{P^t\}$ associated with the Markov process Φ is ψ -irreducible, then we say that Φ is ψ -irreducible. The set of functions $s : X \rightarrow \mathbb{R}_+$ with $\psi(s) = \int s(x) \psi(dx) > 0$ is denoted by \mathcal{B}^+ and all such s are called ψ -positive.

Throughout the paper, we will assume that Φ is ψ -irreducible. Moreover, without loss of generality we assume that ψ is *maximal* in the sense that any other irreducibility measure ψ' is absolutely continuous with respect to ψ [37]. We will also assume that the semigroup $\{P^t : t \in \mathbb{T}\}$ is *aperiodic*, that is, for any $s \in \mathcal{B}^+$ and any initial condition x ,

$$P^t(x, s) > 0 \quad \text{for all } t \text{ sufficiently large.}$$

If the semigroup associated with the Markov process Φ is aperiodic, then we say that Φ is aperiodic.

A measurable subset C of X is called *full* if $\psi(C^c) = 0$ and it is called *absorbing* if $R_\theta(x, C^c) = 0$ for $x \in C$ (for some θ). We recall that, for a ψ -irreducible Φ , a nonempty absorbing set is always full ([37], Proposition 4.2.3).

A function $s \in \mathcal{B}^+$ and a measure ν on \mathcal{B} are called *small* if, for some $\theta > 0$,

$$(19) \quad R_\theta(x, A) \geq s(x) \nu(A), \quad x \in X, A \in \mathcal{B}.$$

In [37], Proposition 5.5.5, it was shown that for a ψ -irreducible Φ , one can always find a θ and a pair (s, ν) satisfying the bound (3.2), such that $s(x) > 0$ for all x and ν is equivalent to the maximal irreducibility measure ψ (in the sense that they are mutually absolutely continuous). A similar construction works in continuous time as well.

If a small function s is of the form $s = \varepsilon \mathbb{1}_C$ for some $\varepsilon > 0$ and $C \in \mathcal{B}$, then the set C is called *small*. We denote by \mathcal{B}_p^+ the set of all small functions $s \in \mathcal{B}^+$ and we denote by \mathcal{M}_p^+ the set of all (positive) small measures ν which satisfy (19) for some $s \in \mathcal{B}_p^+$. Both \mathcal{M}_p^+ and \mathcal{B}_p^+ are positive cones and they are closed under addition.

2.2. *Ergodicity conditions.* Let $V : X \rightarrow [0, \infty]$ be an extended real-valued function with $V(x_0) < \infty$ for at least one $x_0 \in X$. Let S_V denote the (nonempty) set

$$(20) \quad S_V = \{x : V(x) < \infty\}.$$

In most of the results below our assumptions will guarantee that S_V is absorbing, hence full, so that $V(x) < \infty$ a.e. $[\psi]$.

Let L_∞^V denote the vector space of measurable functions $h : X \rightarrow \mathbb{C}$ satisfying

$$\|h\|_V := \sup_{x \in X} \frac{|h(x)|}{V(x)} < \infty.$$

Similarly, L_∞^f will denote the corresponding space for an arbitrary nonnegative (measurable) function f on X . We define the V -norm $\|\hat{P}\|_V$ of an arbitrary kernel $\hat{P} = \hat{P}(x, dy)$ by

$$(21) \quad \|\hat{P}\|_V := \sup \frac{\|\hat{P}h\|_V}{\|h\|_V},$$

where the supremum is over all $h \in L_\infty^V$ with $\|h\|_V \neq 0$.

In what follows, it will be convenient to describe some important properties of Φ in terms of its *generator* \mathcal{A} rather than in terms of its transition semigroup $\{P^t\}$. For a function $g : X \rightarrow \mathbb{C}$, we write $\mathcal{A}g = h$ if for each initial condition $\Phi(0) = x \in X$ the process $\{m(t) : t \in \mathbb{T}\}$ defined by

$$(22) \quad m(t) := \int_{[0,t)} h(\Phi(s)) ds - g(\Phi(t)), \quad t \in \mathbb{T},$$

is a local martingale with respect to the natural filtration $\{\mathcal{F}_t = \sigma(\Phi(s), 0 \leq s \leq t) : t \in \mathbb{T}\}$. In discrete time the generator is simply $\mathcal{A} = P - I$.

Next we introduce two different regularity conditions on Φ , taken from [37]. As we will see, the first one guarantees the validity of ergodic results as in equations (3)–(5), whereas the second one will be used to prove their multiplicative counterparts (6)–(8); see Section 4.

Throughout this paper we assume that the function V is finite for at least one $x \in X$.

(V3) For a function $f : X \rightarrow [1, \infty)$, a probability measure ν on \mathcal{B} , a constant $b < \infty$, a function $s : X \rightarrow (0, 1]$ and a $V : X \rightarrow (0, \infty]$,

$$\begin{aligned} \mathcal{A}V &\leq -f + bs, \\ R &\geq s \otimes \nu. \end{aligned}$$

(V4) For a probability measure ν on \mathcal{B} , some constants $b < \infty$ and $\delta > 0$, a function $s : X \rightarrow (0, 1]$, and a $V : X \rightarrow [1, \infty]$,

$$\begin{aligned} \mathcal{A}V &\leq -\delta V + bs, \\ R &\geq s \otimes \nu. \end{aligned}$$

Note that condition (V4) is stronger than (V3): When (V4) holds, (V3) also holds with $f = V$, $V' = V/\delta$ and $b' = b/\delta$. The assumption that (V4) holds for a Markov chain Φ is the main condition required for most of our results and it will remain in effect essentially for the rest of the paper. To formalize this assumption we introduce the following definition:

GEOMETRIC ERGODICITY. A Markov process Φ is called geometrically ergodic (with Lyapunov function V) if it is ψ -irreducible, aperiodic and satisfies condition (V4) (with this V).

In Section 7, numerous examples are given for which the validity of (V4) is explicitly verified; see also [37], Chapter 16. For comparison, we also introduce the following related condition for continuous-time Markov processes, which we think of as the natural multiplicative analog of (V3):

(mV3) For a function $f : X \rightarrow [1, \infty)$, a probability measure ν on \mathcal{B} , constants $\delta > 0$ and $b < \infty$, a function $s : X \rightarrow (0, 1]$ and a $V : \delta \rightarrow [1, \infty]$,

$$\begin{aligned} \log(e^{-V} \mathcal{A}e^V) &\leq -\delta f + bs, \\ R &\geq s \otimes \nu. \end{aligned}$$

As discussed in the introduction, condition (mV3) is very closely related to the conditions in the well-known Donsker–Varadhan large deviations results. In particular, under the conditions of [52], especially Assumption (3) on page 34, it follows from Theorem 3.3 below that (mV3) is satisfied. Moreover, when the state space is not compact, Varadhan’s conditions imply that (mV3) holds with an unbounded f with compact sublevel sets, an assumption already stronger than (V4) as the following proposition shows. A detailed study of the Markov processes satisfying (mV3) is given in [33], where the analog of (mV3) for discrete-time Markov processes is also given. In the context of diffusions, more specific results can be found in [27]. Proposition 2.1 is proved in [33].

PROPOSITION 2.1. *Suppose Φ is ψ -irreducible and aperiodic. If (mV3) holds, then so does (V4).*

2.3. Ergodic theorems. Under either (V3) or (V4), there exists a unique invariant probability measure π on \mathcal{B} (see below). Given such a π , we define Π as the kernel

$$\Pi = \mathbf{1} \otimes \pi$$

so that $\Pi(x, A) = \pi(A)$, $x \in X$ and $A \in \mathcal{B}$. If $\pi(V) := \int_X \pi(dx)V(x) < \infty$, then Π acts on L_∞^V as a bounded linear operator.

A *fundamental kernel* is a linear operator $Z : L_\infty^f \rightarrow L_\infty^V$ (for some measurable functions $f \geq 1$, $V \geq 1$) that satisfies

$$(23) \quad \mathcal{A}Z = -(I - \Pi).$$

That is, for any $F \in L_\infty^f$, the function $\widehat{F} = ZF \in L_\infty^V$ solves the *Poisson equation*

$$(24) \quad \mathcal{A}\widehat{F} = -F + \pi(F),$$

where $\pi(F) = \int_{\mathcal{X}} \pi(dx) F(x)$. Equivalently, the stochastic process

$$(25) \quad m(t) = \widehat{F}(\Phi(t)) - \widehat{F}(\Phi(0)) + \int_{[0,t)} (F(\Phi(r)) - \pi(F)) dr, \quad t \geq 0,$$

is a local martingale with respect to $\{\mathcal{F}_t\}$.

The following two theorems give equivalent conditions for Φ to be ergodic or geometrically ergodic, respectively. Corollary 2.3 states that a fundamental kernel exists and the ergodic results (3)–(5) given in the Introduction indeed hold as soon as Φ satisfies (V3).

For any $C \in \mathcal{B}$, let τ_C denote the hitting time

$$\tau_C := \inf\{t \geq 1 : \Phi(t) \in C\}.$$

THEOREM 2.2 (Ergodicity). *Suppose that Φ is ψ -irreducible and aperiodic. For any function $f : \mathcal{X} \rightarrow [1, \infty)$ the following are equivalent:*

- (i) *The process Φ is positive recurrent with invariant probability measure π and $\pi(f) < \infty$.*
- (ii) *There exists a small set C such that*

$$\sup_{x \in C} \mathbb{E}_x \left[\int_{[0, \tau_C)} f(\Phi(t)) dt \right] < \infty.$$

- (iii) *Condition (V3) holds with the same f .*

If any of these conditions holds, then the set S_V defined in (20) is absorbing and full, and

$$(26) \quad \sup_{g: |g| \leq f} |P^t(x, g) - \pi(g)| \rightarrow 0, \quad t \rightarrow \infty, \quad x \in S_V.$$

Moreover, for any small measure ν there exists a fundamental kernel Z which is a bounded linear operator,

$$Z : L_\infty^f \rightarrow \{h \in L_\infty^V : \nu(h) = 0\}.$$

If Z' is any other such fundamental kernel, then $\|ZF - Z'F\|_V = 0$, $F \in L_\infty^f$.

PROOF. The discrete-time version of (i)–(iii) is a consequence of the f -norm ergodic theorem in [37]; the continuous-time version follows from [35], Theorem 5.3. The construction of the fundamental kernel and the uniform bound is given in [23], Theorem 2.3. \square

Recall the definition of S_t in (1).

COROLLARY 2.3 (Ergodic theorems). *Let Φ be a ψ -irreducible, aperiodic Markov process that satisfies (V3). If $F \in L_\infty^V$ with f as in (V3), then for $x \in S_V$:*

- (a) $E_x[\frac{1}{t}S_t] \rightarrow \pi(F)$ as $t \rightarrow \infty$.
- (b) *There exists $\widehat{F} \in L_\infty^V$ with $\pi(\widehat{F}) = 0$, so that \widehat{F} solves the Poisson equation*

$$\mathcal{A}\widehat{F} = -F + \pi(F).$$

- (c) *If, in addition, $\pi(V) < \infty$, then \widehat{F} satisfies*

$$(27) \quad \widehat{F}(x) = \lim_{t \rightarrow \infty} E_x[S_t - t\pi(F)].$$

PROOF. The convergence in norm (26) implies the convergence,

$$E_x[F(\Phi(t))] \rightarrow \pi(F), \quad t \rightarrow \infty, \quad x \in S_V,$$

which gives (a). For (b) we can take $\widehat{F} = ZF$, where Z is given in Theorem 2.2.

When $\pi(V) < \infty$ it follows that, for some $b_1 < \infty$,

$$\int_{\mathbb{T}} |E_x[F(\Phi(t))] - \pi(F)| dt \leq b_1 \|F\|_f V(x), \quad x \in S_V.$$

This is given as Theorem 14.0.1 of [37] in discrete time. The continuous-time case follows on considering the skeleton chain $\Phi(\delta k)$, $k = 1, 2, 3, \dots$, as discussed on page 247 of [36]. This implies that one version of the fundamental kernel may be expressed as

$$Z(x, A) = \int_{\mathbb{T}} (P^t(x, A) - \pi(A)) dt, \quad x \in S_V, \quad A \in \mathcal{B},$$

and Z is a bounded linear operator from L_∞^f to L_∞^V . This gives (c). \square

The solution \widehat{F} of the Poisson equation given in Corollary 2.3(b) arises in almost every limit theorem considered below. In particular, it can be used to define the asymptotic variance σ^2 in the central limit theorem; see Theorem 17.4.5 of [37]. In the discrete-time case, $\sigma^2 < \infty$ as soon as $\pi(\widehat{F}^2) < \infty$, and equation (17.44) of [37] gives the representation,

$$\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \text{Var}_x\{S_n\} = E_\pi[\widehat{F}(\Phi(n))^2 - (P\widehat{F}(\Phi(n)))^2], \quad x \in S_V.$$

Next we obtain a characterization of the case when $\sigma^2 = 0$. In discrete time, a similar result was derived in [3] using different methods.

PROPOSITION 2.4 (Variance characterization). *Suppose that Φ satisfies (V3) with $\pi(V^2) < \infty$. Then, for any $F \in L_\infty^f$, the asymptotic variance*

$$(28) \quad \sigma^2 := \lim_{t \rightarrow \infty} \frac{1}{t} \text{Var}_x \{S_t\}$$

exists for any initial condition $\Phi(0) = x \in S_V$. Writing $\bar{F} = F - \pi(F)$, σ^2 satisfies, for all $t > 0$,

$$(29) \quad \sigma^2 = \frac{1}{t} \mathbf{E}_\pi \left[\left(\widehat{F}(\Phi(t)) - \widehat{F}(\Phi(0)) + \int_{[0,t)} \bar{F}(\Phi(s)) ds \right)^2 \right] < \infty.$$

Moreover:

(i) *If $\sigma^2 = 0$, then there exists $G \in L_\infty^V$ satisfying*

$$(30) \quad \int_{[0,t)} \bar{F}(\Phi(s)) ds = G(\Phi(t)) - G(\Phi(0)) \quad a.s. [\pi].$$

When time is discrete, this can also be expressed as $P(x, S_x) = 1$, $x \in S_V$, where $S_x = \{y \in X : G(y) = G(x) - \bar{F}(x)\}$.

(ii) *Conversely, if (30) holds for some $G \in L_\infty^V$, then $\sigma^2 = 0$.*

PROOF. Result (i) is an immediate consequence of (29), which follows from the martingale characterization of \widehat{F} [see (25)]. Result (ii) is immediate from (28). \square

THEOREM 2.5 (Geometric ergodicity). *Suppose that Φ is ψ -irreducible and aperiodic. The following are equivalent:*

(i) *There exists a probability measure π and a $V : X \rightarrow [1, \infty]$ such that P^t converges to π in the V -norm,*

$$P^t \rightarrow \mathbf{1} \otimes \pi, \quad t \rightarrow \infty.$$

(ii) *There exists a small set C and $\varepsilon > 0$ such that*

$$\sup_{x \in C} \mathbf{E}_x[\exp(\varepsilon \tau_C)] < \infty.$$

(iii) *Condition (V4) holds for some $V : X \rightarrow [1, \infty]$.*

If any of these conditions holds, then the set S_V defined in (20) is absorbing and full for any function V satisfying (iii), and there exist constants $b_0 > 0$, B_0 , $B'_0 < \infty$ and an invariant probability measure π on \mathcal{B} such that

$$\begin{aligned} \|P^t - \mathbf{1} \otimes \pi\|_V &\leq B_0 e^{-b_0 t}, \quad t \in \mathbb{T}, \\ |\mathbf{E}_x[S_t - t\pi(F)] - \widehat{F}(x)| &\leq B'_0 \|F\|_V e^{-b_0 t}, \quad F \in L_\infty^V, x \in S_V, t \in \mathbb{T}. \end{aligned}$$

PROOF. The discrete-time result is from [37], Theorem 15.0.1, and the continuous-time version is the main result of [15]. \square

3. The generalized principal eigenvalue. As in the previous section, we assume that Φ is a geometrically ergodic Markov process. We fix throughout this section a bounded (measurable) function $F : X \rightarrow \mathbb{R}$ and a real number $\alpha \in \mathbb{R}$, and we define the semigroup $\{\widehat{P}_\alpha^t : t \in \mathbb{T}\}$ by

$$(31) \quad \widehat{P}_\alpha^t(x, A) = E_x[\exp(\alpha S_t) \mathbb{I}_A(\Phi(t))], \quad t \in \mathbb{T},$$

where S_t is defined as before by (1). In this section we consider the general properties of this positive semigroup; we therefore suppress the dependency on α and F and simply write $\{\widehat{P}^t\}$ for $\{\widehat{P}_\alpha^t : t \in \mathbb{T}\}$.

We say that an arbitrary positive kernel \widehat{P} is *probabilistic* if $\widehat{P}(x, X) = 1$ for all $x \in X$. Similarly, a semigroup $\{\widehat{P}^t\}$ is called *probabilistic* if \widehat{P}^t is probabilistic for all t . Clearly, the semigroup $\{\widehat{P}^t\} = \{\widehat{P}_\alpha^t\}$ is, in general, *nonprobabilistic*. The definitions of ψ -irreducibility and aperiodicity carry over to nonprobabilistic semigroups immediately. (Further extensions to kernels defined for *complex* numbers $\alpha \in \mathbb{C}$ will be treated in Section 4.) Note that $\{\widehat{P}^t\}$ is irreducible and aperiodic as soon as $\{P^t\}$ is. We also define a family of resolvent kernels \widehat{R}_θ for $\theta > 0$ exactly as in (18), with \widehat{P}^t in place of P^t . To ensure that these are finite for all $x \in X$ and a suitable class of $A \in \mathcal{B}$, we usually consider θ 's in the range $\theta > |\alpha| \|F\|_\infty$ (and as before, we write $\widehat{R} = \widehat{R}_1$). With \widehat{R}_θ replacing R_θ , the definitions of small functions, measures and sets carry over verbatim.

Finally, we define the *generator* $\widehat{\mathcal{A}}$ of a general semigroup $\{\widehat{P}^t : t \in \mathbb{T}\}$: We write $\widehat{\mathcal{A}}g = h$ if

$$(32) \quad \widehat{P}^t g(x) = g(x) + \int_{[0,t)} \widehat{P}^s h(x) ds, \quad t \in \mathbb{T}, x \in X.$$

The following resolvent equations will play a central role in a lot of what follows:

$$(33) \quad \widehat{\mathcal{A}}\widehat{R}_\theta = \begin{cases} (e^\theta - 1)(\widehat{R}_\theta - I), & \text{discrete time,} \\ \theta(\widehat{R}_\theta - I), & \text{continuous time.} \end{cases}$$

In continuous time, the resolvent equation can be used to establish the identity

$$(34) \quad \sum_{n=1}^\infty \widehat{R}_\theta^n z^{-n} = \theta z^{-1} \int_{[0,\infty)} \exp(-(1 - z^{-1})\theta t) \widehat{P}^t dt, \quad z \in \mathbb{C},$$

whenever the sum and integral converge absolutely. This is tremendously valuable in consolidating continuous- and discrete-time theory.

Since the semigroup $\{\widehat{P}^t\}$ is ψ -irreducible, the kernel \widehat{R}_θ satisfies the following minorization condition: There are $s \in \mathcal{B}_p^+$ and $\nu \in \mathcal{M}_p^+$ such that

$$\widehat{R}_\theta \geq s \otimes \nu.$$

(Note that, since the semigroup $\{\widehat{P}^t\}$ is derived from $\{P^t\}$, the above domination condition is satisfied with s and ν that are small with respect to $\{P^t\}$.) Let $\{\kappa_n : n \geq 1\}$ denote the positive sequence defined by

$$\kappa_n = \nu(\widehat{R}_\theta)^{n-1} s, \quad n \geq 1.$$

This sequence is supermultiplicative,

$$\begin{aligned}\kappa_{n+m} &= \nu \widehat{R}_\theta^{n-1} \widehat{R}_\theta \widehat{R}_\theta^{m-1} s \\ &\geq \nu (\widehat{R}_\theta)^{n-1} (s \otimes \nu) (\widehat{R}_\theta)^{m-1} s \\ &= \kappa_n \kappa_m,\end{aligned}$$

so there exists some $L(\theta) \in (-\infty, \infty]$ such that

$$\frac{1}{n} \log(\nu \widehat{R}_\theta^n s) = \frac{1}{n} \log(\kappa_{n+1}) \rightarrow L(\theta), \quad n \rightarrow \infty.$$

The constant

$$(35) \quad r_\theta := \exp(-L(\theta))$$

is called the *convergence parameter* for the kernel \widehat{R}_θ [45]. It satisfies

$$(36) \quad \sum_{n=0}^{\infty} [\widehat{R}_\theta^n s(x)] r^n \begin{cases} = \infty, & \text{for all } x \in \mathbf{X}, \text{ if } r > r_\theta, \\ < \infty, & \text{for a.e. } x \in \mathbf{X}[\psi], \text{ if } r < r_\theta. \end{cases}$$

To move from the resolvent back to the original semigroup, we apply the resolvent equations (33). These relationships establish the major part of the following theorem.

THEOREM 3.1 (Generalized principal eigenvalue). *Suppose Φ is ψ -irreducible and aperiodic. Then there is a $\lambda_\circ \in (0, \infty]$ such that, for any $s \in \mathcal{B}_p^+$:*

(i)

$$\int_{\mathbb{T}} \lambda^{-t} \widehat{P}^t s(x) dt \begin{cases} = \infty, & \text{for all } x \in \mathbf{X}, \lambda < \lambda_\circ, \\ < \infty, & \text{for a.e. } x \in \mathbf{X}[\psi], \lambda > \lambda_\circ; \end{cases}$$

(ii)

$$\frac{1}{t} \log(\widehat{P}^t s(x)) \rightarrow \Lambda_\circ := \log(\lambda_\circ) \quad \text{a.e. } x \in \mathbf{X}[\psi], t \rightarrow \infty.$$

PROOF. Result (i) follows from Theorem 3.2 of [45] for discrete-time chains and follows from (34) for the continuous-time case, where we may translate to the discrete-time case using the resolvent \widehat{R}_θ . Then with $\lambda_\theta = r_\theta^{-1}$,

$$(37) \quad \lambda_\circ = \begin{cases} e^\theta + (1 - e^\theta)\lambda_\theta, & \text{discrete time,} \\ \exp(\theta(1 - \lambda_\theta^{-1})), & \text{continuous time.} \end{cases}$$

The second part follows from an argument similar to that used in the proof of Lemma 3.2 of [2]. \square

We call the constant λ_\circ the *generalized principal eigenvalue* (g.p.e.) of the semigroup $\{\widehat{P}^t : t \in \mathbb{T}\}$. This generalizes the corresponding definition of [47] and,

as we will see in Theorem 3.3, λ_o does indeed play the role of an eigenvalue. The interpretation of (ii) is the pinned multiplicative mean ergodic theorem (14) discussed in the Introduction,

$$(38) \quad \frac{1}{t} \log E_x[\exp(\alpha S_t) \mathbb{I}_C(\Phi(t))] \rightarrow \Lambda(\alpha), \quad t \rightarrow \infty,$$

for a.e. $x \in X$ [ψ]. This follows from taking $s = \varepsilon \mathbb{I}_C$ in (ii) with C small and $\varepsilon > 0$.

Theorem 3.1 leaves open what happens in (36) when $\lambda = \lambda_o$. The semigroup $\{\widehat{P}^t\}$ is called:

1. λ_o -transient if

$$\int_{\mathbb{T}} \lambda_o^{-t} v \widehat{P}^t s \, dt < \infty;$$

2. λ_o -recurrent if

$$\int_{\mathbb{T}} \lambda_o^{-t} v \widehat{P}^t s \, dt = \infty;$$

3. λ_o -geometrically recurrent if the function

$$(\lambda_o - z) \int_{\mathbb{T}} (v \widehat{P}^t s) z^{-t} \, dt$$

is analytic in a neighborhood of $z = \lambda_o$.

In (1)–(3), (s, v) is any pair with $s \in \mathcal{B}_p^+$ and $v \in \mathcal{M}_p^+$. The particular small function or small measure chosen is not important [45].

The construction of h in part (ii) of the following lemma is an extension of the *minimal harmonic function* in [45], Proposition 3.13, where here we allow the semigroup $\{P^t\}$ to possibly be transient.

LEMMA 3.2. (i) Suppose that $\{\widehat{P}^t : t \in \mathbb{Z}_+\}$ has g.p.e. $\lambda_o < \infty$ and suppose that the following minorization condition holds for some $s \in \mathcal{B}_p^+$ and $v \in \mathcal{M}_p^+$:

$$\widehat{P} \geq s \otimes v.$$

Then

$$\sum_{t=0}^{\infty} \lambda_o^{-t-1} v(\widehat{P} - s \otimes v)^t s \leq 1$$

with equality if and only if the semigroup is λ_o -recurrent.

(ii) If in (i) we take $\{\widehat{P}^t : t \in \mathbb{Z}_+\}$ to be the probabilistic semigroup $\{P^t : t \in \mathbb{Z}_+\}$, then

$$h(x) := \sum_0^{\infty} (P - s \otimes v)^n s(x) \leq 1,$$

$$\sum_0^{\infty} (P - s \otimes v)^n P s(x) = -s(x) + (1 + v(s))h(x) \leq 2 \quad \text{for all } x \in X.$$

In this case, if $\{P^t : t \in \mathbb{Z}_+\}$ is 1-recurrent, $h(x) = 1$ for a.e. $x \in X$ [ψ].

PROOF. Part (i) is Proposition 5.2 of [45]. The essence of this result is the inversion formula

$$(39) \quad [Iz - \widehat{P}]^{-1} = [Iz - (\widehat{P} - s \otimes v)]^{-1} \left(I + \frac{1}{1 - \kappa} s \otimes v \right),$$

where

$$\kappa = v[Iz - (\widehat{P} - s \otimes v)]^{-1} s.$$

From (39) it may be seen that, for $z > 0$,

$$(40) \quad \kappa = 1 \quad \text{if and only if } v[Iz - \widehat{P}]^{-1} s = \infty.$$

The proof of (ii) is by induction. Define, for $n \geq 0$,

$$h_n = \sum_{t=0}^n (P - s \otimes v)^t s.$$

For $n = 0$, $h_0 = s$ and $s \leq 1$ by assumption. If true for n , then

$$\begin{aligned} h_{n+1}(x) &= (P - s \otimes v)h_n(x) + s(x) \\ &\leq (P - s \otimes v)\mathbf{1}(x) + s(x) \\ &= [P(x, \mathbf{X}) - s(x)v(\mathbf{X})] + s(x) \\ &= 1, \end{aligned}$$

where in the last equation we have again used the fact that $s \leq 1$. It follows that $h(x) = \lim h_n(x) \leq 1$ for all x .

To see the second bound, write

$$(P - s \otimes v)^n P s = (P - s \otimes v)^{n+1} s + (P - s \otimes v)^n [s \otimes v] s.$$

Summing over n gives the desired result. \square

The reason we call the constant λ_\circ a generalized eigenvalue is clarified by the next theorem, where it is shown that if the semigroup $\{\widehat{P}^t : t \in \mathbb{T}\}$ is λ_\circ -recurrent, then there is a function $\check{f} : \mathcal{X} \rightarrow [0, \infty)$ so that $(\check{f}, \lambda_\circ)$ solve the eigenvalue problem

$$(41) \quad \widehat{P} \check{f} = \lambda_\circ \check{f}.$$

Equation (41) is an instance of the *multiplicative Poisson equation*. Conditions for the existence of a solution to (41) based upon Lemma 3.2(i) are well known in the discrete-time case. A candidate solution is given by

$$(42) \quad \check{f} = \sum_{n=0}^{\infty} r_\theta^n \widehat{R}_\theta (\widehat{R}_\theta - s \otimes v)^n s,$$

where (θ, s, v) satisfy $\theta > \lambda_\circ$, $s \in \mathcal{B}_p^+$, $v \in \mathcal{M}_p^+$, $\widehat{R}_\theta \geq s \otimes v$ and r_θ is the convergence parameter defined in (35).

THEOREM 3.3 (Existence of an eigenfunction \check{f}). *Suppose that Φ is ψ -irreducible and that the g.p.e. λ_\circ of the positive semigroup $\{\hat{P}^t\}$ is finite. Then the function \check{f} given in (42) is finite a.e. $[\psi]$ and the following statements are valid:*

(i) *If $\{\hat{P}^t : t \in \mathbb{T}\}$ is λ_\circ -recurrent, then \check{f} solves the multiplicative Poisson equation*

$$(43) \quad \hat{P}^t \check{f} = \lambda_\circ^t \check{f}, \quad t \in \mathbb{T}.$$

(ii) *If $\{\hat{P}^t : t \in \mathbb{T}\}$ is λ_\circ -transient, then for any small function $s \in \mathcal{B}_p^+$, there exists $\delta > 0$ such that*

$$(44) \quad \hat{P} \check{f} = \lambda_\circ \check{f} - \delta s.$$

Hence, in the λ_\circ -transient case there is a solution \bar{f} to the pointwise inequality

$$(45) \quad \hat{P} \bar{f} \leq \lambda_\circ \bar{f}$$

with \bar{f} finite a.e. $[\psi]$ and where the inequality is strict whenever $\bar{f}(x) < \infty$.

(iii) *The solution (42) is minimal and essentially unique: If $\bar{f} : X \rightarrow (0, \infty)$ is any solution to the inequality (45), then there exists $c \in \mathbb{R}_+$ such that $\bar{f}(x) \geq c \check{f}(x)$ for all x .*

If $\{\hat{P}^t : t \in \mathbb{T}\}$ is λ_\circ -recurrent, then we have $\bar{f} = c \check{f}$ a.e. $[\psi]$ and $\bar{f}(x) \geq c \check{f}(x)$ for all x .

PROOF. These results are all based on Theorem 5.1 of [45] in the discrete-time case.

If $\theta > 0$ is taken large enough, then the resolvent \hat{R}_θ satisfies $\nu \hat{R}_\theta s < \infty$ for any small s, ν satisfying the domination condition $\hat{R}_\theta \geq s \otimes \nu$. We then set

$$\check{f}_\theta = \sum_{n=0}^{\infty} r_\theta^n (\hat{R}_\theta - s \otimes \nu)^n s,$$

where r_θ is the convergence parameter for \hat{R}_θ . We have

$$r_\theta (\hat{R}_\theta - s \otimes \nu) \check{f}_\theta = \check{f}_\theta - s$$

and hence

$$(46) \quad \check{f} := \hat{R}_\theta \check{f}_\theta = r_\theta^{-1} \check{f}_\theta - \delta_\theta s,$$

where $\delta_\theta = r_\theta^{-1} - \nu(\check{f}_\theta) \geq 0$. This constant is strictly positive if and only if the semigroup $\{\hat{R}_\theta^n\}$ is r_θ^{-1} -transient [see Lemma 3.2(i)].

Results (i)–(iii) then follow from the resolvent equation in discrete or continuous time. \square

4. Spectral gap and multiplicative mean ergodic theorems. The following assumptions will be held throughout the remainder of this paper:

ASSUMPTION 1. The Markov process Φ is geometrically ergodic with a Lyapunov function $V : X \rightarrow [1, \infty)$ such that $\pi(V^2) < \infty$.

ASSUMPTION 2. The (measurable) function $F : X \rightarrow [-1, 1]$ has zero mean $\pi(F) = 0$ and nontrivial asymptotic variance $\sigma^2 := \lim_t \text{Var}_x\{S_t/\sqrt{t}\} > 0$.

Note that the additional assumption $\pi(V^2) < \infty$ can be made without any loss of generality: (V4) implies (V3) with $f = V$ as discussed above, which implies that $\pi(f) < \infty$ ([37], Theorem 14.0.1). Moreover, Lemma 15.2.9 of [37] says that (V4) also holds with respect to \sqrt{V} (and some, possibly different, small function s), so we can always take V in (V4) such that $\pi(V^2) < \infty$.

Until Section 4.2 we specialize to the discrete-time case for the sake of clarity.

With V as in Assumptions 1 and 2, the *spectrum* $\mathfrak{S}(\widehat{P}) \subset \mathbb{C}$ of a bounded linear operator $\widehat{P} : L_\infty^V \rightarrow L_\infty^V$ is defined to be the set of nonzero $\lambda \in \mathbb{C}$ for which the inverse $(I\lambda - \widehat{P})^{-1}$ does not exist as a bounded linear operator on L_∞^V .

Recall that an arbitrary kernel \widehat{P} acts on functions (on the right-hand side) and on signed measures (on the left-hand side) as in (17). With that in mind, we think of a kernel \widehat{P} as an operator acting on an appropriate function space. The kernel \widehat{P} is a bounded linear operator on L_∞^V provided its V -norm $\|\widehat{P}\|_V$ is finite, since this is precisely the induced operator norm. For an arbitrary linear operator $\widehat{P} : L_\infty^V \rightarrow L_\infty^V$ we continue to define the norm $\|\widehat{P}\|_V$ as in (21). Also we recall that \widehat{P} acts on a suitable space of measures (on the left) as

$$\nu \widehat{P}(A) := \nu(\widehat{P}\mathbb{I}_A), \quad A \in \mathcal{B}.$$

For $\alpha \in \mathbb{C}$ the kernel \widehat{P}_α defined in (31) yields an operator $\widehat{P}_\alpha : L_\infty^V \rightarrow L_\infty^V$ acting via

$$(47) \quad \widehat{P}_\alpha g(x) = \exp(\alpha F(x)) P g(x), \quad x \in X, g \in L_\infty^V.$$

Its spectrum is denoted $\mathfrak{S}_\alpha = \mathfrak{S}(\widehat{P}_\alpha)$. The n -fold composition of the kernel \widehat{P}_α with itself acts on L_∞^V as

$$\widehat{P}_\alpha^n g(x) = \mathbb{E}_x[\exp(\alpha S_n) g(\Phi(n))], \quad x \in X, g \in L_\infty^V, n \geq 1,$$

where $\{S_n : n \geq 1\}$ denote the partial sums (2). Letting \mathcal{M}_1^V denote the space of signed and possibly complex-valued measures μ satisfying $|\mu|(V) < \infty$, we obtain analogously

$$\mu \widehat{P}_\alpha^n(A) = \int \mathbb{E}_x[\exp(\alpha S_n) \mathbb{I}(\Phi(n) \in A)] \mu(dx), \quad A \in \mathcal{B}, \mu \in \mathcal{M}_1^V, n \geq 1.$$

In this section we identify a region $\Omega \subset \mathbb{C}$ such that, for geometrically Markov chains, eigenfunctions $\check{f}_\alpha \in L_\infty^V$ and (positive) eigenmeasures $\check{\mu}_\alpha \in \mathcal{M}_1^V$ exist

for \widehat{P}_α , corresponding to a given eigenvalue $\lambda_\alpha \in \mathcal{S}_\alpha$ and $\alpha \in \Omega$. Suppose that such $\check{f}_\alpha, \check{\mu}_\alpha$ are found and assume that they are normalized so that

$$(48) \quad \check{\mu}_\alpha(\check{f}_\alpha) = \check{\mu}_\alpha(X) = 1.$$

We then let $\widehat{Q}_\alpha : L_\infty^V \rightarrow L_\infty^V$ denote the operator $\widehat{Q}_\alpha = \check{f}_\alpha \otimes \check{\mu}_\alpha$,

$$\widehat{Q}_\alpha g(x) = \check{\mu}_\alpha(g) \check{f}_\alpha(x), \quad g \in L_\infty^V, x \in X.$$

Note that \widehat{Q}_α is a *projection* operator, that is, $\widehat{Q}_\alpha^2 = \widehat{Q}_\alpha$.

The main results of this section are summarized in the following two theorems. In particular, the multiplicative mean ergodic theorem given in (50) will play a central role in the proofs of all the subsequent probabilistic limit theorems.

THEOREM 4.1 (Multiplicative mean ergodic theorem). *Assume that the Markov chain Φ and the functional F satisfy Assumptions 1 and 2. With δ and b as in (V4), define*

$$(49) \quad \bar{a} := \left(\frac{e-1}{2b-\delta} \right) \delta > 0.$$

Then there exists $\bar{\omega} > 0$ such that, for any α in the compact set

$$\Omega = \{\alpha = a + i\omega \in \mathbb{C} : |a| \leq \bar{a}, \text{ and } |\omega| \leq \bar{\omega}\},$$

there is an eigenvalue $\lambda_\alpha \in \mathcal{S}_\alpha$ which is maximal and isolated, that is,

$$|\lambda_\alpha| = \max\{|\lambda| : \lambda \in \mathcal{S}_\alpha\} \quad \text{and} \quad \mathcal{S}_\alpha \cap \{z : |z| \geq |\lambda_\alpha| - \delta_0\} = \{\lambda_\alpha\}$$

for some $\delta_0 > 0$.

Moreover, for any such α , there exist $\check{f}_\alpha \in L_\infty^V$ and $\check{\mu}_\alpha \in \mathcal{M}_1^V$, satisfying (48), and the following statements are valid:

(i) The functions \check{f}_α solve the multiplicative Poisson equation

$$\widehat{P}_\alpha \check{f}_\alpha = \lambda_\alpha \check{f}_\alpha$$

and the $\check{\mu}_\alpha$ are eigenmeasures for the kernels \widehat{P}_α :

$$\check{\mu}_\alpha \widehat{P}_\alpha = \lambda_\alpha \check{\mu}_\alpha.$$

(ii) There exist constants $b_0 > 0$ and $B_0 < \infty$ such that for all $\alpha \in \Omega$, $x \in X$, $n \geq 1$,

$$(50) \quad |\mathbb{E}_x[\exp(\alpha S_n - n\Lambda(\alpha))] - \check{f}_\alpha(x)| \leq B_0 |\alpha| V(x) \exp(-b_0 n),$$

where $\Lambda(\alpha) := \log(\lambda_\alpha)$ is analytic on Ω and S_n are the partial sums defined in (2). More generally, for any $g \in L_\infty^V$,

$$|\mathbb{E}_x[\exp(\alpha S_n - n\Lambda(\alpha))g(\Phi(n))] - \widehat{Q}_\alpha g(x)| \leq B_0 \|g\|_V V(x) \exp(-b_0 n).$$

PROOF. The existence of an isolated, maximal eigenvalue λ_α is given in Proposition 4.12. It is nonzero for $\alpha = a \in [-\bar{a}, \bar{a}]$ by Proposition 4.3, and since it is analytic in α (by Proposition 4.12), we can pick $\bar{\omega} > 0$ small enough such that λ_α is nonzero on Ω .

The existence of an eigenfunction and an eigenmeasure as in (i) follows from Proposition 4.12 combined with Proposition 4.8. To see that $\bar{a} > 0$ note that, under (V4),

$$(51) \quad \pi(V) = \pi(PV) \leq (1 - \delta)\pi(V) + b\pi(s).$$

Hence, $b \geq \delta\pi(V)/\pi(s) \geq \delta$.

To prove the limit theorems in (ii), consider the linear operator

$$U_{(z,\alpha)} = [Iz - (\lambda_\alpha^{-1}\widehat{P}_\alpha - \widehat{Q}_\alpha)]^{-1}.$$

From Proposition 4.8 we can find $\varepsilon_0 > 0$ such that $U_{(z,\alpha)}$ is an analytic function of two variables $(z, \alpha) = (z, a + i\omega)$ on the domain

$$\mathcal{D} = \{|z| > 1 - \varepsilon_0, |a| < \bar{a} + \varepsilon_0, |\omega| < \bar{\omega} + \varepsilon_0\}.$$

We may also assume that $\varepsilon_0 > 0$ is suitably small so that, for some $\bar{b} < \infty$, we have $\|U_{(z,\alpha)}\|_V \leq \bar{b}$ for all $(z, \alpha) \in \mathcal{D}$.

Set $b_0 = -\log(1 - \varepsilon_0) > 0$. The following bound then holds for all $(z, \alpha) \in \mathcal{D}$, $g \in L_\infty^V$, $x \in X$ and $n \geq 1$ by representing $U_{(z,\alpha)}$ as a power series and using the fact that \widehat{Q}_α is a projection operator:

$$\begin{aligned} \bar{b}V(x) &\geq \left| \int_0^{2\pi} \exp(i(n+1)\phi) U_{(\exp(-b_0+i\phi), \alpha)} g(x) d\phi \right| \\ &= \exp((n+1)b_0) |(\lambda_\alpha^{-1}\widehat{P}_\alpha - \widehat{Q}_\alpha)^n g(x)| \\ &= \exp((n+1)b_0) |\lambda_\alpha^{-n}\widehat{P}_\alpha^n g(x) - \widehat{Q}_\alpha g(x)|. \end{aligned}$$

This gives the second bound in (ii). The first one follows from the second since, when $\alpha = 0$ and $g = \mathbf{1}$,

$$|\lambda_\alpha^{-n}\widehat{P}_\alpha^n g(x) - \widehat{Q}_\alpha g(x)| = 0$$

for all $n \geq 1$, $x \in X$. \square

Next we give a weaker multiplicative mean ergodic theorem for all $\alpha = a + i\omega$ in a neighborhood of the $i\omega$ axis. A function $F : X \rightarrow \mathbb{R}$ is called *lattice* if there are $h > 0$ and $0 \leq d < h$, such that

$$(52) \quad \frac{F(x) - d}{h} \text{ is an integer,} \quad x \in X.$$

The minimal h for which (52) holds is called the *span* of F . If the function F can be written as a sum,

$$F = F_0 + F_\ell,$$

where F_ℓ is lattice with span h and F_0 has zero asymptotic variance [recall (28)], then F is called *almost-lattice* (and h is its span). Otherwise, F is called *strongly nonlattice*.

Although these definitions are somewhat different from the ones commonly used when studying the partial sums of independent random variables, in the Markov case they lead to the natural analog of the classical lattice/nonlattice dichotomy. This dichotomy, which is close in spirit to the discussion in [50], is stated in Theorem 4.13.

THEOREM 4.2 (Bounds around the $i\omega$ axis). *Assume that the Markov chain Φ and the functional F satisfy Assumptions 1 and 2.*

(NL) *Suppose that F is strongly nonlattice. For any $0 < \omega_0 < \omega_1 < \infty$, there exist $b_0 > 0$ and $B_0 < \infty$ (possibly different than in Theorem 4.1) such that*

$$(53) \quad \left| \mathbb{E}_x[\exp(\alpha S_n - n\Lambda(\alpha))] \right| \leq B_0 V(x) \exp(-b_0 n), \quad x \in \mathbf{X}, n \geq 1,$$

for all $\alpha = a + i\omega$ with $|a| \leq \bar{a}$ and $\omega_0 \leq |\omega| \leq \omega_1$.

(L) *Suppose that F is almost-lattice with span $h > 0$. For any $\epsilon > 0$, there exist $b_0 > 0$ and $B_0 < \infty$ (possibly different than above and in Theorem 4.1) such that (53) holds for all $\alpha = a + i\omega$ with $|a| \leq \bar{a}$ and $\epsilon \leq |\omega| \leq 2\pi/h - \epsilon$.*

PROOF. By Theorem 4.13 we have the bound $\hat{\xi}_\alpha < \hat{\xi}_a = \lambda_a$ for the range of $\alpha \in \mathbb{C}$ considered in the theorem. This implies that there is an $\epsilon_1 > 0$ and $b_1 < \infty$ such that

$$\| [Iz - e^{-\lambda_a} \hat{P}_\alpha]^{-1} \|_V < b_1$$

for all $|z| \geq 1 - \epsilon_1$ and all α in this range. An argument similar to the proof of Theorem 4.1(ii) then gives the desired bounds. \square

4.1. Spectral radius and spectral gap. Recalling our standing assumptions, Assumptions 1 and 2, we fix the Lyapunov function $V : \mathbf{X} \rightarrow [1, \infty)$ throughout this section.

For complex α we wish to construct $\Lambda(\alpha) \in \mathbb{C}$ satisfying the multiplicative mean ergodic limit,

$$\Lambda(\alpha) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_x[\exp(\alpha S_n)], \quad x \in \mathbf{X}.$$

This requires a generalization of the notion of the g.p.e. in Section 3. The previous definition is meaningless when $\alpha \notin \mathbb{R}$, since the definition of a small set depends on the linear ordering of \mathbb{R} .

Spectral radius. For a bounded linear operator $\widehat{P}: L_\infty^V \rightarrow L_\infty^V$ we define the *spectral radius* of \widehat{P} by

$$(54) \quad \hat{\xi} := \lim_{n \rightarrow \infty} (\|\widehat{P}^n\|_V)^{1/n} = \exp\left(\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\widehat{P}^n\|_V\right).$$

Note that in the above definition \widehat{P} is not assumed to be a positive operator and it is possibly complex-valued. Since $\|\cdot\|_V$ is an operator norm, the sequence $\{\log(\|\widehat{P}^n\|_V) : n \geq 1\}$ is subadditive [48]. Therefore $\hat{\xi}$ always exists, although it may be infinite.

We let $\hat{\xi}_\alpha$ denote the spectral radius of the operator \widehat{P}_α defined in (47). When $\alpha = a$ is real, from the definitions we have that $\hat{\xi}_\alpha \geq \lambda_\alpha$, where λ_α is the g.p.e. of the positive kernel \widehat{P}_α . One of the main goals of this section is to show that the spectral radius $\hat{\xi}_\alpha$ coincides with λ_α for real α in a neighborhood of $\alpha = 0$. We first establish upper and lower bounds:

PROPOSITION 4.3. *Under Assumptions 1 and 2, the spectral radius $\hat{\xi}_\alpha$ of \widehat{P}_α is finite and*

$$\hat{\xi}_\alpha \leq (b+1)e^{|\alpha|}$$

for all $\alpha = a + i\omega \in \mathbb{C}$. Moreover, for $\alpha = a \in \mathbb{R}$,

$$\hat{\xi}_a \geq e^{-a} > 0.$$

PROOF. The function s in (V4) is necessarily bounded by 1. Consequently, under (V4) we have for any $g \in L_\infty^V$ and $\alpha = a + i\omega \in \mathbb{C}$,

$$|\widehat{P}_\alpha g(x)| \leq e^{|\alpha|} \|g\|_V P V \leq e^{|\alpha|} \|g\|_V (1+b)V.$$

This implies that $\|\widehat{P}_\alpha\|_V \leq e^{|\alpha|}(1+b)$. The operator norm $\|\cdot\|_V$ is submultiplicative,

$$\|\widehat{P}_\alpha^n\|_V \leq \|\widehat{P}_\alpha\|_V^n \leq e^{|\alpha|n} (1+b)^n, \quad n \geq 1,$$

giving the upper bound.

When $\alpha = a$ is real, for any $g \in L_\infty^V$, $g \geq 0$, we have

$$\widehat{P}_a^n g(x) \geq e^{-an} P^n g(x).$$

It follows immediately that $\hat{\xi}_a \geq \hat{\xi}_0 e^{-a} = e^{-a}$. \square

Spectral gap and V-uniform operators. Recall the following classical result from [48], page 421:

THEOREM 4.4 (Decomposition theorem). *Let $\widehat{P}: L_\infty^V \rightarrow L_\infty^V$ be a bounded linear operator and suppose that $z_0 \in \mathcal{S}(\widehat{P})$ is isolated, that is, for some $\varepsilon_0 > 0$,*

$$\mathcal{S}(\widehat{P}) \cap D = \{z_0\} \quad \text{where } D = \{z \in \mathbb{C} : |z - z_0| \leq \varepsilon_0\}.$$

Then, the bounded operator on L_∞^V is well defined,

$$\widehat{Q} = \frac{1}{2\pi i} \int_{\partial D} [Iz - \widehat{P}]^{-1} dz,$$

and moreover:

- (i) $\widehat{Q}: L_\infty^V \rightarrow L_\infty^V$ is a projection operator, that is, $\widehat{Q}^2 = \widehat{Q}$;
- (ii) $\widehat{P}\widehat{Q} = \widehat{Q}\widehat{P} = z_0\widehat{Q}$;
- (iii) $\mathcal{S}(\widehat{Q}) = \{1\}$ and $\mathcal{S}(\widehat{P} - z_0\widehat{Q}) \cap D = \emptyset$.

We say that $z_0 \in \mathcal{S}(\widehat{P})$ is a *pole of finite multiplicity* if z_0 is an isolated point in $\mathcal{S}(\widehat{P})$ and the associated projection operator \widehat{Q} can be expressed as a finite linear combination of some $\{s_i\} \subset L_\infty^V$ and $\{v_i\} \subset \mathcal{M}_1^V$:

$$(55) \quad \widehat{Q} = \sum_{i,j=0}^{n-1} m_{i,j} [s_i \otimes v_j].$$

In particular, we call z_0 a *pole of multiplicity 1* if (55) holds for $n = 1$ and also there exists $\varepsilon_0 > 0$ such that

$$\mathcal{S}(\widehat{P} - z_0\widehat{Q}) \subset \{z : |z| \leq \hat{\xi} - \varepsilon_0\},$$

where $\hat{\xi}$ is the spectral radius of \widehat{P} .

We say that \widehat{P} admits a *spectral gap* if there exists $\varepsilon_0 > 0$ such that $\mathcal{S}(\widehat{P}) \cap \{z : |z| \geq \hat{\xi} - \varepsilon_0\}$ is finite and contains only poles of finite multiplicity.

Furthermore, we say that \widehat{P} is *V-uniform* if it admits a spectral gap and also there exists a unique pole $\lambda_\circ \in \mathcal{S}(\widehat{P})$ of multiplicity 1, satisfying $|\lambda_\circ| = \hat{\xi}$. In that case, λ_\circ is called *the generalized principal eigenvalue*, generalizing the previous definition. In particular, if \widehat{P}_α is V-uniform for some $\alpha \in \mathbb{C}$, then we write λ_α for its associated g.p.e.

Much of the development of this section, is based on properties of rank-one operators of the form $\widehat{M} = s_0 \otimes v_0$ for some $s_0 \in L_\infty^V$ and $v_0 \in \mathcal{M}_1^V$. The associated *potential operator* is defined as

$$(56) \quad \widehat{U}_z := [Iz - (\widehat{P} - \widehat{M})]^{-1}, \quad z \in \mathbb{C},$$

whenever the inverse exists. The potential operator is used to construct eigenfunctions and eigenmeasures for a V-uniform operator:

PROPOSITION 4.5. *Suppose that \widehat{P} is V -uniform with g.p.e. λ_\circ and suppose that the associated s_0 and ν_0 in (55) are chosen so that the potential operator \widehat{U}_z in (56) is bounded for z in a neighborhood of $|z| \geq |\lambda_\circ|$. Then, setting $\check{f} = \widehat{U}_{\lambda_\circ} s_0$ and $\check{\mu} = \nu_0 \widehat{U}_{\lambda_\circ}$, we have $\check{f} \in L_\infty^V$ and $\check{\mu} \in \mathcal{M}_1^V$,*

$$\widehat{P}\check{f} = \lambda_\circ \check{f} \quad \text{and} \quad \check{\mu}\widehat{P} = \lambda_\circ \check{\mu}.$$

PROOF. From V -uniformity we know that there exists $\varepsilon_0 > 0$ such that the inverse $[Iz - \widehat{P}]^{-1}$ exists and is bounded as a linear operator on L_∞^V for all $|z| \geq \hat{\xi} - \varepsilon_0$, $z \neq \lambda_\circ$. Moreover, for such z we may apply the inversion formula (39) to obtain the identity

$$(57) \quad [Iz - \widehat{P}]^{-1} = \widehat{U}_z + \frac{(\widehat{U}_z s_0) \otimes (\nu_0 \widehat{U}_z)}{1 - \nu_0 \widehat{U}_z s_0}.$$

Since $\lambda_\circ \in \mathcal{S}(\widehat{P})$ and $\lambda_\circ \notin \mathcal{S}(\widehat{P} - s_0 \otimes \nu_0)$, it follows from this equation that $\nu_0 \widehat{U}_{\lambda_\circ} s_0 = 1$.

Applying $[Iz - \widehat{P}]$ to (57) on the left and s_0 on the right then gives

$$s_0 = [Iz - \widehat{P}]\widehat{U}_z s_0 + [Iz - \widehat{P}] \frac{(\widehat{U}_z s_0)(\nu_0 \widehat{U}_z s_0)}{1 - \nu_0 \widehat{U}_z s_0}.$$

Multiplying both sides by $(1 - \nu_0 \widehat{U}_z s_0)$ and then setting $z = \lambda_\circ$ gives $0 = [I\lambda_\circ - \widehat{P}]\widehat{U}_{\lambda_\circ} s_0$, which shows that \check{f} is an eigenfunction.

The proof that $\check{\mu}$ is an eigenmeasure is completely analogous and follows by applying $[Iz - \widehat{P}]$ to (57) on the right-hand side and ν_0 on the left-hand side. \square

The following proposition provides useful characterizations of V -uniformity.

PROPOSITION 4.6. *The following are equivalent for an operator \widehat{P} with finite spectral radius $\hat{\xi}$:*

(i) \widehat{P} is V -uniform.

(ii) *There exists $\lambda \in \mathbb{C}$ satisfying $|\lambda| = \hat{\xi}$, and $\varepsilon_0 > 0$, $s_0 \in L_\infty^V$ and $\nu_0 \in \mathcal{M}_1^V$ satisfying*

$$\sup\{|z - \lambda| \| [Iz - \widehat{P}] \|_V : |z| \geq \hat{\xi} - \varepsilon_0\} < \infty$$

and

$$\sup\{\| \widehat{U}_z \|_V : |z| \geq \hat{\xi} - \varepsilon_0\} < \infty,$$

where \widehat{U}_z is the potential operator defined in (56) with $\widehat{M} = s_0 \otimes \nu_0$.

(iii) *There exists $\lambda \in \mathbb{C}$ satisfying $|\lambda| = \hat{\xi}$, and $\check{f} \in L_\infty^V$ and $\check{\mu} \in \mathcal{M}_1^V$ such that*

$$\lambda^{-n} \widehat{P}^n \rightarrow \check{f} \otimes \check{\mu}, \quad n \rightarrow \infty,$$

where the convergence is in the V -norm.

PROOF. If (i) holds, then the matrix-inversion formula (39) gives

$$[Iz - \widehat{P} + \lambda_o \check{f} \otimes \check{\mu}]^{-1} = [Iz - \widehat{P}]^{-1} - \frac{\lambda_o}{z(z - \lambda_o)} \check{f} \otimes \check{\mu}.$$

The left-hand side is bounded for $|z| \geq \widehat{\xi} - \varepsilon_0$ under (i). Hence we may set $\check{f} = s_0$ and $\check{\mu} = \nu_0$ to obtain (ii).

The implication (ii) \Rightarrow (i) also follows from the matrix inversion formula (39) since (57) then holds for all $|z| \geq \widehat{\xi} - \varepsilon_0$, $z \neq \lambda$. This implies the limit

$$(58) \quad \widehat{Q} := \lim_{z \rightarrow \lambda} (z - \lambda)[Iz - \widehat{P}]^{-1} = \frac{(\widehat{U}_\lambda s_0) \otimes (\nu_0 \widehat{U}_\lambda)}{\nu_0 \widehat{U}_\lambda^2 s_0}$$

and (i) holds with this \widehat{Q} , and $\lambda_o = \lambda$.

The equivalence of (i) and (iii) follows exactly as in Theorem 4.1(ii). \square

For a probabilistic kernel \widehat{P} , the following proposition says that V -uniformity implies that the chain with transition kernel \widehat{P} is geometrically ergodic. The converse is also true; see Proposition 4.10.

COROLLARY 4.7. *If \widehat{P} is a V -uniform, probabilistic kernel, then the Markov chain with transition kernel \widehat{P} is geometrically ergodic.*

PROOF. Since \widehat{P} is probabilistic, applying the limit result of Proposition 4.6(iii) to the constant function $\mathbf{1}$ implies that $\lambda = 1$ and that \check{f} is constant. By rescaling we can take $\check{f} = \mathbf{1}$ and $\check{\mu}$ to be a probability measure. From Theorem 2.5 it follows that \widehat{P} is geometrically ergodic. \square

Proposition 4.5 applied to the family of kernels $\{\widehat{P}_\alpha\}$ gives the following:

PROPOSITION 4.8. *Suppose that \widehat{P}_{α_0} is V -uniform for a given $\alpha_0 \in \mathbb{C}$. Then there exists $\varepsilon_0 > 0$ such that \widehat{P}_α is V -uniform (with associated g.p.e. λ_α) for all $\alpha \in \mathbb{C}$, $|\alpha - \alpha_0| < \varepsilon_0$. Moreover, for each such α there exist $\check{f}_\alpha \in L_\infty^V$ and $\check{\mu}_\alpha \in \mathcal{M}_1^V$ such that:*

- (i) \check{f}_α solves the multiplicative Poisson equation, $\widehat{P}_\alpha \check{f}_\alpha = \lambda_\alpha \check{f}_\alpha$;
- (ii) $\check{\mu}_\alpha$ is an eigenmeasure for \widehat{P}_α , $\check{\mu}_\alpha \widehat{P}_\alpha = \lambda_\alpha \check{\mu}_\alpha$;
- (iii) the g.p.e. λ_α is an analytic function of α and so is $\check{f}_\alpha(x)$ for any fixed $x \in X$.

PROOF. The existence of eigenvectors in (i) and (ii) is immediate from Proposition 4.5 when $\alpha = \alpha_0$. Define $\widehat{U}_z = \widehat{U}_{z, \alpha}$ by (56) with $\widehat{P} = \widehat{P}_\alpha$, and $\widehat{M} = s_0 \otimes \nu_0$:

$$(59) \quad \widehat{U}_{z, \alpha} := [Iz - (\widehat{P}_\alpha - s_0 \otimes \nu_0)]^{-1}.$$

From V -uniformity we know that \widehat{M} can be chosen so that \widehat{U}_{z,α_0} is a bounded linear operator for z in a neighborhood of λ_{α_0} . Since \widehat{P}_α is continuous in V -norm, it then follows that $\widehat{U}_{z,\alpha}$ is a bounded linear operator for (z, α) in a neighborhood O of $(\lambda_{\alpha_0}, \alpha_0)$. This combined with Proposition 4.5 proves (i) and (ii).

Write $\mathcal{J}(z, \alpha) = \nu_0(\widehat{U}_{z,\alpha}s_0)$, $z \in \mathbb{C}$, $\alpha \in O$, so that

$$\begin{aligned} \mathcal{J}(\lambda_\alpha, \alpha) &= 1, \\ \frac{\partial}{\partial z} \mathcal{J}(z, \alpha) \Big|_{z=\lambda_\alpha} &= \nu_0(\widehat{U}_{z,\alpha}^2 s_0) = \check{\mu}_\alpha(\check{f}_\alpha) \neq 0, \quad \alpha \in O, \end{aligned}$$

where \check{f}_α and $\check{\mu}_\alpha$ are the eigenfunction and the eigenmeasure given in Proposition 4.5. We conclude that λ_α is an analytic function by the implicit function theorem.

The proof that $\check{f}_\alpha(x)$ is analytic in α for $x \in S_V = X$ follows from the expansion

$$(60) \quad \check{f}_\alpha = \widehat{U}_{\lambda_\alpha, \alpha} s_0 = \sum_{n=0}^{\infty} \lambda_\alpha^{-n-1} (\widehat{P}_\alpha - s_0 \otimes \nu_0)^n s_0.$$

This expression for \check{f}_α converges uniformly for $\alpha \in O$ and for each n , the finite sum is analytic, which completes the proof of (iii). \square

The eigenfunction (60) will not in general satisfy the required normalization (48). The following eigenfunction and eigenmeasure do satisfy this condition, and are the *unique* solutions:

$$(61) \quad \check{\mu}_\alpha = \frac{\nu_0 \widehat{U}_{\lambda_\alpha, \alpha}}{\nu_0 \widehat{U}_{\lambda_\alpha, \alpha} \mathbf{1}} \in \mathcal{M}_1^V, \quad \check{f}_\alpha = \frac{\widehat{U}_{\lambda_\alpha, \alpha} s_0}{\check{\mu}_\alpha \widehat{U}_{\lambda_\alpha, \alpha} s_0} \in L_\infty^V.$$

Given such \check{f}_a and λ_a for some real a , we define the *twisted kernel* \check{P}_a by

$$\check{P}_a(x, dy) = \lambda_a^{-1} \check{f}_a^{-1}(x) \widehat{P}_a(x, dy) \check{f}_a(y)$$

[cf. (11) in the Introduction] and we let $\check{V}_a = V/\check{f}_a$. (As we will see below, \check{f}_a is bounded away from zero for real a in the range of interest.) The following proposition describes the relationship between the transition kernels, the eigenfunctions and the eigenmeasures $\{\check{P}_a, \check{f}_a, \check{\mu}_a : a \in \mathbb{R}\}$.

PROPOSITION 4.9. *Suppose that \widehat{P}_{a_0} is V -uniform for a given real a_0 . Then there is an open set $O \subset \mathbb{R}$ containing a_0 such that, for all $a \in O$, with \check{f}_a and $\check{\mu}_a$ given in (61) and with \check{P}_a equal to the associated twisted kernel, we have:*

- (i) *The operator \check{P}_a is \check{V}_a -uniform;*
- (ii) *$\frac{d}{da} \Lambda(a) = \frac{d}{da} \log(\lambda_a) = \check{\pi}_a(F)$, where $\check{\pi}_a$ is the invariant probability measure for \check{P}_a ;*

(iii) $\widehat{F}_a := \frac{d}{da} \log(\check{f}_a)$ is a solution to the Poisson equation,

$$(62) \quad \check{P}_a \widehat{F}_a = \widehat{F}_a - F + \pi_a(F).$$

For $a = 0$, this is the unique solution satisfying $\pi(\widehat{F}) = 0$:

(iv) $\frac{d}{da} \check{f}_a \in L_\infty^V$, $\frac{d}{da} \check{\mu}_a \in \mathcal{M}_1^V$.

(v) $\widehat{F}_a \in L_\infty^{1+\log(V_a)}$.

PROOF. The existence of O follows from Proposition 4.8, and from its proof we know that $\|\widehat{U}_{\lambda_a, a}\|_V < \infty$ when $a \in O$, where $\widehat{U}_{z, a}$ is given in (59).

The linear operators \widehat{P}_a and \check{P}_a are related by the scaling λ_a and a similarity transformation,

$$\check{P}_a = \lambda_a^{-1} (I_{\check{f}_a})^{-1} \widehat{P}_a I_{\check{f}_a},$$

where I_g , for an arbitrary function g , denotes the kernel $I_g(x, \cdot) := g(x)\delta_x(\cdot)$. Hence \widehat{P}_a is V -uniform if and only if \check{P}_a is $(I_{\check{f}_a}^{-1}V)$ -uniform. Result (i) immediately follows.

Consider the unnormalized eigenfunction given in (60). Differentiating the expression $\check{f}_a = \widehat{U}_{\lambda_a, a} s_0$ and applying the quotient rule gives

$$(63) \quad \begin{aligned} \check{f}'_a &= \frac{d}{da} [I\lambda_a - \widehat{P}_a + s_0 \otimes v_0]^{-1} s_0 \\ &= -\widehat{U}_{\lambda_a, a} [I\lambda'_a - I_F \widehat{P}_a] \widehat{U}_{\lambda_a, a} s_0 \\ &= -\widehat{U}_{\lambda_a, a} [(\lambda'_a - \lambda_a F) \check{f}_a] = \lambda_a \widehat{U}_{\lambda_a, a} I_{\check{f}_a} [F - \Lambda'(a)]. \end{aligned}$$

The right-hand side of (63) lies in L_∞^V since $F \in L_\infty$, $\check{f}_a \in L_\infty^V$ and $\widehat{U}_{\lambda_a, a}: L_\infty^V \rightarrow L_\infty^V$ is a bounded linear operator. This proves the first bound in (iv) since the two versions of \check{f}_a are related by a smooth normalization. The proof that $\frac{d}{da} \check{\mu}_a \in \mathcal{M}_1^V$ is identical.

Differentiating both sides of the eigenfunction equation gives

$$F\lambda_a \check{f}_a + \widehat{P}_a \check{f}'_a = \lambda'_a \check{f}_a + \lambda_a \check{f}'_a.$$

Dividing this identity by $\lambda_a \check{f}_a$ shows that (62) does indeed hold. To conclude that $\check{\pi}_a(F) = \Lambda'(a)$ we will show that $\check{\pi}_a(|\widehat{F}_a|) < \infty$. The invariant probability measure $\check{\pi}_a$ may be expressed as

$$\check{\pi}_a = k_a \check{\mu}_a I_{\check{f}_a},$$

where k_a is a normalizing constant. Hence,

$$\check{\pi}_a(\widehat{F}_a) = \check{\pi}_a\left(\left|\frac{\check{f}'_a}{\check{f}_a}\right|\right) = k_a \check{\mu}_a(|\check{f}'_a|) < \infty.$$

Finiteness follows from (iv) and the fact that the eigenmeasure $\check{\mu}_a$ lies in \mathcal{M}_1^V . This proves (ii) and the identity in (iii).

To complete the proof of (iii) we must show that $\pi(\widehat{F}_0) = 0$. This follows from the normalization (48) (assumed to hold for all a) which implies the limits

$$\check{f}_a \rightarrow \mathbf{1}, \quad \check{\mu}_a \rightarrow \pi, \quad a \rightarrow 0.$$

To prove (v) we obtain an alternative expression for \widehat{F}_a . We again consider the unnormalized eigenfunction (60). Observe that a fundamental kernel is derived from $\widehat{U}_{\lambda_a, a}$ through a scaling and a similarity transformation

$$Z_a = \lambda_a I_{\check{f}_a}^{-1} \widehat{U}_{\lambda_a, a} I_{\check{f}_a} = [I - \check{P}_a + s_a \otimes v_a]^{-1}$$

with $s_a = \lambda_a^{-1} \check{f}_a^{-1} s_0$ and $v_a = v_0 I_{\check{f}_a}$. We have $P_a Z_a G = Z_a G - G$ whenever $\pi_a(G) = 0$.

Using (63) then gives

$$\widehat{F}_a = \frac{\check{f}'_a}{\check{f}_a} = Z_a(F - \Lambda'(a)) = Z_a(F - \check{\pi}_a(F)).$$

It again follows that \widehat{F}_a solves the Poisson equation: It is the unique solution in L_∞^V with $v_a(\widehat{F}_a) = 0$.

The desired bound on \widehat{F}_α is obtained as follows. Using Jensen's inequality we know that $\widehat{V}_\alpha = \log(V/\check{f}_\alpha) = \log(\check{V}_\alpha)$ solves a version of (V3),

$$\check{P}_\alpha \widehat{V}_\alpha \leq \widehat{V}_\alpha - \varepsilon + bs,$$

where $\varepsilon > 0$ and b is a finite constant. Using the bound $\check{f}_\alpha \in L_\infty^V$ it follows directly that the function \widehat{V}_α is uniformly bounded below. The bound on \widehat{F}_α then follows from [23], Theorem 2.3. \square

PROPOSITION 4.10. *Suppose that Assumptions 1 and 2 hold. Take $\widehat{P} = R$ and define the potential operator \widehat{U}_z as in (56) with $\widehat{M} = s \otimes v$. Then $B_1 := \|\widehat{U}_1\|_V \leq 2b\delta^{-1}$ and*

$$\|\widehat{U}_z\|_V \leq B_1(1 - |z - 1|B_1)^{-1}, \quad |z - 1| \leq B_1^{-1}, \quad z \in \mathbb{C}.$$

Hence both R and P are V -uniform.

PROOF. Under (V4) we have, by (18),

$$(e - 1)(R - I)V = R(P - I)V \leq -\delta RV + bRs.$$

Rearranging terms then gives

$$(R - I)V \leq -\left(\frac{\delta}{e - 1 + \delta}\right)V + b\left(\frac{1}{e - 1 + \delta}\right)Rs,$$

which we write as

$$(R - s \otimes v)V \leq V - \delta_1 V - v(V)s + b_1 Rs,$$

where $\delta_1 = \delta(e - 1 + \delta)^{-1}$ and $b_1 = b(e - 1 + \delta)^{-1}$.

Iterating gives, for all $n \geq 1$,

$$(R - s \otimes v)^n V \leq V - \delta_1 \sum_{i=0}^{n-1} (R - s \otimes v)^i V + \sum_{i=0}^{n-1} (R - s \otimes v)^i (b_1 Rs - v(V)s).$$

Letting $n \rightarrow \infty$ and applying Lemma 3.2(ii) yields

$$\delta_1 \widehat{U}_1 V \leq V - v(V) + 2b_1 \leq 2b_1 V$$

or $\|\widehat{U}_1\|_V \leq 2b_1/\delta_1 = 2b/\delta$.

To obtain a bound for $z \sim 1$ write

$$\widehat{U}_z = [I(z - 1) + [I - (R - s \otimes v)]]^{-1} = [\widehat{U}_1(z - 1) + I]^{-1} \widehat{U}_1.$$

Provided $\|\widehat{U}_1\|_V |z - 1| < 1$, we can write $\widehat{U}_z = \sum_n (1 - z)^n \widehat{U}_1^{n+1}$ and $\|\widehat{U}_z\|_V \leq B_1/[1 - |z - 1|B_1]$. \square

PROPOSITION 4.11. *Suppose that Assumptions 1 and 2 hold, let $a \in \mathbb{R}$ satisfy $|a| \leq |\log(1 - \delta)|$ and suppose that there exists $g : X \rightarrow (0, \infty)$, satisfying $g \in L_\infty^V$ and $\widehat{P}_a g \leq \lambda_a g$. Then \widehat{P}_a is V -uniform.*

PROOF. The conditions of the proposition imply that there exists $b_1 < \infty$ such that

$$\widehat{P}_a V \leq e^{|a|}(1 - \delta)V + e^{|a|}bs \leq V + b_1s.$$

From the resolvent equation (18) we then have, for some $b_2 < \infty$,

$$\widehat{R}_\theta V \leq V + b_2 \widehat{R}_\theta s,$$

where \widehat{R}_θ is the resolvent kernel defined through \widehat{P}_a .

We also have $\Lambda(a) > 0$ for all $a \neq 0$ under Assumptions 1 and 2 and hence the g.p.e. γ_θ for \widehat{R}_θ is also strictly greater than 1 when $\theta > |a| > 0$ [see (37)]. Choosing $s_0 \in \mathcal{B}_p^+$ and $v_0 \in \mathcal{M}_p^+$ so that $R_\theta \geq s_0 \otimes v_0$, we find that

$$\gamma_\theta^{-1}(\widehat{R}_\theta - s_0 \otimes v_0)V \leq V - \varepsilon V + b_2 \widehat{R}_\theta s_0,$$

where $\varepsilon = 1 - \gamma_\theta^{-1} > 0$. Exactly as in the proof of Proposition 4.10 we conclude that

$$\varepsilon \widehat{U}_{\gamma_\theta} V \leq b_2 \widehat{U}_{\gamma_\theta} \widehat{R}_\theta s_0 \leq 2b_2 \widehat{U}_{\gamma_\theta} s_0,$$

where $\widehat{U}_z = \sum z^{-n-1}(\widehat{R}_\theta - s_0 \otimes v_0)^n$.

From Theorem 3.3(iii) and the conditions of the proposition we know that $\check{f} = \widehat{U}_{\gamma_\theta} s_0$ satisfies $\check{f} \leq cg$ for some constant c and hence $\check{f} \in L_\infty^V$. It follows that $\|\widehat{U}_{\gamma_\theta}\|_V < \infty$, from which V -uniformity of \widehat{R}_θ , and hence of \widehat{P}_a , immediately follow. \square

PROPOSITION 4.12. *Suppose that Assumptions 1 and 2 hold. Then there exists $\varepsilon_0 > 0$, $\bar{b} < \infty$ such that:*

- (i) $S_0 \cap \{z \in \mathbb{C} : |z| \geq 1 - \varepsilon_0\} = \{1\}$;
- (ii) $\| [Iz - (P - \mathbf{1} \otimes \pi)]^{-1} \|_V \leq \bar{b}$ when $|z| \geq 1 - \varepsilon_0$;
- (iii) \widehat{P}_α is V -uniform for all $\alpha = a + i\omega \in \mathbb{C}$ satisfying

$$|\omega| \leq \varepsilon_0 \quad \text{and} \quad |a| \leq \bar{a} := \left(\frac{e-1}{2b-\delta} \right) \delta;$$

moreover, the associated g.p.e. λ_α is an analytic function of α in this range, and so is the corresponding eigenfunction $\check{f}_\alpha(x)$ (for each fixed $x \in \mathbf{X}$);

- (iv) the eigenfunctions \check{f}_a are (uniformly) bounded from below when a is real:

$$\inf_{-\bar{a} \leq a \leq \bar{a}} \check{f}_a(x) > 0, \quad x \in \mathbf{X}.$$

PROOF. Results (i) and (ii) follow immediately from Proposition 4.10. To prove (iii) we must establish an appropriate range of real a for which \widehat{P}_a is V -uniform. From Proposition 4.10 we know that $P = \widehat{P}_0$ is V -uniform.

For any function $G_0 \in L_\infty$, set $g_0 = \exp(G_0)$ and consider the kernel $I_{g_0}R$, where, as before, I_{g_0} denotes the kernel $I_{g_0}(x, \cdot) := g_0(x)\delta_x(\cdot)$. We assume that the convergence parameter for this kernel is equal to 1. It then follows from Proposition 4.10 that the function

$$\check{g}_r(x) = \sum_{k=0}^{\infty} [I_{g_0}(R - s \otimes \nu)]^k I_{g_0}s$$

lies in L_∞^V provided $\|g_0\|_\infty^{-1} > 1 - B_1^{-1}$ and it is clear that $I_{g_0}R$ is in fact V -uniform in this case. Applying Lemma 3.2(i) we know that $\nu(\check{g}_r) = 1$.

The function \check{g}_r solves the eigenfunction equation

$$[I_{g_0}(R - s \otimes \nu)]\check{g}_r = \check{g}_r - I_{g_0}s \quad \implies \quad R\check{g}_r = g_0^{-1}\check{g}_r.$$

Setting $\check{g} = R\check{g}_r = \check{g}_r g_0^{-1}$ and applying the resolvent equation (18) then gives

$$(P - I)\check{g} = (P - I)R\check{g}_r = (e - 1)(R - I)\check{g}_r = (e - 1)\check{g} - (e - 1)g_0\check{g}.$$

Hence \check{g} is the solution to the multiplicative Poisson equation for the function $G = \log(g) := -\log(e - (e - 1)g_0)$. The map $g_0 \mapsto g$ is one to one.

We have already remarked that $\check{g}_r \in L_\infty^V$ provided $\|g_0\|_\infty^{-1} > 1 - B_1^{-1}$, and hence $\check{g} \in L_\infty^V$ whenever g_0 satisfies this bound. If $B_1 \leq e$, then this constraint is trivially satisfied. For $B_1 > e$, equivalently the function g must satisfy

$$(64) \quad \|g\|_\infty < \frac{1}{e - (e - 1)(1 - B_1^{-1})^{-1}} = \frac{B_1 - 1}{B_1 - e}.$$

From the inequality $\log(1 + x) < x$, $x \neq 0$, we obtain

$$\log\left(\frac{B_1 - 1}{B_1 - e}\right) = -\log\left(1 - \frac{e - 1}{B_1 - 1}\right) > \left(\frac{e - 1}{B_1 - 1}\right) \geq \left(\frac{e - 1}{2b - \delta}\right)\delta,$$

where the last inequality uses the bound $B_1 \leq 2b/\delta$.

This gives the sufficient condition $\|G\|_\infty \leq \bar{a}$. Proposition 4.11 implies that $I_g P$ is V -uniform and $\check{g} \in L_\infty^V$ when this uniform bound holds.

The function G falls outside of the class of functions F satisfying Assumptions 1 and 2, since $\Lambda(a) > 0$ for all $a \neq 0$ when $\pi(F) = 0$, and we have already noted that the spectral radius $\xi(g)$ of $I_g P$ is equal to 1. However, given any a , the function $G = aF - \Lambda(a)$ satisfies $\xi(g) = 1$ and $G(x) \leq |a| - \Lambda(a) < |a|$, $x \in X$, so that the normalized function satisfies (64) when $|a| \leq \bar{a}$. This transformation immediately gives the desired conclusion in (iii).

To see (iv), take any $\bar{\lambda}$ satisfying $\bar{\lambda} \geq \max(\lambda_{\bar{a}}, \lambda_{-\bar{a}})$ and set

$$\widehat{G}_a = \sum_0^\infty \bar{\lambda}^{-n-1} \widehat{P}_a^n.$$

By irreducibility we can find $s_0 : X \rightarrow (0, 1)$ and a probability distribution ν_0 on \mathcal{B} satisfying the uniform bound,

$$\widehat{G}_a(x, A) \geq R_\theta(x, A) \geq s_0(x)\nu_0(A), \quad x \in X, A \in \mathcal{B}, a \in [-\bar{a}, \bar{a}],$$

where $\theta = \bar{a} + \log(\bar{\lambda})$. We may assume that ν_0 is equivalent to the irreducibility measure ψ .

It follows that for all $a \in [-\bar{a}, \bar{a}]$ and all x ,

$$(\bar{\lambda} - \lambda_a)^{-1} \check{f}_a(x) = \widehat{G}_a \check{f}_a(x) \geq s_0(x)\nu_0(\check{f}_a) > 0.$$

By continuity of \check{f}_a we obtain the desired uniform bound. \square

We now develop the consequences of the lattice condition. Our main conclusion is contained in Theorem 4.13: The function F is almost-lattice *if and only if* the spectral radius $\hat{\xi}_{i\omega}$ attains its upper bound (i.e., $\hat{\xi}_{i\omega} = 1$) for some $\omega > 0$.

Some of the spectral theory for complex α is most easily developed in a Hilbert space setting. Define $L_2 := \{f : X \rightarrow \mathbb{C} \text{ such that } \|f\|_2^2 = \pi(|f|^2) < \infty\}$ with the natural associated inner product $\langle h, g \rangle = \pi(h^*g)$, $h, g \in L_2$. We note that $V \in L_2$ under our standing assumptions, Assumptions 1 and 2. For any n , the induced operator norm of $\widehat{P}_\alpha^n : L_2 \rightarrow L_2$ may be expressed as

$$\begin{aligned} \|\widehat{P}_\alpha^n\|_2 &= \sup \frac{\|\widehat{P}_\alpha^n g\|_2}{\|g\|_2} \\ &= \sup \{|\mathbb{E}_\pi[h^*(\Phi(0)) \exp(\alpha S_n)g(\Phi(n))]| : \|h\|_2 \leq 1, \|g\|_2 \leq 1\}. \end{aligned}$$

We let $\hat{\gamma}_\alpha$ denote the L_2 -spectral radius,

$$\hat{\gamma}_\alpha := \lim_{n \rightarrow \infty} \|\widehat{P}_\alpha^n\|_2^{1/n}.$$

When $\alpha = i\omega$, the linear operators $\{\widehat{P}_{i\omega}^n\}$ are contractions on L_2 , so that $\widehat{\gamma}_{i\omega} \leq 1$.

Theorem 4.13 provides several characterizations of the almost-lattice condition. It is analogous to the variance characterization given in Proposition 2.4.

THEOREM 4.13 (Characterization of the lattice condition). *The following are equivalent under Assumptions 1 and 2 for any given $\omega > 0$, $-\bar{a} \leq a \leq \bar{a}$:*

- (i) $\widehat{\xi}_{i\omega} = 1$;
- (ii) $\widehat{\gamma}_{i\omega} = 1$;
- (iii) $\widehat{\xi}_{a+i\omega} = \widehat{\xi}_a$;
- (iv) *there exists a bounded function $\Theta : X \rightarrow [0, 2\pi)$ and $d_0 > 0$ such that for a.e. $x \in X$ $[\psi]$,*

$$(65) \quad \begin{aligned} & \exp\left(i\omega \int_{[0,t)} (F(\Phi(s)) - d_0) ds\right) \\ & = \exp(i\Theta(\Phi(t)) - i\Theta(\Phi(0))) \quad \text{a.s. } [P_x]; \end{aligned}$$

- (v) *F is an almost-lattice function whose span is an integer multiple of $2\pi/\omega$.*

PROOF. We first note that by Proposition 2.4 the existence of Θ , ω and d_0 satisfying (iv) is equivalent to the almost-lattice condition (v). To prove the proposition it remains to show that (i)–(iv) are equivalent.

The implications (iv) \Rightarrow (i), (ii), (iii) are obvious since, under (iv), we have for all $n \geq 1$,

$$\widehat{P}_{a+i\omega}^n(x, \cdot) = \exp(ind_0) I_{i\Theta} \widehat{P}_a^n(x, \cdot) I_{i\Theta}^{-1} \quad \text{for a.e. } x \in X [\psi].$$

We now establish implication (i) \Rightarrow (ii). We first note that if $\widehat{\xi}_{i\omega} = 1$ then, from the fact that the V -norm is submultiplicative (as it is an operator norm), we must have $\|\widehat{P}_{i\omega}^n\|_V \geq 1$ for all n . Note also that for any $g \in L_\infty^V$,

$$\begin{aligned} |\widehat{P}_{i\omega}^{n+m} g(x)| &= |\mathbf{E}_x[\exp(\alpha S_n) \mathbf{E}_{\Phi(n)}[\exp(\alpha S_m) g(\Phi(m))]]| \\ &\leq \mathbf{E}_x[|\mathbf{E}_{\Phi(n)}[\exp(\alpha S_m) g(\Phi(m))]]| \\ &\leq \int |\mathbf{E}_y[\exp(\alpha S_m) g(\Phi(m))]| \pi(dy) \\ &\quad + O(V(x) \exp(-b_0 n)), \quad n, m \geq 1, x \in X, \end{aligned}$$

where $b_0 > 0$ exists by V -uniformity of P . This implies the bound

$$(66) \quad \begin{aligned} 1 &= \widehat{\xi} \\ &\leq \liminf_{m \rightarrow \infty} (\sup\{|\mathbf{E}_\pi[h^*(\Phi(0)) \exp(\alpha S_m) g(\Phi(m))]| : \|h\|_\infty \leq 1, \|g\|_V \leq 1\}). \end{aligned}$$

We have already remarked that \widehat{P}_α is a contraction on L_2 . It follows that either

$\|\widehat{P}_\alpha^n\|_2 \rightarrow 0$ geometrically fast or $\|\widehat{P}_\alpha^n\|_2 = 1$ for all n . We may conclude the latter using (66), and this establishes the implication (i) \Rightarrow (ii).

We now show that (ii) implies (iv). The supremum in the definition of $\|\widehat{P}_{i\omega}^n\|_2$ is attained since $\hat{\gamma}_{i\omega} = 1$. To see this, construct for any $N \geq 1$ functions h^N and g^N with L_2 -norm equal to 1 with

$$1 \geq \|h^N\|_2 \|\widehat{P}_{i\omega}^n g^N\|_2 \geq \langle h^N, \widehat{P}_{i\omega}^n g^N \rangle \geq \|\widehat{P}_\alpha^n\|_2 - 1/N = 1 - 1/N.$$

Part of the construction ensures that the inner product above is real-valued. These bounds imply that $\|h^N - \widehat{P}_{i\omega}^n g^N\|_2 \rightarrow 0, N \rightarrow \infty$, which is equivalently expressed as

$$E_\pi[(h^N(\Phi(0)))^* \exp(i\omega S_n) g^N(\Phi(n)) - 1] \rightarrow 0, \quad N \rightarrow \infty.$$

It then follows that $\exp(i\omega S_n) \in \sigma(\Phi(0), \Phi(n))$ and that there exist $h_n, g_n \in L_2$ such that

$$(67) \quad h_n^*(\Phi(0)) \exp(i\omega S_n) g_n(\Phi(n)) = 1 \quad \text{a.s. } [\mathbf{P}_\pi].$$

We may assume without loss of generality that $|g_n(x)| = |h_n(x)| = 1$ for all x since $|\exp(i\omega S_n)| = 1$.

Note that (67) is almost the desired conclusion (iv). In particular, on dividing the expressions for n and $(n + 1)$ we obtain the suggestive identity

$$(68) \quad \exp(i\omega F(\Phi(n))) = \left(\frac{h_{n+1}(\Phi(0))}{h_n(\Phi(0))} \right) \left(\frac{g_n(\Phi(n))}{g_{n+1}(\Phi(n+1))} \right), \quad n \geq 0.$$

To establish (iv) we show that $\{g_n, h_n\}$ may be chosen as follows: $\{g_n\}$ is independent of n , with common value $g \in L_\infty$, and we may construct $\theta_0 \in \mathbb{R}$ such that $h_n = e^{i\theta_0 n} g$ for all n . The required function Θ in (iii) can then be taken as a version of $-\log(g)$.

Applying (67) and appealing to stationarity, we conclude that for any $n, m \geq 1$,

$$h_{n+m}^*(\Phi(0)) \exp(i\omega S_{n+m}) g_{n+m}(\Phi(n+m)) = 1 \quad \text{a.s. } [\mathbf{P}_\pi]$$

and

$$\begin{aligned} & h_m^*(\Phi(n)) \exp \left\{ \left(i\omega \sum_{k=n}^{n+m-1} F(\Phi(k)) \right) g_m(\Phi(n+m)) \right\} \\ &= \vartheta^n [h_m^*(\Phi(0)) \exp(i\omega S_m) g_m(\Phi(m))] \\ &= 1 \quad \text{a.s. } [\mathbf{P}_\pi], \end{aligned}$$

where ϑ^n denotes the n -fold shift operator on the sample space.

Combining (67) with these two identities then gives

$$h_{n+m}^*(\Phi(0)) h_n(\Phi(0)) g_n^*(\Phi(n)) h_m(\Phi(n)) g_m^*(\Phi(n+m)) g_{n+m}(\Phi(n+m)) = 1.$$

On taking conditional expectations with respect to $\Phi(0) = x$ we see that for a.e. $x \in \mathbf{X}$ [ψ],

$$\begin{aligned} h_{n+m}^*(x)h_n(x) &= \mathbb{E}_\pi[g_n(\Phi(n))h_m^*(\Phi(n))g_m(\Phi(n+m))g_{n+m}^*(\Phi(n+m))] \\ &\quad + O(V(x)\exp(-b_0n)) \\ &= \pi(g_n h_m^*)\pi(g_m g_{n+m}^*) + O(V(x)\exp(-b_0n) + \exp(-b_0m)), \quad n, m \geq 1. \end{aligned}$$

Since $|g_n(x)| = |h_n(x)| = 1$ for all x we conclude from Jensen's inequality that for all $n, k \geq 1$,

$$g_n^*(x)h_{n+k}(x) = \pi(g_n^* h_{n+k}) + \epsilon_1(x),$$

$$h_{n+k}^*(x)h_n(x) = \pi(h_{n+k}^* h_n) + \epsilon_2(x),$$

where $|\epsilon_1(x)| + |\epsilon_2(x)| = O(V(x)\exp(-b_0n))$. This, combined with (68), shows that the desired expression can be obtained as an approximation: For any $\epsilon > 0$ we can find a function Θ [of the form $-\log(g_n)$ for large n] and $\theta_0 \in \mathbb{R}$ such that for a.e. $\Phi(0) = x \in \mathbf{X}$,

$$|\exp(i\omega F(\Phi(0))) - \exp(i(\theta_0 + \Theta(\Phi(1)) - \Theta(\Phi(0))))| \leq \epsilon V(x) \quad \text{a.s. } [\mathbf{P}_x].$$

This easily gives (iv).

Finally we show that (iii) implies (iv). Observe first that we have already established the equivalence of (i) and (iv). Moreover, (iii) is equivalent to the statement (i) for the transition kernel \check{P}_a , from which we deduce the implication (iii) \Rightarrow property (iv) for the Markov chain with transition law \check{P}_a . This is equivalent to (iv) for the original Markov chain. \square

4.2. Continuous time. We now translate the definitions and results of the previous section to the continuous-time case. Suppose that $\{\widehat{P}^t : t \in \mathbb{R}_+\}$ is a semigroup of operators on L_∞^V with generator $\widehat{\mathcal{A}}$ and with finite spectral radius given by

$$\widehat{\xi} := \lim_{t \rightarrow \infty} \|\widehat{P}^t\|_V^{1/t}.$$

[Note that the definition of the generator of a positive semigroup given in (32) immediately generalizes to general (not necessarily positive) semigroups.]

Consider the eigenvector equation $\widehat{\mathcal{A}}h = \Lambda h$. The functions h we consider will always be of the form $h = \widehat{R}_\theta h_0$, usually with $h_0 \geq 0$, where \widehat{R}_θ is defined as in (18). When all the integrals are well defined we have the resolvent equation (33), so that

$$\widehat{\mathcal{A}}h = \theta(\widehat{R}_\theta - I)h_0.$$

This identity allows us to lift all of the previous results to the continuous-time setting. In particular, under (V4), Fatou’s lemma implies that the resolvent $R = R_1$ satisfies

$$RV \leq (1 - \delta_1)V + b_1Rs$$

with $\delta_1 = \delta(1 + \delta)^{-1}$ and $b_1 = b(1 + \delta)^{-1}$. It then follows as in the discrete time case that

$$\| [I - R + s \otimes v]^{-1} \|_V \leq 2b_1/\delta_1.$$

Recall the definition of the semigroup $\{\widehat{P}_\alpha^t : t \in \mathbb{R}_+\}$ from (31), where we now allow α to be possibly complex. The next lemma offers an expression for the generator of this semigroup, analogous to the classical Feynman–Kac formula for diffusions. The result is easy to check via the martingale representation (22).

LEMMA 4.14 (Feynman–Kac formula). *The generator $\widehat{\mathcal{A}}_\alpha$ of the semigroup $\{\widehat{P}_\alpha^t : t \in \mathbb{R}_+\}$ satisfies*

$$(69) \quad \widehat{\mathcal{A}}_\alpha = \mathcal{A} + \alpha F,$$

where \mathcal{A} is the generator of $\{P^t\}$.

Although none of the generators we consider is a linear operator on L_∞^V , we may still define the *spectrum* of $\widehat{\mathcal{A}}$, $\mathcal{S}(\widehat{\mathcal{A}}) \subset \mathbb{C}$, as the set of $z \in \mathbb{C}$ such that the inverse $[Iz - \widehat{\mathcal{A}}]^{-1}$ does not exist as a bounded linear operator. However, we relax the definition by restricting the function space on which the inverse is defined: We say $z \in \mathcal{S}(\widehat{\mathcal{A}})$ if and only if $[Iz - \widehat{\mathcal{A}}]^{-1}$ does not exist as a bounded linear operator on the range of the resolvent $\{Rg : g \in L_\infty^V\}$. We have the generalized resolvent equation

$$z[Iz - \widehat{\mathcal{A}}]^{-1} = \widehat{R}_z = \int_{[0, \infty)} ze^{-zt} \widehat{P}^t dt, \quad z \in \mathbb{C},$$

where the integral converges in norm for $z \notin \mathcal{S}(\widehat{\mathcal{A}})$ such that $|e^z| \geq \widehat{\xi}$. The generator $\widehat{\mathcal{A}}$ is called *V-uniform* if it admits a spectral gap and there is a unique pole $\Lambda_\circ \in \mathcal{S}(\widehat{\mathcal{A}})$ of multiplicity 1, satisfying $|\lambda_\circ| = \exp(\Lambda_\circ) = \widehat{\xi}$.

PROPOSITION 4.15. *If $\{\widehat{P}^t : t \in \mathbb{R}_+\}$ has finite spectral radius $\widehat{\xi}$, then:*

(i) *The following statements are equivalent:*

- (a) *The generator $\widehat{\mathcal{A}}$ has eigenvalue $\Lambda_\circ \in \mathbb{C}$ and associated eigenfunction $\check{f} \in L_\infty^V$.*
- (b) *For $\theta > \widehat{\xi}$, the resolvent \widehat{R}_θ has eigenvalue $\lambda_\theta = \theta^{-1}\Lambda_\circ - 1$ and eigenfunction \check{f} .*

(ii) *The following statements are equivalent:*

- (a) $\widehat{\mathcal{A}}$ is V -uniform.
- (b) \widehat{R}_θ is V -uniform.

The proof of Proposition 4.15 is obvious from (34).

Using these identities, the following results may be proved as in Theorems 4.1 and 4.2. The definition of \bar{a} is given in (49).

THEOREM 4.16 (Multiplicative mean ergodic theorem). *Suppose that the Markov process $\Phi = \{\Phi(t) : t \in \mathbb{R}_+\}$ and the functional F satisfy Assumptions 1 and 2, and write $\bar{a} = (\frac{e-1}{2b-\delta})$ as before. Then there exists $\bar{\omega} > 0$ and $\delta_0 > 0$ such that for any $\alpha = a + i\omega \in \mathbb{C}$ with $|a| \leq \bar{a}$ and $|\omega| \leq \bar{\omega}$, there exists $\Lambda(\alpha) \in \mathcal{S}_\alpha$ which is maximal and isolated:*

$$\begin{aligned} \operatorname{Re}(\Lambda(\alpha)) &= \max\{\operatorname{Re}(\Lambda) : \Lambda \in \mathcal{S}_\alpha\} \quad \text{and} \\ \mathcal{S}_\alpha \cap \{z : \operatorname{Re}(z) \leq \operatorname{Re}(\Lambda(\alpha)) - \delta_0\} &= \Lambda(\alpha). \end{aligned}$$

Moreover, for any such α , there exist $\check{f}_\alpha \in L_\infty^V$ and $\check{\mu}_\alpha \in \mathcal{M}_1^V$, satisfying (48), and the following statements hold:

- (i) For all $x \in \mathbf{X}$, $A \in \mathcal{B}$ and $t \in \mathbb{R}_+$,

$$\begin{aligned} \widehat{P}_\alpha^t \check{f}_\alpha(x) &= \lambda_\alpha^t \check{f}_\alpha(x), \\ \check{\mu}_\alpha \widehat{P}_\alpha^t(A) &= \lambda_\alpha^t \check{\mu}_\alpha(A). \end{aligned}$$

The function \check{f}_α is also an eigenfunction for $\widehat{\mathcal{A}}_\alpha$:

$$\widehat{\mathcal{A}}_\alpha \check{f}_\alpha = \Lambda(\alpha) \check{f}_\alpha.$$

- (ii) There exist $b_0 > 0$ and $B_0 < \infty$ such that for all $x \in \mathbf{X}$, $t > 0$,

$$\begin{aligned} |\mathbb{E}_x[\exp(\alpha S_t - t\Lambda(\alpha))g(\Phi(t))] - \widehat{Q}_\alpha g(x)| &\leq B_0 \|g\|_V e^{-b_0 t} V(x), \\ |\mathbb{E}_x[\exp(\alpha S_t - t\Lambda(\alpha))] - \check{f}_\alpha(x)| &\leq B_0 |\alpha| e^{-b_0 t} V(x). \end{aligned}$$

THEOREM 4.17 (Bounds around the $i\omega$ axis). *Assume that the Markov process $\Phi = \{\Phi(t) : t \in \mathbb{R}_+\}$ and the functional F satisfy Assumptions 1 and 2.*

(NL) *Suppose that F is strongly nonlattice. For any $0 < \omega_0 < \omega_1 < \infty$, there exist $b_0 > 0$ and $B_0 < \infty$ (possibly different than above) such that*

$$(70) \quad |\mathbb{E}_x[\exp(\alpha S_t - t\Lambda(a))]| \leq B_0 V(x) \exp(-b_0 t), \quad x \in \mathbf{X}, t > 0,$$

for all $\alpha = a + i\omega$ with $|a| \leq \bar{a}$ and $\omega_0 \leq |\omega| \leq \omega_1$.

(L) *Suppose that F is almost-lattice with span $h > 0$. For any $\epsilon > 0$, there exist $b_0 > 0$ and $B_0 < \infty$ (possibly different than above) such that (70) holds for all $\alpha = a + i\omega$ with $|a| \leq \bar{a}$ and $\epsilon \leq |\omega| \leq 2\pi/h - \epsilon$.*

5. Edgeworth expansions for the CLT. Here we show how the multiplicative mean ergodic theorems of the previous section can be used to obtain Edgeworth expansions for the central limit theorem (CLT) satisfied by the partial sums of a geometrically ergodic Markov chain; see, for example, [37], Chapter 17, for the standard CLT.

Throughout this section we consider a discrete-time Markov chain Φ and a bounded functional $F : X \rightarrow \mathbb{R}$. Recall our standing assumptions, Assumptions 1 and 2, about Φ and F . To avoid repetitions later on, we collect below a number of properties that will be used repeatedly in the proofs of the results in this and the following section. They are proved in the Appendix.

Properties. Assume that the discrete-time Markov chain Φ and the function $F : X \rightarrow \mathbb{R}$ satisfy Assumptions 1 and 2 and let S_n denote the partial sums as before. Choose and fix an arbitrary $x \in X$ and let

$$(71) \quad m_n(\alpha) := \mathbb{E}_x[\exp(\alpha S_n)], \quad n \geq 1, \alpha \in \mathbb{C}.$$

P1. There is a sequence $\{\epsilon_n\}$ such that

$$m_n(\alpha) = \exp(n\Lambda(\alpha))[\check{f}_\alpha(x) + |\alpha|\epsilon_n], \quad n \geq 1,$$

and $|\epsilon_n| \rightarrow 0$ exponentially fast as $n \rightarrow \infty$, uniformly over all $\alpha \in \Omega$ (with Ω as in Theorem 4.1).

P2. If F is strongly nonlattice, then for any $0 < \omega_0 < \omega_1 < \infty$ there is a sequence $\{\epsilon'_n\}$ such that

$$m_n(\alpha) = \exp(n\Lambda(a))\epsilon'_n, \quad n \geq 1,$$

and $|\epsilon'_n| \rightarrow 0$ exponentially fast as $n \rightarrow \infty$, uniformly over all $\alpha = a + i\omega$ with $|a| \leq \bar{a}$ and $\omega_0 \leq |\omega| \leq \omega_1$ (with \bar{a} as in Theorem 4.1).

P3. If F is lattice (or almost-lattice) with span $h > 0$, then for any $\epsilon > 0$, as $n \rightarrow \infty$,

$$\sup_{\epsilon \leq |\omega| \leq 2\pi/h - \epsilon} |m_n(i\omega)| \rightarrow 0 \quad \text{exponentially fast.}$$

P4. The function $\Lambda(\cdot)$ is analytic in Ω with $\Lambda(0) = \Lambda'(0) = 0$ and $\Lambda''(0) = \sigma^2 > 0$. Moreover, $\sigma_a^2 := \Lambda''(a)$ is strictly positive for all real $a \in [-\bar{a}, \bar{a}]$.

P5. The third derivative $\rho_3 := \Lambda'''(0)$ can be expressed as

$$\begin{aligned} \rho_3 &= \mathbb{E}_\pi[F^3(\Phi(0))] + 3 \sum_{\substack{i=-\infty \\ i \neq 0}}^{\infty} \mathbb{E}_\pi[F^2(\Phi(0))F(\Phi(i))] \\ &\quad + 6 \sum_{i,j=1}^{\infty} \mathbb{E}_\pi[F(\Phi(0))F(\Phi(i))F(\Phi(i+j))]. \end{aligned}$$

P6. Let \widehat{F} be the solution of the Poisson equation given by (27) and write

$$\Delta_n := \mathbf{E}_x[S_n] - \widehat{F}(x).$$

Then $|\Delta_n| \rightarrow 0$ exponentially fast as $n \rightarrow \infty$.

P7. The eigenfunction \check{f}_α is analytic in $\alpha \in \Omega$, it satisfies $\check{f}_\alpha|_{\alpha=0} \equiv 1$ and it is strictly positive for real α . Moreover, there is some $\bar{\omega}_0 \in (0, \bar{\omega}]$ (depending on x) such that

$$\delta(i\omega) := |\log \check{f}_{i\omega}(x) - i\omega \widehat{F}(x)| \leq (\text{Const})\omega^2,$$

for all $|\omega| \leq \bar{\omega}_0$, where \widehat{F} is as in P6.

The following two results generalize those in [41, 30, 8].

THEOREM 5.1 (Edgeworth expansion for nonlattice functionals). *Suppose that Φ and the strongly nonlattice functional F satisfy Assumptions 1 and 2 and let $G_n(y)$ denote the distribution function of the normalized partial sums $S_n/\sigma\sqrt{n}$:*

$$G_n(y) := \mathbf{P}_x \left\{ \frac{S_n}{\sigma\sqrt{n}} \leq y \right\}, \quad y \in \mathbb{R}.$$

Then, for all $x \in \mathbf{X}$,

$$(72) \quad G_n(y) = \mathcal{G}(y) + \frac{\gamma(y)}{\sigma\sqrt{n}} \left[\frac{\rho_3}{6\sigma^2}(1-y^2) - \widehat{F}(x) \right] + o(n^{-1/2}), \quad n \rightarrow \infty,$$

uniformly in $y \in \mathbb{R}$, where $\gamma(y)$ denotes the standard Normal density and $\mathcal{G}(y)$ is the corresponding distribution function.

It is perhaps worth noting the way in which the convergence in (72) depends on the initial state x of the Markov chain: This dependence is only manifested via the solution $\widehat{F}(x)$ to the Poisson equation. Also observe that, since (72) holds for all $y \in \mathbb{R}$, the restriction on F being $|F| \leq 1$ can clearly be relaxed to $\|F\|_\infty < \infty$.

For the proof of the theorem—given in the Appendix—it is convenient to consider the zero-mean version of the normalized partial sums,

$$\frac{\bar{S}_n}{\sigma\sqrt{n}} := \frac{S_n - \mathbf{E}_x\{S_n\}}{\sigma\sqrt{n}}.$$

Let $\bar{G}_n(y)$ denote the corresponding distribution function. In the proof we show instead that

$$(73) \quad \bar{G}_n(y) = \mathcal{G}(y) + \frac{\rho_3}{6\sigma^3\sqrt{n}}(1-y^2)\gamma(y) + o(n^{-1/2}), \quad n \rightarrow \infty,$$

uniformly in $y \in \mathbb{R}$. From this it is a straightforward calculation to deduce (72) via a Taylor series expansion and using property P6.

Before stating our next result we recall the following notation. If G is the distribution function of a lattice random variable with values on the lattice

$\{d + kh, k \in \mathbb{Z}\}$, the polygonal approximation $G^\#$ to G is the piecewise-linear distribution function $G^\#(y)$ that agrees with $G(y)$ at the midpoints of the lattice, $y = d + (k + 1/2)h, k \in \mathbb{Z}$, and is linearly interpolated between these points. The function $G^\#$ is precisely the convolution of G with the uniform distribution on $[-h/2, h/2]$.

THEOREM 5.2 (Edgeworth expansion for lattice functionals). *Suppose that F is a lattice functional with span $h > 0$, and assume that F and Φ satisfy Assumptions 1 and 2. With $G_n(y)$ as in Theorem 5.1, let $G_n^\#(y)$ denote its polygonal approximation. Then, for all $x \in \mathbb{X}$,*

$$(74) \quad G_n^\#(y) = \mathcal{G}(y) + \frac{\gamma(y)}{\sigma\sqrt{n}} \left[\frac{\rho_3}{6\sigma^2}(1 - y^2) - \widehat{F}(x) \right] + o(n^{-1/2}), \quad n \rightarrow \infty,$$

uniformly in $y \in \mathbb{R}$. In particular, writing $h_n = h/\sigma\sqrt{n}$, (74) holds with $G_n(y)$ in place of $G_n^\#(y)$ at the points $\{y = (k + 1/2)h_n, k \in \mathbb{Z}\}$ and with $[G_n(y) + G_n(y-)]/2$ in place of $G_n^\#(y)$ at the points $\{y = kh_n, k \in \mathbb{Z}\}$.

The proof is given in the Appendix. As with Theorem 5.1, it is more convenient to prove a version of (74) in terms of $\overline{G}_n^\#(y)$ rather than $G_n^\#(y)$, where $\overline{G}_n^\#$ is the polygonal approximation to \overline{G}_n . In the proof we show that

$$(75) \quad \overline{G}_n^\#(y) = \mathcal{G}(y) + \frac{\rho_3}{6\sigma^3\sqrt{n}}(1 - y^2)\gamma(y) + o(n^{-1/2}), \quad n \rightarrow \infty,$$

uniformly in $y \in \mathbb{R}$. Then (74) follows from (75) in the same way that (72) follows from (73).

Before moving on to large deviations we note that, although we shall not pursue these directions further in this paper, using the multiplicative mean ergodic theorems of Section 4 it is possible to prove higher order Edgeworth expansions, as well as precise local limit theorems for the density (or the pseudo-density, when a density does not exist) of S_n . The Edgeworth expansion proofs follow the same outline as those in the case of independent random variables; compare [17], page 541. For the local limit theorems, one can apply directly the general results of [7], Section 2.

6. Moderate and large deviations. In this section we use the multiplicative mean ergodic theorems of Theorems 4.1 and 4.2 to prove moderate and large deviations results for the partial sums of a Markov chain. As in Section 5, we consider the partial sums $\{S_n\}$ of a bounded functional F of the discrete-time, geometrically ergodic Markov chain Φ .

First we note that the multiplicative mean ergodic theorem together with the analyticity of $\Lambda(\alpha)$ in a neighborhood of the origin (see properties P1 and P4 in the previous section) immediately imply that the partial sums S_n satisfy a moderate deviations principle. We state this MDP, without proof, in Proposition 6.1. Its proof

is based on an application of the Gärtner–Ellis theorem, exactly as in the proof of Theorem 3.7.1 in [13].

PROPOSITION 6.1 (Moderate deviations) [10,11]. *Suppose the Markov chain Φ and the functional F satisfy Assumptions 1 and 2, and let $\{b_n\}$ be a sequence of constants such that*

$$\frac{b_n}{\sqrt{n}} \rightarrow \infty \quad \text{and} \quad \frac{b_n}{n} \rightarrow 0, \quad n \rightarrow \infty.$$

Then, for all $x \in X$ and any measurable $B \subset \mathbb{R}$,

$$\begin{aligned} - \inf_{y \in B^\circ} \left(\frac{y^2}{2\sigma^2} \right) &\leq \liminf_{n \rightarrow \infty} \frac{1}{b_n^2/n} \log P_x \left\{ \frac{S_n}{b_n} \in B \right\} \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{b_n^2/n} \log P_x \left\{ \frac{S_n}{b_n} \in B \right\} \leq - \inf_{y \in \overline{B}} \left(\frac{y^2}{2\sigma^2} \right), \end{aligned}$$

where B° denotes the interior of B and \overline{B} denotes its closure.

Note that the same result holds for the centered random variables $[S_n - E_x\{S_n\}]/b_n$ in place of S_n/b_n .

6.1. *Large deviations for Doeblin chains.* Suppose that Φ is a Doeblin recurrent chain, that is, suppose that for some $m \geq 1$, $\epsilon' > 0$ and a probability measure ν' , we have that $P^m \geq \epsilon' \nu'$. Equivalently, the Doeblin condition can be stated as

$$(76) \quad R \geq \epsilon \nu \quad \text{for some } \epsilon > 0 \text{ and a probability measure } \nu$$

and this, in turn, can be seen to be equivalent to geometric ergodicity with a bounded Lyapunov function V in (V4); see [37], Theorem 16.0.2. Then the state space X is small and the results of [43] can be applied to get large deviations results for the partial sums S_n . For example, for a Doeblin chain with a countable state space X and with $\psi =$ counting measure, the partial sums S_n satisfy a large deviations principle under the distributions P_x for any $x \in X$.

However, the situation is more complicated when Φ is stationary, that is, when $\Phi(0) \sim \pi$. In the following proposition we consider the LDP for the partial sums S_n under the stationary distribution P_π .

PROPOSITION 6.2 (Large deviations). *Suppose the Doeblin chain Φ and the functional F satisfy Assumptions 1 and 2, and let $\bar{a} = (\frac{\epsilon-1}{2-\epsilon})\epsilon$, where ϵ is as in (76).*

(i) *The partial sums S_n satisfy an LDP in a neighborhood of the origin: For any $c \in (0, \Lambda'(\bar{a}))$ and any $c' \in (\Lambda'(-\bar{a}), 0)$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_\pi \{S_n \geq nc\} = -\Lambda^*(c),$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_\pi \{S_n \leq nc'\} = -\Lambda^*(c'),$$

where

$$\Lambda^*(c) := \sup_{-\bar{a} < a < \bar{a}} [ac - \Lambda(a)].$$

(ii) *Part (i) cannot in general be extended to a full LDP on the whole real line.*

PROOF. Integrating the multiplicative mean ergodic theorem in (50) with respect to π and noting that $\pi(\check{f}_a) \in (0, \infty)$ for all $|a| \leq \bar{a}$, we get that

$$\frac{1}{n} \log \mathbb{E}_\pi [\exp(aS_n)] \rightarrow \Lambda(a), \quad n \rightarrow \infty,$$

for all real $a \in [-\bar{a}, \bar{a}]$. Since $\Lambda(a)$ is analytic, (i) follows from the Gärtner–Ellis theorem ([13], Theorem 2.3.6). To see that in the Doeblin case $\bar{a} = (\frac{e-1}{2-\epsilon})\epsilon$, note that in (V4) we can set $V \equiv 1$ and $s \equiv \epsilon$, take $0 < \delta < 1$ to be arbitrary and define $b = \delta/\epsilon$. We then have a version of (V4),

$$PV = V = (1 - \delta)V + bs.$$

Using the definition of \bar{a} given in Theorem 4.1 then gives

$$\bar{a} := \left(\frac{e-1}{2b-\delta}\right)\delta = \left(\frac{e-1}{2(\delta/\epsilon)-\delta}\right)\delta = \left(\frac{e-1}{2-\epsilon}\right)\epsilon.$$

Part (ii) follows from the counterexample in Proposition 5 of [5]. \square

6.2. *Exact large deviations for geometrically ergodic chains.* Next we consider the more general case of geometrically ergodic Markov chains, satisfying our standing assumptions, Assumptions 1 and 2. With \bar{a} as in Theorem 4.1, let (A', A) denote the interval

$$(A', A) := \{\Lambda'(a) : -\bar{a} < a < \bar{a}\}$$

and note that $0 = \pi(F) = \Lambda'(0) \in (A', A)$. Recall the definition of $\Lambda^*(c)$ in Proposition 6.2.

THEOREM 6.3 (Exact large deviations for nonlattice functionals). *Suppose that Φ and the strongly nonlattice functional F satisfy Assumptions 1 and 2, and let $c \in (0, A)$. Then, for all $x \in \mathbb{X}$,*

$$\mathbb{P}_x \{S_n \geq nc\} \sim \frac{\check{f}_a(x)}{a\sqrt{2\pi n\sigma_a^2}} e^{-n\Lambda^*(c)}, \quad n \rightarrow \infty,$$

where a is chosen so that $\Lambda'(a) = c$ and $\sigma_a := \Lambda''(a)$. A corresponding result holds for the lower tail.

It is perhaps worth pointing out that the way in which the large deviations probabilities $\mathbb{P}_x\{S_n \geq nc\}$ depend on the initial state x of the Markov chain is via the solution $f_a(x)$ to the multiplicative Poisson equation.

Although the proof (given next) relies on an application of a general result from [7], the main idea is similar to the proof of the corresponding result for independent random variables [1]: First, as in the case of finite state space [39], we perform a change of measure that maps the transition kernel P to the twisted kernel \check{P}_a . Since Φ is geometrically ergodic, by Proposition 4.12, \hat{P}_a is V -uniform. Therefore \check{P}_a is V_a -uniform by Proposition 4.9 and hence it is geometrically ergodic by Corollary 4.7. Therefore we can apply the Edgeworth expansions of Section 5 and complete the proof along the lines of the corresponding argument in the case of independent random variables; see, for example, [13], Theorem 3.7.4.

PROOF. Choose and fix an arbitrary $x \in X$. The result of the theorem will follow by an application of [7], Theorem 3.3. We consider the moment-generating functions $m_n(\alpha)$ of S_n , defined in (71) for α in the interior of the compact set Ω in Theorem 4.1. [Note that, although our Ω is different from the open disc used in [7], a close examination of the proof of [7], Theorem 3.3, shows that the result continues to hold when the open disc of radius \bar{a} is replaced with the interior $\{\alpha = a + i\omega : |\alpha| < \bar{a}, |\omega| < \bar{\omega}\}$ of the strip Ω , as long as $\bar{\omega} > 0$.]

We will make repeated use of properties P1–P7 stated in Section 5. From the definition of $m_n(\alpha)$ it is easily seen that it is an analytic function of α , and from P1 and P4 it follows that $m_n(\alpha)$ is nonzero on Ω for all n large enough (uniformly in α).

Let $\Lambda_n(\alpha)$ be the normalized log-moment generating function

$$\Lambda_n(\alpha) := \frac{1}{n} \log m_n(\alpha), \quad \alpha \in \Omega,$$

and

$$\Lambda_n^*(c) := \sup_{-\bar{a} < a < \bar{a}} [ac - \Lambda_n(a)], \quad c \in \mathbb{R}.$$

The main step in the proof is the verification of the assumptions of [7], Theorem 3.3. Most of them, plus some other technical properties, are established in the following lemma (proved in the Appendix).

LEMMA 6.4. *Under the assumptions of the theorem:*

(i) *For n large enough there is a unique $a_n \in (0, \bar{a})$ such that $\Lambda_n'(a_n) = c$ and $\Lambda_n^*(c) = a_n c - \Lambda_n(a_n)$.*

(ii) Similarly, there is a unique $a \in (0, \bar{a})$ such that $\Lambda'(a) = c$ and $\Lambda^*(c) = ac - \Lambda(a)$.

(iii) $a_n \rightarrow a$ as $n \rightarrow \infty$ and, in fact, $a_n - a = O(\frac{1}{n})$.

(iv) $\Lambda_n''(a_n) \rightarrow \sigma_a^2$ as $n \rightarrow \infty$.

(v) $\Lambda_n^*(c) \rightarrow \Lambda^*(c)$ as $n \rightarrow \infty$ and, in fact,

$$\Lambda_n^*(c) = \Lambda^*(c) - \frac{1}{n} \log \check{f}_a(x) + o\left(\frac{1}{n}\right).$$

The theorem follows from [7], Theorem 3.3, upon verifying condition (c) of [7], page 1685. For that, it suffices to show that for all $0 < \omega_0 < \omega_1 < \infty$,

$$\sup_{\omega_0 \leq |\omega| \leq \omega_1} \left| \frac{m_n(a' + i\omega)}{m_n(a')} \right| = o(n^{-1/2})$$

uniformly in a' in a neighborhood of a . However, the above convergence actually takes place exponentially fast, as can be easily verified using properties P1 and P2 from Section 5. \square

THEOREM 6.5 (Exact large deviations for lattice functionals). *Suppose that Φ and the lattice functional F satisfy Assumptions 1 and 2, and assume that F has span $h > 0$. Let $\{c_n\}$ be a sequence of real numbers in $(\epsilon, A - \epsilon)$ for some $\epsilon > 0$ and assume (without loss of generality) that, for each n , c_n is in the support of S_n . Then, for all $x \in X$,*

$$(77) \quad \begin{aligned} &P_x\{S_n \geq nc_n\} \\ &\sim \frac{h}{(1 - \exp(-ha_n))\sqrt{2\pi n\Lambda_n''(a_n)}} \exp(-n\Lambda_n^*(c_n)), \quad n \rightarrow \infty, \end{aligned}$$

where each $a_n \in (0, \bar{a})$ is chosen so that $\Lambda_n'(a_n) = c_n$. A corresponding result holds for the lower tail.

Note that in the lattice case we have given a slightly more general version of the result given in Theorem 6.3. If it turns out to be the case that the c_n converge to some $c \in (\epsilon, A - \epsilon)$, so that the corresponding a_n converge to some $a \in (0, \bar{a})$ at a rate $O(1/n)$. Then applying Lemma 6.4 as before, from (77) we obtain,

$$P_x\{S_n \geq nc_n\} \sim \frac{h\check{f}_a(x)}{(1 - \exp(-ha))\sqrt{2\pi n\sigma_a^2}} \exp(-n\Lambda^*(c)), \quad n \rightarrow \infty,$$

where $\sigma_a^2 = \Lambda''(a)$.

PROOF OF THEOREM 6.5. Choose and fix an arbitrary $x \in X$. The proof parallels that of Theorem 6.3, relying on an application of Theorem 3.5 from [7]. A close examination of its proof in [7] shows that, as in the case of Theorem 3.3

above, Theorem 3.5 remains valid if we replace the open disc of radius \bar{a} by the interior of the strip Ω . Proceeding as in the proof of Theorem 6.3, we now need to verify condition (c') on page 1686 of [7]. For that, it suffices to show that for that for all $\omega_0 \in (0, \pi/h)$,

$$\sup_{\omega_0 < |\omega| \leq \pi/h} \left| \frac{m_n(a' + i\omega)}{m_n(a')} \right| = o(n^{-1/2})$$

uniformly in $a' \in (\epsilon, A - \epsilon)$. Using properties P1 and P3 from Section 5, it is easy to see that the above convergence actually takes place exponentially fast, and this completes the proof. \square

7. Examples.

7.1. *Countable state space models.* Let Φ be a discrete-time Markov chain with a countable set X of states and let ψ be counting measure. Suppose Φ is irreducible in the usual sense that $R(x, y) > 0$ for all $x, y \in X$. Then \mathbb{I}_θ is a small function for any $\theta \in X$, with associated small measure $\nu = P(\theta, \cdot)$. Using this small function and measure in Lemma 3.2(i) leads to the characterization of $\Lambda(a)$ for real a ,

$$(78) \quad \Lambda(a) = \inf \left\{ \Lambda : \mathbb{E}_\theta \left[\exp \left(\sum_{k=0}^{\tau_\theta - 1} [aF(\Phi_k) - \Lambda] \right) \right] \leq 1 \right\};$$

see [2] for details. When the infimum is attained and we may justify differentiation with respect to a , then

$$1 = \mathbb{E}_\theta \left[\exp \left(\sum_{k=0}^{\tau_\theta - 1} [aF(\Phi_k) - \Lambda(a)] \right) \right] \implies 0 = \mathbb{E}_\theta \left[\sum_{k=0}^{\tau_\theta - 1} [F(\Phi_k) - \Lambda'(0)] \right].$$

This gives a more transparent proof of the identity $\Lambda'(0) = \pi(F)$.

The simple queue. For our purposes, the simplest interesting example of a countable state space chain is the $M/M/1$ queue. This is the reflected random walk Φ on $X = \{0, 1, 2, \dots\}$, with

$$P(x, x+1) = p, \quad P(x, (x-1)_+) = q, \quad x \in X,$$

where $p + q = 1$. We assume that $\rho = p/q < 1$ so that the chain is positive recurrent. As we show next:

1. Φ is geometrically ergodic.
2. It is not Doeblin recurrent.
3. With $F = \mathbb{I}_{0^c} - \pi(0^c)$, the multiplicative mean ergodic theorem (50) does not hold for all real α .

It is also not hard to show that Φ does not satisfy (mV3) for any f with finite sublevel sets, so that, in view of the discussion in Section 2.2, the Donsker–Varadhan conditions do not apply. More importantly, as Wu [56] recently showed, not just the conditions but also the large deviations conclusions of the Donsker–Varadhan theory fail in this case. Therefore, this example does not fall under any of the standard conditions known to imply large deviations results.

Below we also show that our central technical result, the multiplicative mean ergodic theorem (50), cannot in general be extended to hold on the entire real line.

First note that one can compute directly the expectations

$$(79) \quad \mathbb{E}_x[r^{\tau_0}] \begin{cases} < \infty, & 0 \leq r \leq \bar{\beta}, \\ = \infty, & r > \bar{\beta}, \end{cases}$$

where $\bar{\beta} = (4qp)^{-1/2} > 1$. To construct a Lyapunov function, consider $V(x) = r_0^x$ for $r_0 > 1$:

$$(80) \quad PV(x) = \begin{cases} (pr_0 + qr_0^{-1})V(x), & x \geq 1, \\ (pr_0 + q)V(0), & x = 0. \end{cases}$$

Choosing a minimal value for $(1 - \delta) := (pr_0 + qr_0^{-1})$ gives $r_0 = \rho^{-1/2}$ and a solution to (V4):

$$PV(x) = \sqrt{4pq}V(x) = \bar{\beta}^{-1}V(x), \quad x \geq 1.$$

It easily follows that with $r = \bar{\beta}$, $\mathbb{E}_x[r^{\tau_0}] = V(x) = \rho^{-x/2}$, $x \geq 1$. This gives the finite bound in (79) and shows that Φ is geometrically ergodic with Lyapunov function V .

The easiest way to see that Φ is not Doeblin recurrent is to notice that, in k time steps, Φ cannot visit more than its $2k$ neighboring states, which implies that the state space is not small; see [37], Theorem 16.0.2.

Now let $F = \mathbb{I}_{0^c} - \pi(0^c)$. Using the characterization (78) with $\theta = 0$, we find that $\Lambda(a)$ is the unique solution to the fixed-point equation

$$\mathbb{E}_0[\exp\{(\pi_0 a - \Lambda(a))\tau_0\}] = e^a,$$

where $\pi_0 := \pi(0)$. It follows from (79) that $\exp\{\pi_0 a - \Lambda(a)\} \leq \bar{\beta}$ for all $a \in \mathbb{R}$. Also, from the fixed-point equation it follows that if $a^* := \log \mathbb{E}_0[\bar{\beta}^{\tau_0}]$, then $\Lambda(a^*) = \pi_0 a^* - \log \bar{\beta}$. However, since $\exp\{\pi_0 a^* - \Lambda(a^*)\} = \bar{\beta}$ and $(\pi_0 a - \Lambda(a))$ is nondecreasing in a , from (79) we conclude that $\Lambda(a) = a - \log \bar{\beta}$ for all $a \geq a^*$ and hence $\Lambda''(a) = 0$ for $a \geq a^*$. [To see that $(\pi_0 a - \Lambda(a))$ is nonincreasing, simply recall from Proposition 4.9 that $\Lambda'(a) = \pi(a)$ so that $\Lambda'(a) \leq \sup_x F(x) = \pi_0$.] However, as we saw in property P4, the multiplicative mean ergodic theorem (50) implies that $\Lambda''(a) > 0$ for all a for which it is valid; therefore it cannot be valid for real $a \geq a^*$.

A plot of $\Lambda(a)$ for $F = \mathbb{I}_{0^c} - \pi(0^c)$ is shown in Figure 1.

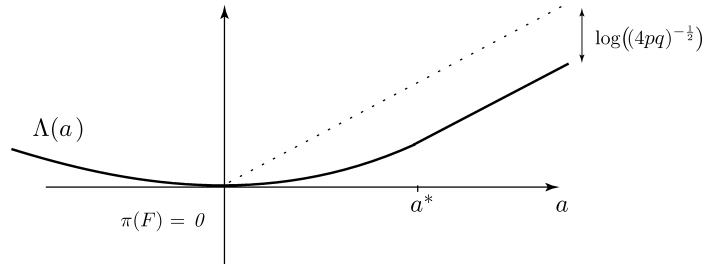


FIG. 1. The solid curve shows the log-moment generating function $\Lambda(a)$, $a \in \mathbb{R}$, for the $M/M/1$ queue with $F = \mathbb{I}_{0^c} - \pi(0^c)$. It is strictly convex for $a < a^*$ and it is linear for $a \geq a^*$.

7.2. *Diffusions.* Consider an elliptic diffusion on a manifold X . We assume that Φ is nonexplosive, so that the sample paths are continuous on $[0, \infty)$ with probability 1. It is then strong Feller and ψ -irreducible, where ψ is Lebesgue measure on X (see, e.g., [47]) and compact subsets of X are small.

Consider the special case where $X = \mathbb{R}^n$ and the diffusion term is constant,

$$\mathcal{A} = h \cdot \nabla_x + \frac{1}{2} \sigma^2 \Delta,$$

where Δ denotes the Laplacian. If $\check{f}_a \in C^2$ solves the multiplicative Poisson equation for some $a \in \mathbb{R}$, we may consider the *twisted* process Φ_a , that is, the Markov process with transition semigroup defined as before,

$$\check{P}_a^t(x, dy) = \lambda_a^{-t} \check{f}_a^{-1}(x) \widehat{P}_a^t(x, dy) \check{f}_a(y), \quad t > 0.$$

If (69) holds, then the generator $\check{\mathcal{A}}_a$ of Φ_a is given by

$$\begin{aligned} (81) \quad \check{\mathcal{A}}_a &= (h + \nabla_x \sigma^2 \check{F}_\alpha) \cdot \nabla_x + \frac{1}{2} \sigma^2 \Delta^2 \\ &= \mathcal{A} + \sigma^2 \nabla_x \check{F}_\alpha \cdot \nabla_x, \end{aligned}$$

where $\check{F}_\alpha = \log(\check{f}_a)$. Note that the twisted process has the same diffusion term as the original—only the drift is affected by the twisting.

Reflected Brownian motion. Diffusions with reflection are currently a popular model in the operations research area. Consider for example a two-dimensional reflected Brownian motion (RBM) Φ on $X = \mathbb{R}_+^2$ with normal reflection on each boundary. We show below that (when the drift is negative) Φ is geometrically ergodic, but it is not Doeblin recurrent and it does not satisfy (mV3) for any f with compact sublevel sets (for the same reasons as in the reflected random walk example above).

Within the interior of X , the sample paths are identical to those of the affine stochastic differential equation model,

$$d\Phi_i = -\delta_i dt + dW_i, \quad i = 1, 2,$$

where $\mathbf{W} = (W_1, W_2)$ is a standard Brownian motion on \mathbb{R}^2 and the drift term δ_i is positive for each i . A characterization of the generator can be found in [55].

Suppose that $V : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is smooth and suppose that the boundary conditions

$$(82) \quad \begin{aligned} \frac{\partial}{\partial x_1} V &\leq \frac{\partial}{\partial x_2} V, & x_1 = 0, \\ \frac{\partial}{\partial x_1} V &\geq \frac{\partial}{\partial x_2} V, & x_2 = 0, \end{aligned}$$

are satisfied. Then, with $L = -\delta \cdot \nabla + \frac{1}{2} \Delta$, the process

$$m(t) = V(\Phi(t)) - \int_{[0,t)} LV(\Phi(s)) ds$$

is a supermartingale.

A candidate Lyapunov function is the quadratic $V_0(x) = \|x\|^2$, since $LV_0 = -2\delta \cdot x + 1$ is negative for large x and the boundary conditions (82) are satisfied. The supermartingale property implies that

$$\mathbb{E}_x[V_0(\Phi(t))] \leq -\delta_0 \int_{[0,t)} \mathbb{E}[\Phi_1(s) + \Phi_2(s)] ds + t,$$

where $\delta_0 = 2 \min(\delta_1, \delta_2)$. This may be seen as a generalization of (V3), with $f(x)$ equal to a norm on \mathbb{R}^2 . To obtain a version of (V4), first consider $V_1 = \sqrt{V_0}$. We have, for some $\delta_1 > 0$ and some $B < \infty$,

$$\mathbb{E}_x[V_1(\Phi(t))] \leq -\delta_1 t, \quad 0 \leq t \leq 1, \quad \|x\| \geq B.$$

Finally, setting $V(x) = \exp(\beta V_1(x))$, we can find $\beta > 0$ sufficiently small such that

$$\mathbb{E}_x[V(\Phi(t))] \leq \exp(-\beta t)V(x), \quad 0 \leq t \leq 1, \quad \|x\| \geq B.$$

We conclude that the RBM is geometrically ergodic, provided the reflection is normal and the drift is negative. Therefore, for any bounded functional F , from Theorems 6.3 and 6.5 we get precise large deviations bounds for the time averages $\{S_t\}$, at least in some interval around the mean $\pi(F)$ of F . Moreover, in view of recent results in [6] [where a detailed study of large deviations properties of RBM (in one dimension) is performed], we should not expect the limit theorems, Theorems 6.3 and 6.5, to hold on the whole real line.

In general, geometric ergodicity depends upon the interaction of the drift vector and the reflection vectors along the boundaries. In multidimensional models it is not always obvious how to choose an appropriate Lyapunov function, but one can devise numerical methods to search for a quadratic V_0 satisfying the required constraints; see [34, 49].

APPENDIX

PROOF OF P1–P7. P1, P2, P3 and the analyticity of $\Lambda(\alpha)$ follow from the multiplicative mean ergodic theorems in Theorems 4.1 and 4.2.

To establish P4, first note that $\Lambda(0) = 0$ follows from the uniform convergence in Theorem 4.1. Similarly it follows that $\Lambda'(0) = \pi(F) = 0$ and that $\Lambda''(0) = \lim_n (1/n) \text{Var}_x(S_n) = \sigma^2 > 0$ by Assumptions 1 and 2 and Proposition 2.4.

On considering the kernel \check{P}_a for real $a \in \Omega$, since $\Lambda''(0) > 0$, $\Lambda''(a) \geq 0$ for all such a and $\pi_a(F) = \Lambda'(a)$ by Proposition 4.9, it follows that $\Lambda'(a) = \check{\pi}_a(F) > 0$ for all nonzero $a \in \Omega$. Now, if $\Lambda''(a) = \sigma_a^2$ is zero, then by (30) in Proposition 2.4 it follows that $\pi(F - \check{\pi}_a(F)) = 0$. This is impossible since $\check{\pi}_a(F) > 0$.

The exponential convergence in P6 is given in Theorem 2.5. The analyticity of \check{f}_α is stated in Theorem 4.12 and $\check{f}_\alpha|_{\alpha=0} \equiv 1$ by P1. Proposition 4.9 combined with Proposition 4.12 gives P7.

Property P5 requires more work. For a neighborhood \mathcal{O} of zero the function \check{f}_α given below is a constant times the normalized eigenfunction given in (61):

$$\check{f}_\alpha = H_\alpha^{-1} \mathbf{1}, \quad H_\alpha = I\lambda_\alpha - \widehat{P}_\alpha + \mathbf{1} \otimes \pi, \quad \alpha \in \mathcal{O}.$$

It is the unique solution in L_∞^V satisfying $\pi(\check{f}_\alpha) = 1$. Hence, for all k ,

$$\pi\left(\frac{d^k}{d\alpha^k} \check{f}_\alpha\right) = 0, \quad \alpha \in \mathcal{O}.$$

We have a form of the quotient rule,

$$\check{f}'_\alpha = -H_\alpha^{-1} H'_\alpha H_\alpha^{-1} \mathbf{1},$$

and after repeated differentiation we obtain

$$\begin{aligned} \frac{d^3}{d\alpha^3} \check{f}_\alpha &= -6H_\alpha^{-1} H'_\alpha H_\alpha^{-1} H'_\alpha H_\alpha^{-1} H'_\alpha H_\alpha^{-1} \mathbf{1} \\ &\quad + 3H_\alpha^{-1} H'_\alpha H_\alpha^{-1} H''_\alpha H_\alpha^{-1} \mathbf{1} \\ &\quad + 3H_\alpha^{-1} H''_\alpha H_\alpha^{-1} H'_\alpha H_\alpha^{-1} \mathbf{1} \\ &\quad - H_\alpha^{-1} H'''_\alpha H_\alpha^{-1} \mathbf{1}. \end{aligned}$$

Evaluating at $\alpha = 0$, we have $H_0^{-1} = Z = [I - P + \Pi]^{-1}$ and $\frac{d^k}{d\alpha^k} H_\alpha|_{\alpha=0} = [I\lambda_0^{(k)} - (I_F)^k P]$, $k \geq 1$. Using $\Pi Z = \Pi$ and $Z\mathbf{1} = P\mathbf{1} = \mathbf{1}$ then gives

$$\begin{aligned} 0 = \pi\left(\frac{d^3}{d\alpha^3} \check{f}_\alpha\right) &= 6\Pi I_F P Z I_F P Z F \\ &\quad + 3\Pi I_F P Z (F^2 - \sigma^2) \\ &\quad + 3\Pi (I_F^2 - \sigma^2) P Z I_F \\ &\quad + \Pi (F^3 - \lambda_0'''). \end{aligned}$$

The proof is then complete on interpreting these formulae, since $\Lambda'''(0) = \lambda_0'''$ and

$$PZG(x) = \pi(G) + \sum_{k=1}^{\infty} E_x[G(\Phi(k)) - \pi(G)]$$

for any function $G \in L_{\infty}^V$. \square

PROOF OF THEOREM 5.1. We follow closely Feller's argument in the proof of Theorem 1 in [17], page 539, leading to the statement (73). Choose and fix $x \in X$ arbitrary. For $n \geq 1$, define

$$(83) \quad M_n(\alpha) := E_x[\exp(\alpha \bar{S}_n)] = m_n(\alpha) \exp(-\alpha E_x\{S_n\}), \quad \alpha \in \mathbb{C},$$

and the distribution functions

$$(84) \quad \Psi_n(y) := \mathcal{G}(y) - \frac{\rho_3}{6\sigma^3\sqrt{n}}(y^2 - 1)\gamma(y), \quad y \in \mathbb{R},$$

with corresponding characteristic functions

$$(85) \quad \phi_n(\omega) := \exp\left(\frac{-\omega^2}{2}\right) \left(1 + \frac{\rho_3(i\omega)^3}{6\sigma^3\sqrt{n}}\right), \quad \omega \in \mathbb{R}.$$

Let $\epsilon > 0$ be arbitrary. Choose A large enough so that $A > 24(\epsilon\pi)^{-1}|\Psi_n'(y)|$ for all $y \in \mathbb{R}$, $n \geq 1$. From Esseen's smoothing lemma given in [17], page 538, with $T = A\sqrt{n}$ we get that

$$(86) \quad |\bar{G}_n(y) - \Psi_n(y)| \leq \frac{1}{\pi} \int_{-A\sqrt{n}}^{A\sqrt{n}} \left| M_n\left(\frac{i\omega}{\sigma\sqrt{n}}\right) - \phi_n(\omega) \right| \frac{d\omega}{|\omega|} + \frac{\epsilon}{\sqrt{n}}, \quad y \in \mathbb{R}.$$

To prove (73) it suffices to show that this integral is $o(n^{-1/2})$.

We first consider the integral in the range $B\sqrt{n} \leq |\omega| \leq A\sqrt{n}$, with $0 < B < \min\{\sigma\bar{\omega}_0, A\}$ to be chosen later (where $\bar{\omega}_0$ is as in P7). Applying the change of variables $t = \omega/(\sigma\sqrt{n})$, this integral is bounded above by

$$\frac{\sigma}{B\pi} \int_{B/\sigma \leq |t| \leq A/\sigma} |m_n(it)| dt + \frac{\sigma}{B\pi} \int_{B/\sigma \leq |t| \leq A/\sigma} |\phi_n(\sigma\sqrt{n}t)| dt.$$

The second integrand converges to zero exponentially fast, uniformly over t in that range, and the first integrand converges to zero exponentially fast by P2. Therefore, the above expression is certainly no larger than $o(n^{-1/2})$.

Next we consider the integral in (86) in the range $|\omega| \leq B\sqrt{n}$. From the definition of M_n and by properties P1 and P6, after the change of variables $t = \omega/(\sigma\sqrt{n})$ this equals

$$\frac{1}{\pi} \int_{|t| \leq B/\sigma} \left| \exp(-it\Delta_n - it\hat{F}(x)) \exp(n\Lambda(it)) [\check{f}_{it}(x) + it\epsilon_n] - \phi_n(t\sigma\sqrt{n}) \right| \frac{dt}{|t|}.$$

Expanding $\Lambda(it)$ in a Taylor series around zero yields

$$\begin{aligned} & \frac{1}{\pi} \int_{-B/\sigma}^{B/\sigma} \exp\left(-\frac{1}{2}nt^2\sigma^2\right) \\ & \times \left| \exp\left\{-it\Delta_n + \frac{n}{6}(it)^3\Lambda'''(it) + \log(\check{f}_{it} + it\epsilon_n) - it\widehat{F}\right\} \right. \\ & \left. - 1 - \frac{n\rho_3}{6}(it)^3 \Big| \frac{dt}{|t|} \right. \end{aligned}$$

for some real $s = s(t)$ with $|s| < B/\sigma$. Noting that

$$\log(\check{f}_{it}(x) + it\epsilon_n) - it\widehat{F} = \delta(it) + \log\left(1 + \frac{it\epsilon_n}{\check{f}_{it}}\right),$$

where $\delta(\cdot)$ is as in P7, the second exponent in the above integrand can be written as

$$\frac{n}{6}(it)^3\Lambda'''(it) - it\Delta_n + \delta(it) + it\epsilon_n''(it),$$

where $\epsilon_n''(it) := [\log(1 + (it\epsilon_n/\check{f}_{it}))]/(it)$ and

$$(87) \quad |\epsilon_n''(it)| \rightarrow 0 \quad \text{exponentially fast, } n \rightarrow \infty,$$

uniformly in $|t| \leq B/\sigma$ (by P1). Therefore, the integral we wish to bound is

$$\begin{aligned} & \frac{1}{\pi} \int_{-B/\sigma}^{B/\sigma} \exp\left(-\frac{1}{2}nt^2\sigma^2\right) \\ (88) \quad & \times \left| \exp\left\{\frac{n}{6}(it)^3\Lambda'''(is) - it\Delta_n + it\epsilon_n''(it) + \delta(it)\right\} - 1 - \frac{n\rho_3}{6}(it)^3 \Big| \frac{dt}{|t|} \right. \end{aligned}$$

To show that this is $o(n^{-1/2})$ we will apply the simple inequality from [17], page 534,

$$(89) \quad |e^\alpha - 1 - \beta| \leq (|\alpha - \beta| + \frac{1}{2}|\beta|^2)e^\gamma,$$

where $\gamma \geq \max\{|\alpha|, |\beta|\}$. First we choose B small enough so that the following four bounds hold for all $|t| < B/\sigma$:

- (a) $|\Lambda'''(it) - \rho_3| < 6\epsilon$,
- (b) $\frac{B}{6\sigma^3}|\Lambda'''(it)| \leq \frac{1}{4}$,
- (c) $|\delta(it)| \leq \frac{\epsilon}{2}$,
- (d) $\frac{B\rho_3}{6\sigma^3} \leq \frac{1}{4}$,

where (a) and (b) are possible by the analyticity of $\Lambda(\cdot)$ and the definition of ρ_3 in P5, and (c) is possible because of P7. Then, writing

$$\alpha := \frac{n}{6}(it)^3\Lambda'''(is) - it\Delta_n + it\epsilon_n''(it) + \delta(it)$$

and

$$\beta := \frac{n\rho_3}{6}(it)^3$$

using (b), (c) and (87), we can bound

$$\begin{aligned} |\alpha| &\leq nt^2\sigma^2 \frac{B}{6\sigma^3} \sup_{|t| < B/\sigma} |\Lambda'''(it)| + |t| [|\Delta_n| + |\epsilon_n''(it)|] + |\delta(it)| \\ &\leq \frac{1}{4}nt^2\sigma^2 + \frac{|t|}{n} + \frac{\epsilon}{2} \\ &\leq \frac{1}{4}nt^2\sigma^2 + \epsilon, \end{aligned}$$

where the last two inequalities are valid after taking n large enough. Similarly, using (a), (b), (87) and P7,

$$\begin{aligned} |\alpha - \beta| &\leq \frac{n}{6}|t|^3|\Lambda'''(is) - \rho_3| + \frac{|t|}{n} + (\text{Const})t^2 \\ &\leq \epsilon n|t|^3 + \frac{|t|}{n} + (\text{Const})t^2 \end{aligned}$$

for n large enough and, using (d),

$$|\beta| \leq \frac{1}{4}nt^2\sigma^2.$$

Applying inequality (89) with $\gamma := \frac{1}{4}nt^2\sigma^2$ and in conjunction with the last three bounds, the integral in (88) is bounded above by

$$\frac{1}{\pi} \int_{-B/\sigma}^{B/\sigma} \exp\left(-\frac{1}{4}nt^2\sigma^2 + \epsilon\right) \left[\epsilon n|t|^3 + (\text{Const})t^2 + \frac{|t|}{n} + \frac{1}{2} \left(\frac{n\rho_3|t|^3}{6} \right) \right] \frac{dt}{|t|}$$

and changing variables back to $\omega = t(\sigma\sqrt{n})$,

$$\begin{aligned} &\frac{e^\epsilon}{\pi} \int_{-B\sqrt{n}}^{B\sqrt{n}} e^{-\omega^2/4} \left[\epsilon \left(\frac{\omega^2}{\sqrt{n}\sigma^3} \right) + (\text{Const}) \frac{|\omega|}{n\sigma^2} + \frac{1}{n^{3/2}\sigma} + \frac{\rho_3|\omega|^5}{72n\sigma^2} \right] d\omega \\ &\leq \frac{\epsilon}{\sqrt{n}} \left(\frac{e^\epsilon}{\pi} \int_{-\infty}^{\infty} \omega^2 e^{-\omega^2/4} d\omega \right) + O\left(\frac{1}{n}\right). \end{aligned}$$

Since ϵ was arbitrary, this shows that the integral in (88) is $o(n^{-1/2})$ and the proof is complete. \square

PROOF OF THEOREM 5.2. We follow closely Feller's argument in the proof of Theorem 2 in [17], page 540. Choose and fix an arbitrary $x \in X$. Let $\epsilon > 0$ arbitrary and let Ψ_n be the distribution function (84). Recall that $G_n^\# = G_n * U[-h_n/2, h_n/2]$ and $\overline{G}_n^\# = \overline{G}_n * U[-h_n/2, h_n/2]$. Proceeding as in [17],

page 540, along equations (4.9) and (4.10) (with Ψ_n in place of G and $\Psi_n^\# := \Psi_n * U[-h_n/2, h_n/2]$ in place of $G^\#$), we obtain, after taking $A > 0$ large enough,

$$(90) \quad \begin{aligned} |\overline{G}_n^\#(y) - \Psi_n(y)| &\leq \frac{1}{\pi} \int_{-A\sqrt{n}}^{A\sqrt{n}} \left| M_n\left(\frac{i\omega}{\sigma\sqrt{n}}\right) - \phi_n(\omega) \right| \frac{|s_n(\omega)|}{|\omega|} d\omega \\ &+ \frac{\epsilon}{\sqrt{n}} + O\left(\frac{1}{n}\right), \quad y \in \mathbb{R}, \end{aligned}$$

where M_n and ϕ_n are defined in (83) and (85), and

$$s_n(\omega) := \frac{\sin(\frac{1}{2}h_n\omega)}{\frac{1}{2}h_n\omega}, \quad \omega \in \mathbb{R}.$$

To prove (75) it suffices to show that the integral in (90) is $o(n^{-1/2})$. We separately consider the integral over $|\omega| \leq B\sqrt{n}$ and over $B\sqrt{n} \leq |\omega| \leq A\sqrt{n}$ for some conveniently chosen $B < \min\{\sigma\overline{\omega}, A\}$, where $\overline{\omega}$ is as in P7. Noting that $|s_n(\omega)| \leq 1$ for all ω , the integral in the former range can be shown to be $o(n^{-1/2})$ as in the nonlattice case.

Therefore, it remains to show that

$$(91) \quad \begin{aligned} &\int_{B\sqrt{n} \leq |\omega| \leq A\sqrt{n}} \left| M_n\left(\frac{i\omega}{\sigma\sqrt{n}}\right) - \phi_n(\omega) \right| \frac{|s_n(\omega)|}{|\omega|} d\omega \\ &\leq \int_{B\sqrt{n} \leq |\omega| \leq A\sqrt{n}} \left| m_n\left(\frac{i\omega}{\sigma\sqrt{n}}\right) \right| \frac{|s_n(\omega)|}{|\omega|} d\omega + \int_{B\sqrt{n} \leq |\omega| \leq A\sqrt{n}} \frac{|\phi_n(\omega)|}{|\omega|} d\omega \\ &= o(n^{-1/2}). \end{aligned}$$

The last integral above is easily seen to decay exponentially in n (as in the proof of Theorem 5.1) and hence we concentrate on the former integral, which, after the change of variables $t = \omega/(\sigma\sqrt{n})$, becomes

$$\frac{2}{h} \int_{B/\sigma \leq |t| \leq A/\sigma} |m_n(it)| \left| \sin\left(\frac{1}{2}ht\right) \right| \frac{dt}{t^2}.$$

Notice that $|\sin(\frac{1}{2}ht)|$ and $m_n(it)$ are periodic functions of t with period $2\pi/h$. Consider, without loss of generality, the range of $t \in [B/\sigma, A/\sigma]$ (the case of negative t is similar). Let $(k - 1)$ denote the number of full periods of length $2\pi/h$ in that interval. Then, since $|\sin(y)/y| \leq 1$ for all real y ,

$$\begin{aligned} &\frac{2}{h} \int_{B/\sigma \leq t \leq A/\sigma} |m_n(it)| \left| \sin\left(\frac{1}{2}ht\right) \right| \frac{dt}{t^2} \\ &\leq \frac{k\sigma}{B} \int_{B/\sigma}^{2\pi/h - B/\sigma} |m_n(it)| dt + \frac{k\sigma^2}{B^2} \int_{-B/\sigma}^{B/\sigma} |m_n(it)| |t| dt, \end{aligned}$$

where the first integral converges to zero exponentially fast by P3. Using P1 to bound $m_n(it)$ and expanding $\Lambda(it)$ in a Taylor series, the second integral is

$$C \int_{-B/\sigma}^{B/\sigma} |t| |\exp\{n\Lambda(it)\}| dt \leq C' \int_{-\infty}^{\infty} |t| \left| \exp\left\{-\frac{1}{2}nt^2\sigma^2\right\} \right| dt = \frac{C''}{n},$$

for some constants C , C' and C'' . This establishes (91) and completes the proof. \square

PROOF OF LEMMA 6.4. Part (ii) is immediate by the choice of c and property P4. For part (i) note that by the uniform convergence of $\Lambda_n(a)$ to $\Lambda(a)$ (property P1), we also have convergence of their derivatives, so for n large enough we can pick a_n as claimed, and since $\Lambda_n''(a)$ eventually will be strictly positive for all $a \in (-\bar{a}, \bar{a})$, this a_n is unique.

For part (iii) recall that $\Lambda_n'(a_n) = c = \Lambda'(a)$, so the fact that $a_n \rightarrow a$ as $n \rightarrow \infty$ follows by the uniform convergence of the functions Λ_n' . Moreover, expanding $\Lambda_n'(a)$ around $a = a_n$ and using P1,

$$\begin{aligned} 0 &= \Lambda'(a) - \Lambda_n'(a_n) \\ &= [\Lambda_n'(a) - \Lambda_n'(a_n)] - [\Lambda'(a) - \Lambda_n'(a)] \\ &= [(a - a_n)\Lambda_n''(a_n) + O(a - a_n)^2] + \frac{d}{da} \left[\frac{1}{n} \log(\check{f}_a(x) + a\epsilon_n \exp\{-n\Lambda(a)\}) \right] \\ &= [(a - a_n)\Lambda_n''(a_n) + O(a - a_n)^2] + O\left(\frac{1}{n}\right) + O(\epsilon_n), \end{aligned}$$

where in the last step we used P7. Taking n large enough so that $\{\Lambda_n''(a_n)\}$ is a bounded sequence, bounded away from zero from below, this implies that $(a_n - a) = O(1/n)$.

Part (iv) is an immediate consequence of P4 and of the uniform convergence in P1. Finally for (v) we have from (i), (ii) and P1,

$$\begin{aligned} \Lambda_n^*(c) &= a_n c - \Lambda(a_n) - \frac{1}{n} \log[\check{f}_a(x) + a\epsilon_n] \\ &= \Lambda^*(c) + (a_n - a)c + (\Lambda(a) - \Lambda(a_n)) - \frac{1}{n} \log \check{f}_{a_n} \\ &\quad - \frac{1}{n} \log\left(1 + \frac{a_n \epsilon_n}{\check{f}_{a_n}}\right) \end{aligned}$$

and, using (iii) and P7,

$$\Lambda_n^*(c) = \Lambda^*(c) + O\left(\frac{1}{n^2}\right) - \frac{1}{n} \log \check{f}_a + O\left(\frac{1}{n^2}\right) + O(\epsilon_n)$$

as required. \square

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DIVISION OF APPLIED MATHEMATICS
AND DEPARTMENT OF COMPUTER SCIENCE
BROWN UNIVERSITY
BOX F, 182 GEORGE STREET
PROVIDENCE, RHODE ISLAND 02912
E-MAIL: yiannis@dam.brown.edu
www.dam.brown.edu/people/yiannis/

DEPARTMENT OF ELECTRICAL
AND COMPUTER ENGINEERING
AND THE COORDINATED SCIENCES LABORATORY
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS 61801
E-MAIL: s-meyn@uiuc.edu
www.black.csl.uiuc.edu/~meyn/