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A CHARACTERIZATION OF MULTIVARIATE REGULAR VARIATION

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We establish the equivalence between the multivariate regular variation of a random vector and the univariate regular variation of all linear combinations of the components of such a vector. According to a classical result of Kesten [*Acta Math.* **131** (1973) 207–248], this result implies that stationary solutions to multivariate linear stochastic recurrence equations are regularly varying. Since GARCH processes can be embedded in such recurrence equations their finite-dimensional distributions are regularly varying.

1. Introduction. Much of the early development of regular variation in the multivariate setting had its genesis in extreme value theory. There is a natural connection between limit theory of component maxima of iid random vectors and multivariate regular variation. Excluding some special degenerate cases, a random vector with positive components is in the maximum domain of attraction of a multivariate extreme value distribution with the same Fréchet marginals if and only if the vector has a distribution which is regularly varying; for details see, for example, [17], [18], Chapter 5. Early on, multivariate regular variation has been used in the theory of summation of iid random vectors to characterize the domains of attraction of stable distributions; see [20]. More recently, multivariate regular variation has been found to be a key concept in problems that go beyond extreme value theory. In particular, it has been used to describe the weak limits of point processes constructed from stationary sequences of random vectors; see [6, 7]. These weak convergence results provided the key ingredients for deriving the weak limiting behavior of sample autocovariances and sample autocorrelations of stationary sequences of random variables with regularly varying tails. In fact, one can apply a continuous mapping argument to obtain the weak convergence of these and other sum-type functionals of a stationary sequence from the weak convergence of appropriately chosen point processes, where the joint distributions of the points constituting the process satisfy a multivariate regular variation condition.

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In the literature one can find various equivalent formulations of regular variation for tail probabilities associated with a random vector. Basrak [1] has documented several of these equivalences. The most commonly used condition is the following: The d-dimensional random vector $\mathbf{X} = (X_1, \dots, X_d)'$ and its distribution are said to be *regularly varying with index* $\alpha \geq 0$ if there exists a random vector $\mathbf{\Theta}$ with values in $\mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$ with respect to the norm $|\cdot|$, such that, for all $t > 0$,

$$\frac{P(|\mathbf{X}| > tu, \mathbf{X}/|\mathbf{X}| \in \cdot)}{P(|\mathbf{X}| > u)} \stackrel{v}{\to} t^{-\alpha} P(\mathbf{\Theta} \in \cdot) \qquad \text{as } u \to \infty. \tag{1.1}$$

The symbol $\stackrel{v}{\to}$ stands for vague convergence on $\mathbb{S}^{d-1}$; vague convergence of measures is treated in detail in [12]. See [8] and [18], Chapter 5, for background on multivariate regular variation.

The aim of this note is to establish another characterization of multivariate regular variation that not only further illuminates the notion of regular variation, but has useful applications in its own right. In its simplest terms, this characterization states that the vector $\mathbf{X}$ is regularly varying if and only if every linear combination, $(\mathbf{t}, \mathbf{X})$, $\mathbf{t} \in \mathbb{R}^d$, is regularly varying, where $(\cdot, \cdot)$ denotes the usual inner product in $\mathbb{R}^d$. This result is in the spirit of the analogous characterization of a multivariate normal random vector and the use of the Cramér–Wold device for establishing weak convergence for a sequence of random vectors. This characterization of multivariate regular variation is known in some special cases, such as when $\mathbf{X}$ has nonnegative components with right tail of power law type and $\alpha \in (0, 2)$ (see [13], Corollary on page 236). It is noteworthy that such a general characterization for regular variation is conspicuously absent from the literature.

The precise formulation of the condition that every linear combination of the random vector is regularly varying is given by:

There exists an $\alpha > 0$ and a slowly varying function L such that, for all $\mathbf{x}$,

$$\lim_{u\to\infty} \frac{P((\mathbf{x}, \mathbf{X}) > u)}{u^{-\alpha} L(u)} = w(\mathbf{x}) \text{ exists,} \tag{1.2}$$

and there exists one $\mathbf{x}_0 \neq \mathbf{0}$ with $w(\mathbf{x}_0) > 0$.

Our basic motivation for studying the equivalence between (1.1) and (1.2) comes from a classical result of Kesten [13]; see also Goldie [11] for an alternate derivation of Kesten's result. They considered the tail behavior of solutions to stochastic recurrence equations of type $\mathbf{X}_t = \mathbf{A}_t \mathbf{X}_{t-1} + \mathbf{B}_t$, where $((\mathbf{A}_t, \mathbf{B}_t))$ is an iid sequence, $\mathbf{A}_t$ are random $d \times d$ matrices and $\mathbf{B}_t$ are d-dimensional random vectors. Stationary GARCH processes can be embedded in this type of stochastic recurrence equations. It turns out that, under mild (but rather technical) conditions on the distribution of $(\mathbf{A}_1, \mathbf{B}_1)$, the random vector $\mathbf{X}_1$ satisfies condition (1.2). However, to apply the well-developed limit theory for sample autocovariance and autocorrelation functions and extreme values, it is crucial that $\mathbf{X}_1$ is regularly varying in the sense of (1.1); see [2, 6, 7, 15] for various analyses of GARCH and bilinear processes.

Several equivalences between (1.1) and (1.2) for various choices of α are given in the following theorem.

THEOREM 1.1. *Let* $\mathbf{X}$ *be a random vector in* $\mathbb{R}^d$:

(i) *If the random vector* $\mathbf{X}$ *is regularly varying with index* $\alpha > 0$ *in the sense of condition* (1.1), *then* (1.2) *holds with the same* α.

(ii) *If* $\mathbf{X}$ *satisfies condition* (1.2), *where the* α *is positive and noninteger*, *then* $\mathbf{X}$ *is regularly varying with index* α *and the distribution of* $\mathbf{\Theta}$ *is uniquely determined*.

(iii) *If* $\mathbf{X}$ *assumes values in* $[0,\infty)^d$ *and satisfies* (1.2) *for* $\mathbf{x} \in [0,\infty)^d \backslash \{\mathbf{0}\}$, *where* $\alpha > 0$ *is a noninteger*, *then* (1.1) *holds with index* α *and the distribution of* $\mathbf{\Theta}$ *is uniquely determined*.

(iv) *If* $\mathbf{X}$ *assumes values in* $[0,\infty)^d$ *and satisfies* (1.2), *where* α *is an odd integer*, *then* (1.1) *holds with index* α *and the distribution of* $\mathbf{\Theta}$ *is uniquely determined*.

There are a few caveats to this theorem. First, we believe that part (iv) remains valid even if α in (1.2) is an even integer, but to date, an argument has not been provided. Second, it is critical that, for integer values of α, the linear combination of $\mathbf{X}$ involves both positive and negative coefficients even if the components of $\mathbf{X}$ are assumed to be nonnegative. Without this restriction counterexamples to this theorem are easy to construct; see Section 2. Moreover, for the case $\alpha = 1$, Kesten ([13], Remark 4) indicates that, for general $\mathbb{R}^d$-valued random vectors, condition (1.2) need not imply (1.1). Meerschaert and Scheffler ([14], Example 6.1.35) show that one can find regularly varying vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ with values in $\mathbb{R}^2$ and index $\alpha = 1$ for which $(x, \mathbf{X}_1)$ and $(x, \mathbf{X}_2)$ have the same limits $w(\mathbf{x})$ in (1.2), but the distributions of the corresponding vectors $\mathbf{\Theta}_1$ and $\mathbf{\Theta}_2$ on the unit sphere, corresponding to $\mathbf{X}_1$ and $\mathbf{X}_2$, are not the same.

2. Proof of theorem. For ease of presentation, we assume throughout that the vector $\mathbf{x}_0$ in (1.2) is given by the vector of 1's $\mathbf{1} = (1, \dots, 1)'$, and that

$$P\big((\mathbf{1}, \mathbf{X}) > u\big) = u^{-\alpha} L(u).$$

(i) Define the family of sets $\{W_{\mathbf{x}},\ \mathbf{x} \in \mathbb{R}^d\}$ by

$$W_{\mathbf{x}} = \big\{\mathbf{y} \in \mathbb{R}^d\colon\ (\mathbf{x}, \mathbf{y}) > 1\big\}.$$

The quotient in (1.2) may be written as $P(\mathbf{X} \in uW_{\mathbf{x}})/P(\mathbf{X} \in uW_{\mathbf{1}})$, which has a limit by the vague convergence in (1.1).

(ii) Define the family of measures

$$m_t = \frac{P(\mathbf{X} \in t\cdot)}{P(\mathbf{X} \in tW_{\mathbf{1}})}, \qquad t \geq 1,$$

on the space $\overline{\mathbb{R}}^d \setminus \{\mathbf{0}\}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$. On this space bounded sets are those that are bounded away from $\mathbf{0}$. We first note that this family of measures is tight. That is, for all bounded Borel sets B on $\overline{\mathbb{R}}^d \setminus \{\mathbf{0}\}$,

$$\sup_{t\geq 1} m_t(B) < \infty.$$

To see this, for any bounded B, there exist $\mathbf{x}_1, \dots, \mathbf{x}_k$ such that $B \subset \bigcup_{j=1}^k W_{\mathbf{x}_j}$ and hence

$$\sup_{t\geq 1} m_t(B) \leq \sup_{t\geq 1} \frac{\sum_{j=1}^k P(\mathbf{X} \in tW_{\mathbf{x}_j})}{P(\mathbf{X} \in tW_{\mathbf{1}})} < \infty$$

by (1.2).

LEMMA 2.1. *If μ is any subsequential vague limit of (m_t), then for any $\mathbf{x} \neq \mathbf{0}$,*

$$\mu(uW_{\mathbf{x}}) = w(\mathbf{x})u^{-\alpha} \qquad \textit{for all } u > 0. \tag{2.1}$$

Moreover, for any $\varepsilon > 0$ and any nonzero vector $\mathbf{x}$,

$$\int_{|(\mathbf{x},\mathbf{y})|>\varepsilon} |(\mathbf{x}, \mathbf{y})|^\gamma \mu(d\mathbf{y}) < \infty \qquad \textit{for all } \gamma < \alpha$$

and

$$\int_{|(\mathbf{x},\mathbf{y})|<\varepsilon} |(\mathbf{x}, \mathbf{y})|^\gamma \mu(d\mathbf{y}) < \infty \qquad \textit{for all } \gamma > \alpha.$$

PROOF. Identity (2.1) follows directly from (1.2). To show the first bound, we have

$$\int_{|(\mathbf{x},\mathbf{y})|>\varepsilon} |(\mathbf{x}, \mathbf{y})|^\gamma \mu(d\mathbf{y}) = \int_{|(\mathbf{x},\mathbf{y})|>\varepsilon} \int_0^{|(\mathbf{x},\mathbf{y})|} \gamma v^{\gamma-1}\, dv\, \mu(d\mathbf{y}),$$

which by Fubini and (2.1) is equal to

$$\begin{aligned} &\int_0^\varepsilon \mu\big(|(\mathbf{x}, \mathbf{y})| > \varepsilon\big)\gamma v^{\gamma-1}\, dv + \int_\varepsilon^\infty \mu\big(|(\mathbf{x}, \mathbf{y})| > v\big)\gamma v^{\gamma-1}\, dv \\ &\quad \leq \text{const}\, \varepsilon^\gamma + \int_\varepsilon^\infty \text{const}\, \gamma v^{-\alpha+\gamma-1}\, dv < \infty \end{aligned}$$

for $\gamma < \alpha$. The proof of the second bound is similar. □

By tightness there exists a subsequential vague limit of the family (m_t); see [12]. To complete the proof of part (ii) of the theorem, it then suffices to show that any two such limits, μ_1 and μ_2, are the same. So now suppose α is between the two integers $2n-2$ and $2n$ for some $n \geq 1$. Define the two measures ν_1 and ν_2 by

$$\begin{aligned}\nu_j(A) &= 2^n \int_A \big(1-\cos\big(2(\mathbf{1},\mathbf{y})\big)\big)^n \mu_j(d\mathbf{y})\\ &= (-1)^n \int_A \big(e^{i(\mathbf{1},\mathbf{y})} - e^{-i(\mathbf{1},\mathbf{y})}\big)^{2n} \mu_j(d\mathbf{y}), \qquad j=1,2.\end{aligned}$$

Since the integrand is bounded and of order $|(\mathbf{1},\mathbf{y})|^{2n}$ for $\mathbf{y}$ near the origin, it follows from Lemma 2.1 that these measures are finite. Also, $\nu_1(\overline{\mathbb{R}}^d \setminus \{\mathbf{0}\}) = \nu_2(\overline{\mathbb{R}}^d \setminus \{\mathbf{0}\})$, which follows from the fact that μ_1 and μ_2 agree on all sets of the form $W_{\mathbf{x}}$.

We now show that the characteristic functions of ν_1 and ν_2 agree, from which we conclude that the two measures are the same. To this end, for an arbitrary $\mathbf{x} \in \mathbb{R}^d$, consider

$$\begin{aligned}\int_{\mathbb{R}^d} e^{i(\mathbf{x},\mathbf{y})}\nu_j(d\mathbf{y}) &= (-1)^n \int_{\mathbb{R}^d} e^{i(\mathbf{x},\mathbf{y})} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} e^{i(k\mathbf{1},\mathbf{y})} e^{-i((2n-k)\mathbf{1},\mathbf{y})} \mu_j(d\mathbf{y})\\ &= (-1)^n \int_{\mathbb{R}^d} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} e^{i(\mathbf{x}-2n\mathbf{1}+2k\mathbf{1},\mathbf{y})} \mu_j(d\mathbf{y}).\end{aligned}$$

Using the following identity for binomial coefficients (see [19], Section 1.2):

$$\sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} k^m = 0 \qquad \text{for any } 0 \leq m < \ell \text{ and } \ell \geq 2,$$

and setting

$$e_m(z) = e^{iz} - 1 - iz - \cdots - \frac{i^m}{m!} z^m,$$

the above integral for $\alpha \in (2n-1, 2n)$ can be written as

$$\begin{aligned}\int_{\mathbb{R}^d} e^{i(\mathbf{x},\mathbf{y})}\nu_j(d\mathbf{y}) &= (-1)^n \int_{\mathbb{R}^d} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} e_{2n-1}\big((\mathbf{x}-2n\mathbf{1}+2k\mathbf{1},\mathbf{y})\big) \mu_j(d\mathbf{y})\\ &= (-1)^n \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \int_{\mathbb{R}^d} e_{2n-1}\big((\mathbf{x}-2n\mathbf{1}+2k\mathbf{1},\mathbf{y})\big) \mu_j(d\mathbf{y}).\end{aligned}$$

Since $e_{2n-1}((\mathbf{x}-2n\mathbf{1}+2k\mathbf{1},\mathbf{y}))$ is of order $|(\mathbf{x}-2n\mathbf{1}+2k\mathbf{1},\mathbf{y})|^{2n-1}$ at ∞ and $|(\mathbf{x}-2n\mathbf{1}+2k\mathbf{1},\mathbf{y})|^{2n}$ at the origin, the integrals on the right-hand side are finite by Lemma 2.1, which also justifies the interchange of summation and integration. By virtue of the integrands's dependence only on the inner-product $(\mathbf{x} - 2n\mathbf{1} + 2k\mathbf{1}, \mathbf{y})$, the integrals must be equal for $j = 1, 2$. For the case $\alpha \in (2n-2, 2n-1)$ the function e_{2n-1} is replaced by e_{2n-2} and the same calculations as above apply. This shows $\nu_1 = \nu_2$. An elementary argument shows that $\mu_j(\{\mathbf{y}\colon (\mathbf{1}, \mathbf{y}) = c\}) = 0$ for all $c \neq 0$ and hence μ_j has zero measure on the zeros of the function $1 - \cos(2(\mathbf{1}, \mathbf{y}))$. It follows that the measures μ_1 and μ_2 are equal.

(iii) The proof of this part is nearly the same as above, only using Laplace transforms instead of characteristic functions. In this case, the measure ν_j is defined by

$$\nu_j(A) = \int_A \bigl(1 - e^{-(\mathbf{1},\mathbf{y})}\bigr)^{2n} \mu_j(d\mathbf{y}).$$

(iv) Before embarking on the proof of this part, we first establish two lemmas, the first of which may be of independent interest. It is a partial converse to Breiman's lemma which states that if $Y > 0$ is a regularly varying random variable with index α and $Z > 0$ is independent of Y with $EZ^\gamma < \infty$ for some $\gamma > \alpha$, then ZY is regularly varying with index α. Specifically,

$$\frac{P(ZY > x)}{P(Y > x)} \to EZ^\alpha \qquad \text{as } x \to \infty. \tag{2.2}$$

LEMMA 2.2. *Let N be a standard normal random variable which is independent of the nonnegative random variable Y. If $(NY)_+$ is regularly varying with index $\alpha > 0$, then Y is regularly varying with the same index.*

PROOF. By the regular variation assumption, there exists a slowly varying function $L(x)$ such that, for $x > 0$,

$$\begin{aligned} L(x)x^{-\alpha} = P(NY > x) &= \int_0^\infty P(Y > x/z)\varphi(z)\,dz \\ &= x\int_0^\infty \frac{P(Y > 1/\sqrt{2s})}{\sqrt{2s}} \frac{e^{-x^2 s}}{\sqrt{2\pi}}\,ds, \end{aligned}$$

where $\varphi(z)$ is the standard normal density function. This implies that

$$\widehat{U}(x) = \int_0^\infty e^{-xs} U(ds) \sim \sqrt{2\pi}\, x^{-(\alpha+1)/2} L(\sqrt{x}) \qquad \text{as } x \to \infty,$$

where

$$U(z) = \int_0^z \frac{P(Y > 1/\sqrt{2s})}{\sqrt{2s}}\,ds.$$

An application of Karamata's Tauberian theorem (see [10], XIII, Section 5) yields that

$$U(s) \sim \sqrt{2\pi} L(1/\sqrt{s}) s^{(\alpha+1)/2} / \Gamma\bigl(1 + (\alpha+1)/2\bigr) \qquad \text{as } s \downarrow 0.$$

Since

$$U(y) = \int_0^{\sqrt{2y}} P(Y > 1/z)\,dz$$

and the integrand is monotone in z, an application of the monotone density theorem (see [4], Theorem 1.7.2.b) yields that

$$P(Y > x) \sim 2^{-\alpha/2}\sqrt{\pi}\,L(x)x^{-\alpha}/\,\Gamma\big((\alpha+1)/2\big) \qquad \text{as } x \to \infty. \qquad \square$$

If the random vector $\mathbf{X}$ is regularly varying with index $\alpha > 0$ in the sense of (1.1), then it is not difficult to show that, for any $p > 0$, the vector

$$\mathbf{X}^p = (|X_1|^p, \ldots, |X_d|^p)'$$

is regularly varying with index α/p, and by virtue of part (1) of the theorem, $\mathbf{X}^p$ satisfies (1.2) with index α/p in (1.2). The following lemma establishes a similar result under the assumption that (1.2) holds.

LEMMA 2.3. *If* $\mathbf{X}$ *is a nonnegative-valued vector satisfying* (1.2) *with index* α *and a slowly varying function* L, *then the vector* $\mathbf{X}^2$ *satisfies* (1.2) *with index* $\alpha/2$ *and a slowly varying function* $\widetilde{L}$.

PROOF. Let $\mathbf{N} = (N_1, \ldots, N_d)'$ be a vector of iid standard normal random variables independent of $\mathbf{X}$. Since $N_1^2(\mathbf{x}^2, \mathbf{X}^2)$ and $(N_1x_1X_1 + \cdots + N_dx_dX_d)^2$ are equal in distribution, we have, for any $\mathbf{x} \neq \mathbf{0}$ and $x > 0$,

$$P\big(N_1^2(\mathbf{x}^2, \mathbf{X}^2) > x^2\big) = 2P(N_1x_1X_1 + \cdots + N_dx_dX_d > x),$$

where $\mathbf{x}^2 = (x_1^2, \ldots, x_d^2)$. Then, by (1.2),

$$f_x = \frac{P(N_1x_1X_1 + \cdots + N_dx_dX_d > x \mid \mathbf{N})}{P((\mathbf{1}, \mathbf{X}) > x)} \xrightarrow{\text{a.s.}} f = w(x_1N_1, \ldots, x_dN_d).$$

Let g_x and g be the dominating functions for f_x and f, respectively, given by

$$g_x = \frac{P((|N_1x_1| + \cdots + |N_dx_d|)(\mathbf{1}, \mathbf{X}) > x \mid \mathbf{N})}{P((\mathbf{1}, \mathbf{X}) > x)} \xrightarrow{\text{a.s.}} g = (|x_1N_1| + \cdots + |x_dN_d|)^\alpha,$$

where the limit follows from (1.2). An application of (2.2) yields

$$Eg_x \to Eg \qquad \text{as } x \to \infty.$$

An appeal to Pratt's lemma (see [16]; cf. [18], page 289) gives

$$Ef_x = \frac{P(N_1x_1X_1 + \cdots + N_dx_dX_d > x)}{P((\mathbf{1}, \mathbf{X}) > x)} \to Ef = Ew(x_1N_1, \ldots, x_dN_d),$$

whence

$$\lim_{x\to\infty} \frac{P(N_1^2(\mathbf{x}^2, \mathbf{X}^2) > x^2)}{P((\mathbf{1}, \mathbf{X}) > x)} = 2Ew(x_1 N_1, \dots, x_d N_d). \tag{2.3}$$

The right-hand expectation is positive for all $\mathbf{x} \neq 0$ with all nonzero components. To see this, it suffices to show that $w(\mathbf{x}) > 0$ for all $\mathbf{x}$ with positive coefficients. For ease of argument, assume $d = 2$. Then, for any positive x_1 and x_2,

$$\begin{aligned} w(2x_1, 0) + w(0, 2x_2) &= \lim_{x\to\infty} \frac{P(x_1 X_1 > x/2) + P(x_2 X_2 > x/2)}{P(X_1 + X_2 > x)} \\ &\geq \lim_{x\to\infty} \frac{P(x_1 X_1 + x_2 X_2 > x)}{P(X_1 + X_2 > x)} = w(x_1, x_2) \\ &\geq \lim_{x\to\infty} \frac{P(x_2 X_2 > x)}{P(X_1 + X_2 > x)} = w(0, x_2). \end{aligned}$$

If $w(0, 1) > 0$, then $w(x_1, x_2) > 0$ for all positive x_1, x_2. If $w(0, 1) = 0$, we must have $w(1, 0) > 0$, and the same argument as above (interchanging the roles of X_1 and X_2) gives that $w(x_1, x_2) > 0$ for positive x_1, x_2.

Hence the right-hand expression in (2.3) is positive for all choices of $\mathbf{x} \neq \mathbf{0}$. Since $(\mathbf{1}, \mathbf{X})$ is regularly varying with index $\alpha > 0$, $N_1^2(\mathbf{x}^2, \mathbf{X}^2)$ is regularly varying with index $\alpha/2$, and Lemma 2.2 implies that $(\mathbf{x}^2, \mathbf{X}^2)$ is regularly varying with index $\alpha/2$, and so (1.2) holds for $\mathbf{X}^2$ for any $\mathbf{x} \neq \mathbf{0}$ with nonnegative components. □

Now we are ready to proceed with the proof of part (iv) of the theorem. Assume (1.2) holds for the nonnegative-valued random vector $\mathbf{X}$ with $\alpha = 2n + 1$ for some integer $n \geq 0$. By Lemma 2.3, $\mathbf{X}^2$ satisfies (1.2) for any $\mathbf{x} \neq \mathbf{0}$ with nonnegative components. Moreover, the corresponding index in (1.2) is $\alpha/2 = n + 0.5$, which is noninteger. Applying part (iii) of the theorem, we conclude that $\mathbf{X}^2$ is regularly varying with index $\alpha/2$, and an easy argument shows that $\mathbf{X}$ is regularly varying with index α. This concludes the proof of part (iv). □

COUNTEREXAMPLE. Here we give an example of two positive-valued random vectors that have different limits in (1.1) with $\alpha = 2$, yet have the same limits in (1.2) for all nonnegative $\mathbf{x}$. To construct the example, let Θ_1 and Θ_2 be two random variables defined on $(0, \pi/2)$ with unequal distribution functions such that

$$E \sin^2(\Theta_1) = E \sin^2(\Theta_2) \quad \text{and} \quad E \sin(2\Theta_1) = E \sin(2\Theta_2). \tag{2.4}$$

The existence of two such random variables satisfying (2.4) is easy to verify. Now define the measure μ_i on $(0, \infty) \times [0, 2\pi)$ by

$$\mu_i(dr, d\theta) = (2r^{-3} dr) \times P(\Theta_i \in d\theta).$$

For $i = 1, 2$, let $\mathbf{X}_i = (R\cos\Theta_i, R\sin\Theta_i)'$, where (R, Θ_i) has distribution given by μ_i restricted to the set $(1, \infty) \times (0, \pi/2)$. For $\mathbf{x} = (x_1, x_2)' \in [0, \infty)^2$, we have

$$\begin{aligned} x^2 P\big((\mathbf{x}, \mathbf{X}_i) > x\big) &= x^2 P(x_1 R\cos\Theta_i + x_2 R\sin\Theta_i > x) \\ &= x^2 \int_1^\infty P(x_1\cos\Theta_i + x_2\sin\Theta_i > x/r) 2r^{-3}\,dr \\ &= \int_0^{x^2} P\big((x_1\cos\Theta_i + x_2\sin\Theta_i)^2 > v\big)\,dv \\ &\to x_1^2 E\cos^2\Theta_i + x_1x_2 E\sin(2\Theta_i) + x_2^2 E\sin^2(\Theta_i), \end{aligned}$$

as $x \to \infty$. By (2.4), the right-hand side is the same for $i = 1, 2$ and all $x_1, x_2 \geq 0$. It follows that $\mathbf{X}_1$ and $\mathbf{X}_2$ have the same limit in (1.2) for all $x_1, x_2 \geq 0$. On the other hand, a routine calculation shows that

$$\frac{P(|\mathbf{X}_i| > tu, \mathbf{X}_i/|\mathbf{X}_i| \in \cdot)}{P(|\mathbf{X}_i| > u)} \xrightarrow{v} t^{-2} P\big((\cos(\Theta_i), \sin(\Theta_i))' \in \cdot\big) \qquad \text{as } u \to \infty,$$

which has distinct limits for $i = 1, 2$.

3. Applications.

3.1. *Stochastic recurrence equations.* We mentioned in the Introduction that linear stochastic recurrence equations

$$\mathbf{X}_n = \mathbf{A}_n\mathbf{X}_{n-1} + \mathbf{B}_n, \qquad n \in \mathbb{Z}, \tag{3.1}$$

where $((\mathbf{A}_n, \mathbf{B}_n))$ is an iid sequence of $d \times d$ random matrices $\mathbf{A}_n$ and d-dimensional random vectors $\mathbf{B}_n$, were the motivating examples for considering different characterizations of multivariate regular variation. Stationary causal solutions to (3.1) satisfy a general regular variation condition in the sense of (1.2). This follows from a fundamental result of Kesten [13], which we present here in a modified form (a combination of his Theorems 3 and 4). In these results, $\|\cdot\|$ denotes the operator norm defined in terms of the Euclidean norm $|\cdot|$.

THEOREM 3.1. *Let* $(\mathbf{A}_n)$ *be an iid sequence of* $d \times d$ *matrices with nonnegative entries and let* $(\mathbf{B}_n)$ *be nonnegative-valued* d-*dimensional vectors. Assume that the following conditions hold*:

(a) *For some* $\varepsilon > 0$, $E\|\mathbf{A}_1\|^\varepsilon < 1$.
(b) $\mathbf{A}_1$ *has no zero rows a.s.*
(c) *The set*

$$\big\{\ln\rho(\mathbf{a}_n\cdots\mathbf{a}_1) \colon n \geq 1, \mathbf{a}_n\cdots\mathbf{a}_1 > 0 \textit{ and } \mathbf{a}_n\cdots\mathbf{a}_1 \in \textit{ the support of } P_{\mathbf{A}_1}\big\}$$

generates a dense group, where $\rho(\mathbf{C})$ *is the spectral radius of the matrix* $\mathbf{C}$ *and* $\mathbf{C} > 0$ *means that all entries of this matrix are positive*.

(d) *There exists a* $\kappa_0 > 0$ *such that*

$$E\left(\min_{i=1,\dots,d} \sum_{j=1}^{d} A_{ij}\right)^{\kappa_0} \geq d^{\kappa_0/2}$$

and

$$E(\|\mathbf{A}_1\|^{\kappa_0} \ln^+ \|\mathbf{A}_1\|) < \infty.$$

Then the following statements hold:

(i) *There exists a unique solution* $\kappa_1 \in (0, \kappa_0]$ *to the equation*

$$0 = \lim_{n\to\infty} \frac{1}{n} E \ln \|\mathbf{A}_n \cdots \mathbf{A}_1\|^{\kappa_1}.$$

(ii) *There exists a unique strictly stationary causal solution* $(\mathbf{X}_n)$ *to the stochastic recurrence equation* (3.1).

(iii) *If* $E|\mathbf{B}_1|^{\kappa_1} < \infty$, *then* $\mathbf{X}_1$ *satisfies the following regular variation condition*:

$$\begin{aligned} &\text{For all } \mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}, \qquad \lim_{u\to\infty} u^{\kappa_1} P((\mathbf{x}, \mathbf{X}_1) > u) = w(\mathbf{x}) \quad \text{exists} \\ &\qquad\qquad \text{and is positive for all nonnegative-valued vectors } \mathbf{x} \neq 0. \end{aligned} \tag{3.2}$$

Clearly, (3.2) is a special case of (1.2). An appeal to Theorem 1.1 immediately gives the following result.

COROLLARY 3.2. *Under the assumptions of Theorem* 3.1, *the marginal distribution of the unique strictly stationary causal solution* $(\mathbf{X}_n)$ *of the stochastic recurrence equation* (3.1) *is regularly varying in the following sense*. *If the value* κ_1 *in* (3.2) *is not an even integer*, *then there exist a positive constant c and a random vector* $\mathbf{\Theta}$ *with values in the unit sphere* $\mathbb{S}^{d-1}$ *such that*

$$u^{\kappa_1} P(|\mathbf{X}_1| > tu, \mathbf{X}_1/|\mathbf{X}_1| \in \cdot) \xrightarrow{v} ct^{-\kappa_1} P(\mathbf{\Theta} \in \cdot) \qquad \text{as } u \to \infty.$$

This result is crucial for the understanding of the finite-dimensional distributions of GARCH processes which are used for modeling stock returns in the econometrics literature. The above corollary is directly applicable to GARCH processes since they can be embedded in multivariate stochastic recurrence equations of type (3.1); see [9], Section 8.4, [7] and [15] for some special cases and see [3] for the case of general GARCH processes.

3.2. *Point process convergence and maximum domains of attraction of multivariate extreme value distributions.* Regular variation conditions of type (1.1) are used for the characterization of maximum domains of attraction of extreme value distributions (see [18], Chapter 5) and are crucial assumptions for the weak convergence of point processes. In what follows, we mention a few results which follow from the characterization of multivariate regular variation given in Theorem 1.1.

Let $(\mathbf{X}_n)$ be an iid sequence of d-dimensional random vectors satisfying the regular variation condition (1.1) for some $\alpha > 0$. Define the sequence of positive numbers a_n by

$$P(|\mathbf{X}_1| > a_n) \sim n^{-1} \qquad \text{as } n \to \infty.$$

Let μ be the measure on $\overline{\mathbb{R}}^d \backslash \{\mathbf{0}\}$ which is determined by the vague limit in (1.1); that is, for any measurable set of the form $(t, \infty) \times S$ in the product space $(0, \infty) \times \mathbb{S}^{d-1}$,

$$\mu\big(\mathbf{x}\colon\ (|\mathbf{x}|, \mathbf{x}/|\mathbf{x}|) \in (t, \infty) \times S\big) = t^{-\alpha} P(\mathbf{\Theta} \in S).$$

It is well known (see [17]) that the sequence of point processes

$$\sum_{t=1}^{n} \varepsilon_{\mathbf{X}_t/a_n} \stackrel{d}{\to} N = \sum_{j=1}^{\infty} \varepsilon_{\mathbf{\Gamma}_j}, \tag{3.3}$$

where $\stackrel{d}{\to}$ denotes convergence in distribution in the space of point measures on $\overline{\mathbb{R}}^d \backslash \{\mathbf{0}\}$ endowed with the vague topology and N is a Poisson random measure (PRM) on $\overline{\mathbb{R}}^d \backslash \{\mathbf{0}\}$ with mean measure μ. Moreover, multivariate regular variation of $\mathbf{X}_1$ is also necessary for (3.3); see [17], Corollary 3.2. If the multivariate points $\mathbf{X}_t/a_n$ in (3.3) are replaced by linear combinations $(\mathbf{x}, \mathbf{X}_t)/a_n$ for some $\mathbf{x} \neq \mathbf{0}$, then $\lim_{n\to\infty} nP((\mathbf{x}, \mathbf{X}_1)/a_n > u) = w(\mathbf{x})$ exists. The same argument as for (3.3) gives that

$$\sum_{t=1}^{n} \varepsilon_{(\mathbf{x}, \mathbf{X}_t)/a_n} \stackrel{d}{\to} \sum_{j=1}^{\infty} \varepsilon_{(\mathbf{x}, \mathbf{\Gamma}_j)},$$

and the limit is again a PRM with corresponding mean measure. The converse, as recorded in the corollary below, is also true by Theorem 1.1.

COROLLARY 3.3. *Assume that* $\mathbf{X}_1$ *satisfies one of the following conditions*:

(a) $\mathbf{X}_1$ *satisfies* (1.2) *for some positive noninteger* α.

(b) $\mathbf{X}_1$ *assumes values in* $[0, \infty)^d \backslash \{\mathbf{0}\}$ *and satisfies* (1.2) *for some odd integer* α.

Then, *for every* $\mathbf{x} \neq \mathbf{0}$,

$$\sum_{t=1}^{n} \varepsilon_{(\mathbf{x}, \mathbf{X}_t)/a_n} \stackrel{d}{\to} N_{\mathbf{x}}, \tag{3.4}$$

where $N_{\mathbf{x}}$ is a PRM whose mean measure depends on $\mathbf{x}$. *Moreover*, (3.4) *implies that there exists a PRM with mean measure determined by the vague limit of* $nP(a_n^{-1}\mathbf{X}_1\cdot)$.

This result can be applied to the limit behavior of extreme order statistics. For example, $a_n^{-1}\max_{t=1,\dots,n}\mathbf{X}_t$ has a nondegenerate limit distribution if and only if $a_n^{-1}\max_{t=1,\dots,n}(\mathbf{x},\mathbf{X}_t)$ has a limit for all $\mathbf{x}\neq 0$ which is nondegenerate for some $\mathbf{x}$.

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