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THE ASYMPTOTIC ELASTICITY OF UTILITY FUNCTIONS AND OPTIMAL INVESTMENT IN INCOMPLETE MARKETS[1]

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The paper studies the problem of maximizing the expected utility of terminal wealth in the framework of a general incomplete semimartingale model of a financial market. We show that the *necessary and sufficient* condition on a utility function for the validity of several key assertions of the theory to hold true is the requirement that the asymptotic elasticity of the utility function is strictly less than 1.

1. Introduction. A basic problem of mathematical finance is the problem of an economic agent who invests in a financial market so as to maximize the expected utility of his terminal wealth. In the framework of a continuous-time model the problem was studied for the first time by Merton in two seminal papers, [27] and [28] (see also [29] as well as [32] for a treatment of the discrete-time case). Using the methods of stochastic optimal control, Merton derived a nonlinear partial differential equation (Bellman equation) for the value function of the optimization problem. He also produced the closed-form solution of this equation, when the utility function is a power function, the logarithm or of the form $1 - e^{-\eta x}$ for some positive η.

The Bellman equation of stochastic programming is based on the requirement of Markov state processes. The modern approach to the problem of expected utility maximization, which permits us to avoid the assumption of Markov asset prices, is based on duality characterizations of portfolios provided by the set of martingale measures. For the case of a *complete* financial market, where the set of martingale measures is a singleton, this "martingale" methodology was developed by Pliska [30], Cox and Huang [4, 5] and Karatzas, Lehoczky and Shreve [22]. It was shown that the marginal utility of the terminal wealth of the optimal portfolio is, up to a constant, equal to the density of the martingale measure; this key result naturally extends the classical Arrow–Debreu theory of an optimal investment derived in a one-step, finite probability space model.

Considerably more difficult is the case of incomplete financial models. It was studied in a discrete-time, finite probability space model by He and Pearson [16] and in a continuous-time diffusion model by He and Pearson [17] and by Karatzas, Lehoczky, Shreve and Xu [21]. The central idea here is to solve a

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dual variational problem and then to find the solution of the original problem by convex duality, similarly to the case of a complete model. In a discrete-time, finite probability space model the solution of the dual problem is always a martingale measure. We shall see in Section 5 that this assertion is not true in a general continuous time setting any more.

In this paper we consider the problem of expected utility maximization in an incomplete market, where asset prices are semimartingales. A subtle feature of this model is that the extension to the set of local martingales is no longer sufficient; to have a solution to the dual variational problem one should deal with a properly defined set of supermartingales. The basic goal of the current paper is to study the expected utility maximization problem under *minimal* assumptions on the model and on the utility function. Our model is very general: we only assume that the value function of the utility maximization problem is finite and that the set of martingale measures is not empty, which is intimately related with the assumption that the market is arbitrage-free. Depending on the assumptions on the *asymptotic elasticity* of the utility function, we split the main result into two theorems: for Theorem 2.1 we do not need any assumption; for Theorem 2.2 we must assume that the asymptotic elasticity of the utility function is less than 1. We provide counterexamples, which show that this assumption is *minimal* for the validity of Theorem 2.

The paper is organized as follows . In Section 2 we formulate the main Theorems 2.1 and 2.2. These theorems are proved in Section 4, after studying an abstract version of the problem of expected utility maximization in Section 3. The counterexamples are collected in Section 5, and in Section 6 we have assembled some basic facts on the notion of asymptotic elasticity. We are indebted to an anonymous referee for a careful reading and pertinent remarks.

2. The formulation of the theorems. We consider a model of a security market which consists of $d+1$ assets, one bond and d stocks. We denote by $S=(S^i)_{1\le i\le d}$ the price process of the d stocks and suppose that the price of the bond is constant. The latter assumption does not restrict the generality of the model, because we always may choose the bond as numéraire (compare, e.g., [8]). The process S is assumed to be a semimartingale on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, P)$. As usual in mathematical finance, we consider a finite horizon T, but we remark that our results can also be extended to the case of an infinite horizon.

A (self-financing) portfolio Π is defined as a pair (x, H), where the constant x is the initial value of the portfolio and $H=(H^i)_{i\le d}$ is a predictable S-integrable process specifying the amount of each asset held in the portfolio. The value process $X=(X_t)_{0\le t\le T}$ of such a portfolio Π is given by

$$X_t = X_0 + \int_0^t H_u\, dS_u, \qquad 0\le t\le T. \tag{2.1}$$

For $x\in\mathbb{R}_+$, we denote by $\mathcal{X}(x)$ the family of wealth processes with nonnegative capital at any instant, that is, $X_t\ge 0$ for all $t\in[0,T]$, and with initial value X_0 equal to x,

$$\mathcal{X}(x) = \{X \geq 0\colon X \text{ is defined by (2.1) with } X_0 = x\}.$$

DEFINITION 2.1. A probability measure $Q \sim P$ is called an *equivalent local martingale measure* if any $X \in \mathcal{X}(1)$ is a local martingale under Q.

If the process S is bounded (resp. locally bounded), then under an equivalent local martingale measure Q (in the sense of the above definition) the process S is a martingale (resp. a local martingale) and vice versa. If S fails to be locally bounded, the situation is more complicated. We refer to [10], Proposition 4.7, for a discussion of this case and the notion of sigma-martingales.

The family of equivalent local martingale measures will be denoted by $\mathcal{M}^e(S)$ or shortly by $\mathcal{M}$. We assume throughout that

$$\mathcal{M} \neq \emptyset. \tag{2.2}$$

This condition is intimately related to the absence of arbitrage opportunities on the security market. See [7], [10] for a precise statement and references.

We also consider an economic agent in our model which has a utility function $U\colon (0,\infty) \to \mathbb{R}$ for wealth. For a given initial capital $x > 0$, the goal of the agent is *to maximize the expected value from terminal wealth* $E[U(X_T)]$. The value function of this problem is denoted by

$$u(x) = \sup_{X \in \mathcal{X}(x)} E[U(X_T)]. \tag{2.3}$$

Hereafter we will assume, similary as in [21], that the function U is strictly increasing, strictly concave, continuously differentiable and satisfies the Inada conditions

$$\begin{aligned} U'(0) &= \lim_{x\to 0} U'(x) = \infty, \\ U'(\infty) &= \lim_{x\to\infty} U'(x) = 0. \end{aligned} \tag{2.4}$$

In the present paper we only consider utility functions defined on $\mathbb{R}_+$, that is, taking the value $-\infty$ on $(-\infty, 0)$; the treatment of utility functions which assume finite values on all of $\mathbb{R}$, such as the exponential utility $U(x) = 1 - e^{\eta x}$, requires somewhat different arguments.

To exclude the trivial case we shall assume throughout the paper that

$$u(x) = \sup_{X\in\mathcal{X}(x)} E\big[U(X_T)\big] < \infty \quad \text{for some } x > 0. \tag{2.5}$$

Intuitively speaking, the value function $u(x)$ can also be considered as a kind of utility function, namely the expected utility of the agent at time T, provided that he or she starts with an initial endowment $x \in \mathbb{R}_+$ and invests optimally in the assets, modeled by $S = (S_t)_{0\leq t\leq T}$, during the time interval $[0, T]$.

It is rather obvious that $u(x)$ is strictly increasing and concave. A basic theme of the present paper will be to investigate under which conditions u also satisfies the other requirements of a utility function.

A. Questions of a "qualitative" nature.

1. Is the value function $u(x)$ again a utility function satisfying the assumptions (2.4), that is, increasing, strictly concave, continuously differentiable and satisfying $u'(0) = \infty$, $u'(\infty) = 0$?
2. Does the optimal solution $\hat{X} \in \mathcal{X}(x)$ in (2.3) exist?

Not too surprisingly, the answer to the second question is "no" in general. Maybe more surprisingly, the answer to the first question is also negative and the two questions will turn out to be intimately related. The key concept to answer the above questions is the following regularity condition on the utility function U.

DEFINITION 2.2. A utility function $U(x)$ has *asymptotic elasticity* strictly less than 1, if

$$AE(U) = \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1.$$

To the best of our knowledge, the notion of the asymptotic elasticity of a utility function has not been defined in the literature previously.

We refer to Section 6 below for equivalent reformulations of this concept, related notions which have been investigated previously in the literature [21] and its relation to the Δ_2-condition in the theory of Orlicz spaces. For the moment we only note that many popular utility functions like $U(x) = \ln(x)$ or $U(x) = x^\alpha/\alpha$, for $\alpha < 1$, have asymptotic elasticity strictly less than one. On the other hand, a function $U(x)$ equaling $x/\ln(x)$, for x sufficiently large, is an example of a utility function with $AE(U) = 1$.

One of the main results of this paper (Theorem 2.2) asserts that for a utility function U the condition $AE(U) < 1$ is *necessary and sufficient* for a positive answer to both questions (1) and (2) above [if we allow $S = (S_t)_{0\le t\le T}$ to vary over all financial markets satisfying the above requirements]. In fact, for question (1) we can prove a stronger result: either $U(x)$ satisfies $AE(U) < 1$, in which case $AE(u) < 1$ too, and, in fact, $AE(u)_+ \le AE(U)_+$; or $AE(U) = 1$, in which case there exists a continuous $\mathbb{R}$-valued process $S = (S_t)_{0\le t\le T}$ inducing a *complete market*, such that $u(x)$ fails to be strictly concave and to satisfy $AE(u) < 1$ in a rather striking way: $u(x)$ is a straight line with slope 1, for $x \ge x_0$. Economically speaking the marginal utility of the value function $u(x)$ is eventually constant to 1 (while the marginal utility of the original utility function $U(x)$ decreases to zero, for $x \to \infty$). We shall discuss the economic interpretation of this surprising phenomenon in more detail in Note 5.2 below.

We now turn to more quantitative aspects.

B. Questions of a "quantitative" nature.

1. How may we calculate the value function $u(x)$?
2. How may we calculate the optimal solution $\hat{X} \in \mathcal{X}(x)$ in (2.3), provided this solution exists?

A well-known tool (compare [2], [21] and the references given therein) to answer these questions is the passage to the conjugate function,

$$V(y) = \sup_{x>0}\big[U(x) - xy\big], \qquad y > 0. \tag{2.6}$$

The function $V(y)$ is the Legendre-transform of the function $-U(-x)$ (see, e.g., [31]). It is well known (see, e.g., [31]) that if $U(x)$ satisfies the hypotheses stated in (2.4) above, then $V(y)$ is a continuously differentiable, decreasing, strictly convex function satisfying $V'(0) = -\infty$, $V'(\infty) = 0$, $V(0) = U(\infty)$, $V(\infty) = U(0)$ and the following bidual relation:

$$U(x) = \inf_{y>0}\big[V(y) + xy\big], \qquad x > 0.$$

We also note that the derivative of $U(x)$ is the inverse function of the negative of the derivative of $V(y)$ which, following [21], we also denote by I,

$$I := -V' = (U')^{-1}. \tag{2.7}$$

The Legendre transform is very useful in answering Question B above (compare [2], [30]). We first illustrate this in the case of a complete market, which is technically easier to handle, so suppose that there is a *unique* equivalent local martingale measure Q for the process S. We then find that the function

$$v(y) = E\left[V\left(y\frac{dQ}{dP}\right)\right] \tag{2.8}$$

is the conjugate function of $u(x)$, which provides a satisfactory answer to the first part of Question B. We resume the situation of a complete market, which to a large extent is well known, in the subsequent theorem.

THEOREM 2.0 (Complete case). *Assume that* (2.2), (2.4) *and* (2.5) *hold true and, in addition, that* $\mathscr{M} = \{Q\}$. *Then*:

(i) $u(x) < \infty$, *for all* $x > 0$, *and* $v(y) < \infty$, *for* $y > 0$ *sufficiently large. Letting* $y_0 = \inf\{y: v(y) < \infty\}$, *the function* $v(y)$ *is continuously differentiable and strictly convex on* (y_0, ∞). *Defining* $x_0 = \lim_{y\searrow y_0}(-v'(y))$, *the function* u *is continuously differentiable on* $(0,\infty)$ *and strictly concave on* $(0, x_0)$. *The value functions* u *and* v *are conjugate*;

$$v(y) = \sup_{x\geq 0}\big[u(x) - xy\big], \qquad y > 0,$$
$$u(x) = \inf_{y\geq 0}\big[v(y) + xy\big], \qquad x > 0.$$

The functions u' *and* v' *satisfy*

$$u'(0) = \lim_{x\to 0} u'(x) = \infty, \qquad v'(\infty) = \lim_{y\to\infty} v'(y) = 0.$$

(ii) *If* $x < x_0$, *then the optimal solution* $\hat{X}(x) \in \mathcal{X}(x)$ *is given by*

$$\hat{X}_T(x) = I\left(y\frac{dQ}{dP}\right),$$

for $y < y_0$, *where* x *and* y *are related via* $y = u'(x)$ [*equivalently* $x = -v'(y)$] *and* $\hat{X}(x)$ *is a uniformly integrable martingale under* Q.

(iii) *For* $0 < x < x_0$ *and* $y > y_0$ *we have*

$$u'(x) = E\left[\frac{\hat{X}_T(x)U'(\hat{X}_T(x))}{x}\right], \qquad v'(y) = E\left[\frac{dQ}{dP}V'\left(y\frac{dQ}{dP}\right)\right].$$

The above theorem dealing with the complete case will essentially be a corollary of the subsequent two theorems on the incomplete case, that is, the case when $\mathcal{M}$ is not necessarily reduced to a singleton $\{Q\}$. In this more general setting, we dualize the optimization problem (2.3); we define the family $\mathcal{Y}(y)$ of nonnegative semimartingales Y with $Y_0 = y$ and such that, for any $X \in \mathcal{X}(1)$, the product XY is a supermartingale,

$$\mathcal{Y}(y) = \{Y \geq 0\colon Y_0 = y \text{ and } XY = (X_tY_t)_{0\leq t\leq T} \text{ is a supermartingale, for all } X \in \mathcal{X}(1)\}.$$

In particular, as $\mathcal{X}(1)$ contains the process $X \equiv 1$, any $Y \in \mathcal{Y}(y)$ is a supermartingale. Note that the set $\mathcal{Y}(1)$ contains the density processes of the equivalent local martingale measures $Q \in \mathcal{M}^e(S)$.

We now define the value function of the dual problem by

$$v(y) = \inf_{Y\in\mathcal{Y}(y)} E[V(Y_T)]. \tag{2.9}$$

We shall show in Lemma 4.3 below that in the case of a complete market the functions $v(y)$ defined in (2.8) and (2.9) coincide, that is, (2.9) extends (2.8) to the case of not necessarily complete markets.

The functions u and $-v$, defined in (2.3) and (2.9), clearly are concave. Hence we may define u' and v' as the right-continuous versions of the derivatives of u and v. Similarly as in Definition 2.2 we define the *asymptotic elasticity* $AE(u)$ of the value function u by

$$AE(u) = \limsup_{x\to\infty} \frac{xu'(x)}{u(x)}.$$

The following theorems are the principal results of the paper.

THEOREM 2.1 (Incomplete case, general utility function U). *Assume that* (2.2), (2.4) *and* (2.5) *hold true*. *Then*:

(i) $u(x) < \infty$, *for all* $x > 0$, *and there exists* $y_0 > 0$ *such that* $v(y)$ *is finitely valued for* $y > y_0$. *The value functions* u *and* v *are conjugate*,

$$v(y) = \sup_{x>0}[u(x) - xy], \qquad y > 0,$$

$$u(x) = \inf_{y>0}[v(y) + xy], \qquad x > 0.$$

The function u is continuously differentiable on $(0,\infty)$ and the function v is strictly convex on $\{v<\infty\}$.
The functions u' and v' satisfy

$$u'(0)=\lim_{x\to 0}u'(x)=\infty, \qquad v'(\infty)=\lim_{y\to\infty}v'(y)=0.$$

(ii) *If $v(y)<\infty$, then the optimal solution $\hat{Y}(y)\in\mathscr{Y}(y)$ to* (2.9) *exists and is unique.*

THEOREM 2.2 (Incomplete case, $AE(U)<1$). *We now assume in addition to the conditions of Theorem* 2.1 *that the asymptotic elasticity of U is strictly less than one. Then in addition to the assertions of Theorem* 2.1 *we have*:

(i) *$v(y)<\infty$, for all $y>0$. The value functions u and v are continuously differentiable on $(0,\infty)$ and the functions u' and $-v'$ are strictly decreasing and satisfy*

$$u'(\infty)=\lim_{x\to\infty}u'(x)=0, \qquad -v'(0)=\lim_{y\to 0}-v'(y)=\infty.$$

The asymptotic elasticity $AE(u)$ of u also is less then 1 *and, more precisely,*

$$AE(u)_+\le AE(U)_+<1,$$

where x_+ denotes $\max\{x,0\}$.

(ii) *The optimal solution $\hat{X}(x)\in\mathfrak{X}(x)$ to* (2.3) *exists and is unique. If $\hat{Y}(y)\in\mathscr{Y}(y)$ is the optimal solution to* (2.9), *where $y=u'(x)$, we have the dual relation*

$$\hat{X}_T(x)=I\big(\hat{Y}_T(y)\big), \qquad \hat{Y}_T(y)=U'\big(\hat{X}_T(x)\big).$$

Moreover, the process $\hat{X}(x)\hat{Y}(y)$ is a uniformly integrable martingale on $[0,T]$.

(iii) *We have the following relations between u', v' and $\hat{X}$, $\hat{Y}$, respectively,*

$$u'(x)=E\left[\frac{\hat{X}_T(x)U'(\hat{X}_T(x))}{x}\right], \qquad v'(y)=E\left[\frac{\hat{Y}(y)V'(\hat{Y}(y))}{y}\right].$$

(iv)

$$v(y)=\inf_{Q\in\mathscr{M}}E\left[V\left(y\frac{dQ}{dP}\right)\right],$$

where dQ/dP denotes the Radon–Nikodym derivative of Q with respect to P on $(\Omega,\mathscr{F}_T)$.

The proofs of the above theorems will be given in Section 4 below.

As Examples 5.2 and 5.3 in Section 5 will show, the requirement $AE(U)<1$ is the *minimal* condition on the utility function U which implies *any* of the assertions (i), (ii), (iii) or (iv) of Theorem 2.2.

As mentioned in the introduction, it is crucial for Theorems 2.1 and 2.2 to hold true in the present gererality to consider the classes $\mathscr{Y}(y)$ of *supermartingales* and we shall see in Example 5.1′ below that, in general, $\mathscr{Y}(1)$ cannot be replaced by the smaller class of $\mathscr{M}^{\text{loc}}$ of *local martingales* considered in [21],

$$\mathscr{M}^{\text{loc}} = \big\{Y \text{ strictly positive local martingale,} \\ \text{s.t. } (X_t Y_t)_t \text{ is a local martingale for any } X \in \mathcal{X}(1)\big\}.$$

Note, however, that we obtain from the obvious inclusions $\mathscr{M} \subseteq \mathscr{M}^{\text{loc}} \subseteq \mathscr{Y}(1)$ in the setting of Theorem 2.2(iv) that

$$v(y) = \inf_{Y \in \mathscr{M}} E[V(yY)] = \inf_{Y \in \mathscr{M}^{\text{loc}}} E[V(yY)],$$

where we identify a measure $Q \in \mathscr{M}$ with its Radon–Nikodym derivate $Y = dQ/dP$.

Let us also point out that it follows from the uniqueness of the solution to 2.9 [established in Theorem 2.1(ii)] that in the case $\hat{Y} \notin \mathscr{M}$ (resp. $\hat{Y} \notin \mathscr{M}^{\text{loc}}$) (see Examples 5.1 and 5.1′ below) there is no solution to the problem

$$\inf_{Y \in \mathscr{M}} E[V(yY)] \text{ (resp. } \inf_{Y \in \mathscr{M}^{\text{loc}}} E[V(yY)].$$

Let us now comment on the economic interpretation of assertions (ii) and (iv) of Theorem 2.2; we start by observing that Theorem 2.1(ii) states that the infimum $\hat{Y}(y)$ to the optimization problem (2.9) exists and is unique (even without any assumption on the asymptotic elasticity of U). If we are lucky and, for fixed $y > 0$, the random variable $\hat{Y}_T(y)/y$ is the density of a probability measure $\hat{Q}$, that is, $d\hat{Q}/dP = \hat{Y}_T(y)/y$, then clearly $\hat{Q}$ is an equivalent local martingale measure, that is, $\hat{Q} \in \mathscr{M}^e(S)$, and we may use $\hat{Q}$ as a pricing rule for derivative securities via the expectation operator $E_{\hat{Q}}[\cdot]$. This choice of an equivalent martingale measure, which allows a nice economic interpretation as “pricing by the marginal rate of substitution,” has been proposed and investigated by Davis [6].

However, even for a “nice” utility function such as $U(x) = \ln(x)$ and for a “nice,” that is, continuous, process $(S_t)_{0 \le t \le T}$ it may happen that we fail to be lucky: in Section 5 we shall give an example (see 5.1) satisfying the assumptions of Theorem 2.2 such that $\hat{Y}(y)$ is a local martingale but fails to be uniformly integrable, that is, $E[\hat{Y}_T(y)/y] < 1$. Hence, defining the measure $\hat{Q}$ by $d\hat{Q}/dP = \hat{Y}_T(y)/y$, we only obtain a measure with total mass less than 1. At first glance the pricing operator $E_{\hat{Q}}[\cdot]$ induced by $\hat{Q}$ seems completely useless; for example, if we apply it to the bond $B_t \equiv 1$, we obtain a price $E_{\hat{Q}}[1] = E[\hat{Y}(y)/y] < 1$, which seems to imply arbitrage opportunities.

But assertion (ii) of Theorem 2.2 still contains a positive message: the optimal investment process $\hat{X}(x)$, where $x = -v'(y)$, is such that $(\hat{X}_t(x) \cdot \hat{Y}_t(y))_{0 \le t \le T}$ is a uniformly integrable martingale.

This implies that, by taking $(\hat{X}_t(x))_{0 \le t \le T}$ as numéraire (instead of the original numéraire $B_t \equiv 1$), we may remedy the above deficiency of $\hat{Q}$ (we refer to [8] for related results on this well known “change of numéraire” technique).

Let us go through the argument in a more formal way: first note that it follows from Theorem 2.2(ii) that $\hat{X}_T(x)$ is strictly positive. Hence we may consider the $\mathbb{R}^{d+2}$-valued semimartingale $\tilde{S} = (1, 1/\hat{X}(x), S^1/\hat{X}(x), \dots, S^d/\hat{X}(x))$; in other words, we consider the process $(\hat{X}(x), 1, S^1, \dots, S^d)$ expressed in units of the process $\hat{X}(x)$. The process $\tilde{Z}_t = \hat{X}_t(x)\hat{Y}_t(y)/xy$ is the density process of a true probability measure $\tilde{Q}$, where $d\tilde{Q}/dP = \hat{X}_T(x)\hat{Y}_T(y)/xy$. The crucial observation is that $\tilde{Q}$ is an equivalent local martingale measure for the $\mathbb{R}^{d+2}$-valued process $\tilde{S}$ (see [8]). Hence by expressing the stock price process S not in terms of the original bond but rather in terms of the new numéraire $\hat{X}(x)$, in other words by passing from S to $\tilde{S}$, we have exhibited an equivalent martingale measure $\tilde{Q}$ for the process $\tilde{S}$ such that the pricing operator $E_{\tilde{Q}}[\cdot]$ makes perfect sense. The above observed fact, that for the original bond $B_t \equiv 1$, which becomes the process $1/\hat{X}_t(x)$ under the numéraire $\hat{X}(x)$, we get

$$E_{\tilde{Q}}\big[1/\hat{X}_T(x)\big] = E\big[\hat{Y}_T(y)/xy\big] < 1/x = 1/\hat{X}_0(x)$$

now may be interpreted that the original bond simply is a silly investment from the point of view of an investor using $\hat{X}$ as numéraire, but this fact does not permit any arbitrage opportunities if we use $E_{\tilde{Q}}[\cdot]$ as a pricing operator for derivative securities.

Summing up, under the assumptions of Theorem 2.2 the optimization problem (2.9) leads to a consistent pricing rule $E_{\tilde{Q}}[\cdot]$, provided we are ready to change the numéraire from $B_t \equiv 1$ to $\hat{X}_t(x)$.

Another positive message of Theorem 2.2 in this context is assertion (iv): although it may happen that $\hat{Y}_T(y)/y$ does not induce an element $\hat{Q} \in \mathscr{M}^e(S)$ (without changing the numéraire) we know at least that $\hat{Y}_T(y)/y$ may be approximated by dQ/dP, where Q ranges in $\mathscr{M}^e(S)$. We shall see in Example 5.3 below that this assertion, too, breaks down as soon as we drop the assumption $AE(U) < 1$.

3. The abstract version of the theorems. We fix the notation

$$\mathscr{C}(x) = \big\{g \in L^0_+(\Omega, \mathscr{F}, P)\colon 0 \le g \le X_T, \text{for some } X \in \mathscr{X}(x)\big\}, \tag{3.1}$$

$$\mathscr{D}(y) = \big\{h \in L^0_+(\Omega, \mathscr{F}, P)\colon 0 \le h \le Y_T, \text{for some } Y \in \mathscr{Y}(y)\big\}. \tag{3.2}$$

In other words, we pass from the sets of *processes* $\mathscr{X}(x), \mathscr{Y}(y)$ to the sets $\mathscr{C}(x), \mathscr{D}(y)$ of *random variables* dominated by the final outcomes X_T, Y_T, respectively. We simply write $\mathscr{C}$, $\mathscr{D}$, $\mathscr{X}$, $\mathscr{Y}$ for $\mathscr{C}(1)$, $\mathscr{D}(1)$, $\mathscr{X}(1)$, $\mathscr{Y}(1)$ and observe that

$$\mathscr{C}(x) = x\mathscr{C} = \{xg\colon g \in \mathscr{C}\} \quad \text{for } x > 0, \tag{3.3}$$

and the analogous relations for $\mathscr{D}(y)$, $\mathscr{X}(x)$ and $\mathscr{Y}(y)$.

The duality relation between $\mathscr{C}$ and $\mathscr{D}$ (or equivalently between $\mathscr{X}$ and $\mathscr{Y}$) is a basic theme in mathematical finance (see, e.g., [1, 7, 18, 21, 24]). In the previous work in the literature, mainly the duality between $\mathscr{C}$ and the Radon–Nikodym densities dQ/dP of equivalent martingale measures (resp. local martingale measures) Q was considered which, in the case of a bounded process S (resp. a locally bounded process S), form a subset $\tilde{\mathscr{D}}$ of the set $\mathscr{D}$ considered here. The novel feature of the present approach is that we have chosen the definition of the processes Y in $\mathscr{Y}$ in such a way to get a perfect bipolar relation between the sets $\mathscr{C}$ and $\mathscr{D}$. This is the content of Proposition 3.1.

Recall that a subset $\mathscr{C}$ of $L^0_+(\Omega, \mathscr{F}, P)$ is called *solid*, if $0 \le f \le g$ and $g \in \mathscr{C}$ implies that $f \in \mathscr{C}$.

PROPOSITION 3.1. *Suppose that the $\mathbb{R}^d$-valued semimartingale S satisfies* (2.2). *Then the sets $\mathscr{C}$, $\mathscr{D}$ defined in* (3.1) *and* (3.2) *have the following properties*:

(i) *$\mathscr{C}$ and $\mathscr{D}$ are subsets of $L^0_+(\Omega, \mathscr{F}, P)$ which are convex, solid and closed in the topology of convergence in measure.*

(ii)

$$g \in \mathscr{C} \textit{ iff } E[gh] \le 1 \quad \textit{for all } h \in \mathscr{D} \quad \textit{and}$$
$$h \in \mathscr{D} \textit{ iff } E[gh] \le 1 \quad \textit{for all } g \in \mathscr{C}.$$

(iii) *$\mathscr{C}$ is a bounded subset of $L^0(\Omega, \mathscr{F}, P)$ and contains the constant function $\mathbb{1}$.*

The proof of Proposition 3.1 is postponed to Section 4 and presently we only note that the crucial assertion is the “bipolar” relation given by (ii). Also note that (ii) and (iii) imply that $\mathscr{D}$ is contained in the unit ball of $L^1(\Omega, \mathscr{F}, P)$, a fact which will frequently be used in the sequel.

For the remainder of this section we only shall assume that $\mathscr{C}$ and $\mathscr{D}$ are two subsets of $L^0_+(\Omega, \mathscr{F}, P)$ verifying the assertions of Proposition 3.1 [and not necessarily defined by (3.1) and (3.2) above]. Again we denote by $\mathscr{C}(x)$ and $\mathscr{D}(y)$ the sets $x\mathscr{C}$ and $y\mathscr{D}$. We shall reformulate Theorems 2.1 and 2.2 in this “abstract setting” and prove them only using the properties of $\mathscr{C}$ and $\mathscr{D}$ listed in Proposition 3.1.

Let $U = U(x)$ and $V = V(y)$ be the functions defined in Section 2 and consider the following optimization problems, which are the “abstract versions” of (2.3) and (2.9):

$$u(x) = \sup_{g \in \mathscr{C}(x)} E[U(g)], \tag{3.4}$$

$$v(y) = \inf_{h \in \mathscr{D}(y)} E[V(h)]. \tag{3.5}$$

If $\mathscr{C}(x)$ and $\mathscr{D}(y)$ are defined by (3.1) and (3.2), the two above value functions coincide with the ones defined in (2.3) and (2.9). As in (2.5) we assume throughout this section that

$$u(x) < \infty \quad \text{for some } x > 0. \tag{3.6}$$

Again the value functions u and $-v$ clearly are concave. We denote by u' and v' the right-continuous versions of the derivatives of u and v. We now can state the "abstract version" of Theorem 2.1.

THEOREM 3.1. *Assume that the sets $\mathscr{C}$ and $\mathscr{D}$ satisfy the assertions of Proposition* 3.1. *Assume also that the utility function U satisfies* (2.4) *and that* (3.6) *holds true. Then*:

(i) $u(x) < \infty$, *for all $x > 0$ and there exists $y_0 > 0$ such that $v(y)$ is finitely valued for $y > y_0$. The value functions u and v are conjugate,*

$$v(y) = \sup_{x>0}\left[u(x) - xy\right], \qquad y > 0, \tag{3.6}$$

$$u(x) = \inf_{y>0}\left[v(y) + xy\right], \qquad x > 0. \tag{3.7}$$

The function u is continuously differentiable on $(0,\infty)$ *and the function v is strictly convex on* $\{v < \infty\}$.
The functions u' and $-v'$ satisfy

$$u'(0) = \lim_{x\to 0} u'(x) = \infty, \qquad v'(\infty) = \lim_{y\to\infty} v'(y) = 0.$$

(ii) *If $v(y) < \infty$, then the optimal solution $\hat{h}(y) \in \mathscr{D}(y)$ to* (3.5) *exists and is unique*.

The proof of Theorem 3.1 will be broken into several lemmas. We will often use the following simple result; see, for example, [7], Lemma A1.1 as well as Lemma 4.2 below for a more sophisticated version of this result.

LEMMA 3.1. *Let $(f^n)_{n\geq 1}$ be a sequence of nonnegative random variables. Then there is a sequence $g^n \in \operatorname{conv}(f^n, f^{n+1}, \ldots)$, $n \geq 1$, which converges almost surely to a variable g with values in* $[0,\infty]$.

Let us denote by V^+ and V^- the positive and negative parts of the function V defined in (2.6).

LEMMA 3.2. *Under the assumptions of Theorem* 3.1, *for any $y > 0$, the family $(V^-(h))_{h\in\mathscr{D}(y)}$ is uniformly integrable, and if $(h^n)_{n\geq 1}$ is a sequence in $\mathscr{D}(y)$ which converges almost surely to a random variable h, then $h \in \mathscr{D}(y)$ and*

$$\liminf_{n\to\infty} E\left[V(h^n)\right] \geq E[V(h)]. \tag{3.9}$$

PROOF. Assume that $V(\infty) < 0$ (otherwise there is nothing to prove). Let ϕ: $(-V(0), -V(\infty)) \to (0,\infty)$ denote the inverse of $-V$. The function ϕ is strictly increasing,

$$\begin{aligned} E\left[\phi(V^-(h))\right] &\leq \left[E\phi(-V(h))\right] + \phi(0) \\ &= E[h] + \phi(0) \leq y + \phi(0) \qquad \forall\, h \in \mathscr{D}(y), \end{aligned}$$

and by (2.7) and the l'Hospital rule,

$$\lim_{x\to -V(\infty)} \frac{\phi(x)}{x} = \lim_{y\to\infty} \frac{y}{-V(y)} = \lim_{y\to\infty} \frac{1}{I(y)} = \infty.$$

The uniform integrability of the sequence $(V^-(h^n))_{n\geq 1}$ now follows from noting that $(h^n)_{n\leq 1}$ remains bounded in $L^1(P)$ [Proposition 3.1 (ii), (iii)] and by applying the de la Vallée–Poussin theorem.

Let $(h^n)_{n\leq 1}$ be a sequence in $\mathscr{D}(y)$ which converges almost surely to a variable h. It follows from the uniform integrability of the sequence $(V^-(h^n))_{n\geq 1}$ that

$$\lim_{n\to\infty} E\big[V^-(h^n)\big] = E[V^-(h)]$$

and from Fatou's lemma that

$$\liminf_{n\to\infty} E\big[V^+(h^n)\big] \geq E[V^+(h)].$$

This implies (3.9). Finally, we note that h is an element of $\mathscr{D}(y)$ because, according to Proposition 3.1, the set $\mathscr{D}(y)$ is closed under convergence in probability. □

We are now able to prove assertion (ii) of Theorem 3.1.

LEMMA 3.3. *In addition to the assumptions of Theorem* 3.1, *assume that* $v(y) < \infty$. *Then the optimal solution* $\hat{h}(y)$ *to the optimization problem* (3.5) *exists and is unique. As a consequence* $v(y)$ *is strictly convex on* $\{v < \infty\}$.

PROOF. Let $(g^n)_{n\geq 1}$ be a sequence in $\mathscr{D}(y)$ such that

$$\lim_{n\to\infty} E\big[V(g^n)\big] = v(y).$$

By Lemma 3.1 there exists a sequence $h^n \in \operatorname{conv}(g^n, g^{n+1}, \ldots)$, $n \geq 1$, and a variable $\hat{h}$ such that $h^n \to \hat{h}$ almost surely. From the convexity of the function V we deduce that

$$E\big[V(h^n)\big] \leq \sup_{m\geq n} E\big[V(g^m)\big],$$

so that

$$\lim_{n\to\infty} E\big[V(h^n)\big] = v(y).$$

We deduce from Lemma 3.1 that

$$E\big[V(\hat{h})\big] \leq \lim_{n\to\infty} E\big[V(h_n)\big] = v(y)$$

and that $\hat{h} \in \mathscr{D}(y)$. The uniqueness of the optimal solution follows from the strict convexity of the function V. As regards the strict convexity of v

fix $y_1 < y_2$ with $v(y_1) < \infty$: note that $(\hat{h}(y_1) + \hat{h}(y_2))/2$ is an element of $\mathscr{D}((y_1 + y_2)/2)$ and therefore, using again the strict convexity of V,

$$v\left(\frac{y_1 + y_2}{2}\right) \le E\left[V\left(\frac{\hat{h}(y_1) + \hat{h}(y_2)}{2}\right)\right] < \frac{v(y_1) + v(y_2)}{2}. \qquad \square$$

We now turn to the proof of assertion (i) of Theorem 3.1. Since the value function u defined in (3.4) clearly is concave and $u(x_0) < \infty$, for some $x_0 > 0$, we have $u(x) < \infty$, for all $x > 0$.

LEMMA 3.4. *Under the assumptions of Theorem* 3.1 *we have*

$$v(y) = \sup_{x>0}\left[u(x) - xy\right] \quad \text{for each } y > 0. \tag{3.10}$$

PROOF. For $n > 0$ we define $\mathscr{B}_n$ to be the positive elements of the ball of radius n of $\mathbf{L}^\infty(\Omega, \mathscr{F}, P)$, that is,

$$\mathscr{B}_n = \left\{g\colon 0 \le g \le n\right\}.$$

The sets $\mathscr{B}_n$ are $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-compact. Noting that, by item (iii) of Proposition 3.1, $\mathscr{D}(y)$ is a closed convex subset of $L^1(\Omega, \mathscr{F}, P)$ we may use the minimax theorem (see, e.g., [33], Theorem 45.8) to get the following equality, for n fixed:

$$\sup_{g\in\mathscr{B}_n} \inf_{h\in\mathscr{D}(y)} E[U(g) - gh] = \inf_{h\in\mathscr{D}(y)} \sup_{g\in\mathscr{B}_n} E[U(g) - gh].$$

From the dual relation [item (ii) of Proposition 3.1] between the sets $\mathscr{C}(x)$ and $\mathscr{D}(y)$, we deduce that $g \in \mathscr{C}(x)$ if and only if

$$\sup_{h\in\mathscr{D}(y)} E[gh] \le xy.$$

It follows that

$$\begin{aligned}\lim_{n\to\infty} \sup_{g\in\mathscr{B}_n} \inf_{h\in\mathscr{D}(y)} E[U(g) - gh] &= \sup_{x>0} \sup_{g\in\mathscr{C}(x)} E[U(g) - xy] \\ &= \sup_{x>0}[u(x) - xy].\end{aligned}$$

On the other hand,

$$\inf_{h\in\mathscr{D}(y)} \sup_{g\in\mathscr{B}_n} E[U(g) - gh] = \inf_{h\in\mathscr{D}(y)} E[V^n(h)] \triangleq v^n(y),$$

where

$$V^n(y) = \sup_{0<x\le n} [U(x) - xy].$$

Consequently, it is sufficient to show that

$$\lim_{n\to\infty} v^n(y) = \lim_{n\to\infty} \inf_{h\in\mathscr{D}(y)} E[V^n(h)] = v(y), \qquad y > 0. \tag{3.11}$$

Evidently, $v^n \le v$, for $n \ge 1$. Let $(h^n)_{n\ge 1}$ be a sequence in $\mathscr{D}(y)$ such that

$$\lim_{n\to\infty} E\big[V^n(h^n)\big] = \lim_{n\to\infty} v^n(y).$$

Lemma 3.1 implies the existence of a sequence $f^n \in \operatorname{conv}(h^n, h^{n+1}, \dots)$, which converges almost surely to a variable h. We have $h \in \mathscr{D}(y)$, because the set $\mathscr{D}(y)$ is closed under convergence in probability. Since $V^n(y) = V(y)$ for $y \ge I(1) \ge I(n)$, we deduce from Lemma 3.4 that the sequence $(V^n(f^n))^-$, $n \ge 1$, is uniformly integrable. Similarly as in the proof of the previous lemma, the convexity of V^n and Fatou's lemma now imply

$$\lim_{n\to\infty} E\big[V^n(h^n)\big] \ge \liminf_{n\to\infty} E\big[V^n(f^n)\big] \ge E[V(h)] \ge v(y),$$

which proves (3.11). □

LEMMA 3.5. *Under the assumptions of Theorem* 3.1, *we have*

$$\lim_{x\to 0} u'(x) = \infty, \qquad \lim_{y\to\infty} v'(y) = 0. \tag{3.12}$$

PROOF. By the duality relation (3.10), the derivatives u' and v' of the value functions u and v satisfy

$$\begin{aligned} -v'(y) &= \inf\{x\colon u'(x) \le y\}, \qquad y > 0,\\ u'(x) &= \inf\{y\colon : -v'(y) \le x\}, \qquad x > 0. \end{aligned}$$

It follows that the assertions (3.12) are equivalent. We shall prove the second one. The function $-v$ is concave and increasing. Hence there is a finite positive limit

$$-v'(\infty) \triangleq \lim_{y\to\infty} -v'(y).$$

Since the function $-V$ is increasing and $-V'(y) = I(y)$ tends to 0 as y tends to ∞, for any $\varepsilon > 0$ there exists a number $C(\varepsilon)$ such that

$$-V(y) \le C(\varepsilon) + \varepsilon y \qquad \forall\, y > 0.$$

By this, the $L^1(P)$-boundedness of $\mathscr{D}$ (3.8) and l'Hospital's rule,

$$\begin{aligned} 0 \le -v'(\infty) &= \lim_{y\to\infty} \frac{-v(y)}{y} = \lim_{y\to\infty} \sup_{h\in\mathscr{D}(y)} E\left[\frac{-V(h)}{y}\right]\\ &\le \lim_{y\to\infty} \sup_{h\in\mathscr{D}(y)} E\left[\frac{C(\varepsilon)+\varepsilon h}{y}\right] \le \lim_{y\to\infty} E\left[\frac{C(\varepsilon)}{y} + \varepsilon\right] = \varepsilon. \end{aligned}$$

Consequently, $-v'(\infty) = 0$. □

PROOF OF THEOREM 3.1. It suffices to remark that we obtain from the assumption $u(x_0) < \infty$, for some $x_0 > 0$, and the concavity of U that $u(x) < \infty$, for all $x > 0$ and that u is concave. Formula (3.8) now follows from Lemma 3.4 and the general bidual property of the Legendre-transform (see, e.g., [31], Theorem III.12.2).

The continuous differentiability of u follows from the strict convexity of v on $\{v < \infty\}$ again by general duality results ([31], Theorem V.26.3). □

In the setting of Theorem 3.1 we still prove, for later use, the following result.

LEMMA 3.6. *Under the hypotheses of Theorem* 3.1, *let* $(y^n)_{n\geq 1}$ *be a sequence of positive numbers which converges to a number* $y > 0$ *and assume that* $v(y_n) < \infty$ *and* $v(y) < \infty$. *Then* $\hat{h}(y^n)$ *converges to* $\hat{h}(y)$ *in probability and* $V(\hat{h}(y^n))$ *converges to* $V(\hat{h}(y))$ *in* $\mathbf{L}^1(\Omega, \mathscr{F}, P)$.

PROOF. If $\hat{h}(y^n)$ does not converge to $\hat{h}(y)$ in probability, then there exists $\varepsilon > 0$ such that

$$\limsup_{n\to\infty} P\big(\big|\hat{h}(y^n) - \hat{h}(y)\big| > \varepsilon\big) > \varepsilon.$$

Moreover, since by item (iii) of Proposition 3.1 we have $E\hat{h}(y^n) \leq y^n$ and $E\hat{h}(y) \leq y$, we may assume (by possibly passing to a smaller $\varepsilon > 0$) that

$$\limsup_{n\to\infty} P\big(\big|\hat{h}(y^n) + \hat{h}(y)\big| \leq 1/\varepsilon; \big|\hat{h}(y^n) - \hat{h}(y)\big| > \varepsilon\big) > \varepsilon. \tag{3.13}$$

Define

$$h^n = \tfrac{1}{2}\big(\hat{h}(y^n) + \hat{h}(y)\big), \qquad n \geq 1.$$

From the convexity of the function V we have

$$V(h^n) \leq \tfrac{1}{2}\big(V\big(\hat{h}(y^n)\big) + V\big(\hat{h}(y)\big)\big)$$

and from (3.13) and the strict convexity of V we deduce the existence of $\eta > 0$ such that

$$\limsup_{n\to\infty} P\big\{V(h^n) \leq \tfrac{1}{2}\big(V\big(\hat{h}(y^n)\big) + V\big(\hat{h}(y)\big)\big) - \eta\big\} > \eta.$$

Hence

$$\begin{aligned} E\big[V(h^n)\big] &\leq \tfrac{1}{2}\big(E\big[V\big(\hat{h}(y^n)\big)\big] + E\big[V\big(\hat{h}(y)\big)\big]\big) - \eta^2 \\ &= \tfrac{1}{2}\big(v(y^n) + v(y)\big) - \eta^2. \end{aligned}$$

The function v is convex and therefore continuous on the set $\{v < \infty\}$. It follows that

$$\limsup_{n\to\infty} E\big[V(h^n)\big] \leq v(y) - \eta^2.$$

By Lemma 3.1 we can construct a sequence $g^n \in \operatorname{conv}(h^n, h^{n+1}, \ldots)$, $n \geq 1$, which converges almost surely to a variable g. It follows from Lemma 3.2 and the convexity of V that $g \in \mathscr{D}(y)$ and

$$E[V(g)] = E\Big[\liminf_{n\to\infty} V(g^n)\Big] \leq \liminf_{n\to\infty} E\big[V(g^n)\big]$$
$$\leq \liminf_{n\to\infty} E\big[V(h^n)\big] \leq v(y) - \eta^2,$$

which contradicts the definition of $v(y)$. Therefore $\hat{h}(y^n)$ converges to $\hat{h}(y)$ in probability as n tends to ∞.

By Lemma 3.2 the sequence $(V^-(\hat{h}(y^n)))_{n\geq 1}$ is uniformly integrable. Consequently, $V(\hat{h}(y^n))$ converges to $V(\hat{h}(y))$ in $\mathbf{L}^1(\Omega, \mathscr{F}, P)$ if

$$\lim_{n\to\infty} EV\big(\hat{h}(y^n)\big) = V\big(\hat{h}(y)\big).$$

which in turn follows from the continuity of the value function v on the set $\{v < \infty\}$. $\square$

We now state the abstract version of Theorem 2.2.

THEOREM 3.2. *In addition to the assumptions of Theorem* 3.1, *we also suppose that the asymptotic elasticity of the utility function U is strictly less than one*, *that is*,

$$AE(U) = \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1.$$

Then in addition to the assertions of Theorem 3.1 *we have*:

(i) $v(y) < \infty$, *for all* $y > 0$. *The value functions u and v are continuously differentiable on* $(0, \infty)$ *and the functions u' and $-v'$ are strictly decreasing and satisfy*

$$u'(\infty) = \lim_{x\to\infty} u'(x) = 0, \qquad -v'(0) = \lim_{y\to 0} -v'(y) = \infty.$$

The asymptotic elasticity $AE(u)$ of u is less than or equal to the asymptotic elasticity of the utility function U,

$$AE(u)_+ \leq AE(U)_+ < 1,$$

where x_+ denotes $\max\{x, 0\}$.

(ii) *The optimal solution $\hat{g}(x) \in \mathscr{C}(x)$ to* (3.4) *exists and is unique*. *If $\hat{h}(y) \in \mathscr{D}(y)$ is the optimal solution to* (3.5), *where $y = u'(x)$, we have the dual relation*

$$\hat{g}(x) = I\big(\hat{h}(y)\big), \hat{h}(y) = U'\big(\hat{g}(x)\big).$$

Moreover,

$$E\big[\hat{g}(x)\hat{h}(y)\big] = xy.$$

(iii) *We have the following relations between* u', v' *and* $\hat{g}$, $\hat{h}$, *respectively*,

$$u'(x) = E\left[\frac{\hat{g}(x)U'(\hat{g}(x))}{x}\right], \qquad v'(y) = E\left[\frac{\hat{h}(y)V'(\hat{h}(y))}{y}\right].$$

Again, the proof of Theorem 3.2 will be broken into several steps. As regards some useful results pertaining to the asymptotic elasticity, we have assembled them in Section 6 below and we shall freely use them in the sequel.

As observed in Section 6 we may assume without loss of generality that $U(\infty) = V(0) > 0$. We start with an analogue to Lemma 3.6 above.

LEMMA 3.7. *Under the hypotheses of Theorem* 3.2, *let* $(y_n)_{n=1}^{\infty}$ *be a sequence of positive numbers tending to* $y > 0$. *Then* $V'(\hat{h}(y_n))\hat{h}(y_n)$ *tends to* $V'(\hat{h}(y))$ $\hat{h}(y)$ *in* $L^1(\Omega, \mathscr{F}, P)$.

PROOF. By Lemma 3.6 the sequence $\hat{h}(y_n)$ tends to $\hat{h}(y)$ in probability, hence by the continuity of V' we conclude that $V'(\hat{h}(y_n))\hat{h}(y_n)$ tends to $V'(\hat{h}(y))\hat{h}(y)$ in probability.

In order to obtain the conclusion we have to show the uniform integrability of the sequence $V'(\hat{h}(y_n))\hat{h}(y_n)$. At this point we use the hypothesis that the asymptotic elasticity of U is less then one, which by Lemma 6.3(iv) implies the existence of $y_0 > 0$ and a constant $C < \infty$ such that

$$-V'(y) < C\frac{V(y)}{y} \quad \text{for } 0 < y < y_0.$$

Hence the sequence of random variables $(V'(\hat{h}(y_n))\hat{h}(y_n)\mathbb{1}_{\{\hat{h}(y_n)<y_0\}})_{n=1}^{\infty}$ is dominated in absolute value by the sequence $(C|V(\hat{h}(y_n))|\mathbb{1}_{\{\hat{h}(y_n)<y_0\}})_{n=1}^{\infty}$ which is uniformly integrable by Lemma 3.6.

As regards the remaining part $(V'(\hat{h}(y_n))\hat{h}(y_n)\mathbb{1}_{\{\hat{h}(y_n)\geq y_0\}})_{n=1}^{\infty}$, the uniform integrability follows as in the proof of Lemma 3.2 from the fact that $(\hat{h}(y_n))_{n=1}^{\infty}$ is bounded in $L^1(\Omega, \mathscr{F}, P))$ and $\lim_{y\to\infty} V'(y) = 0$. $\square$

REMARK 3.1. For later use we remark that, given the setting of Lemma 3.7 and in addition a sequence $(\mu_n)_{n=1}^{\infty}$ of real numbers tending to 1 , we still may conclude that $V'(\mu_n\hat{h}(y_n))\hat{h}(y_n)$ tends to $V'(\hat{h}(y))\hat{h}(y)$ in $L^1(\Omega, \mathscr{F}, P)$. Indeed, it suffices to remark that it follows from Lemma 6.3 that, for fixed $0 < \mu < 1$, we can find a constant $\tilde{C} < \infty$ and $y_0 > 0$ such that

$$-V'(\mu y) < \tilde{C}\frac{V(y)}{y} \quad \text{for } 0 < y < y_0.$$

Plugging this estimate into the above proof yields the conclusion.

LEMMA 3.8. *Under the assumptions of Theorem* 3.2, *the value function* v *is finitely valued and continuously differentiable on* $(0, \infty)$, *the derivative* v' *is* *strictly increasing and satisfies*

$$-yv'(y) = E\big[\hat{h}(y)I(\hat{h}(y))\big]. \tag{3.14}$$

PROOF. Observe that $-yv'(y) = \lim_{\lambda\to 1}(v(y)-v(\lambda y)/(\lambda-1))$, provided the limit exists. We shall show

$$\limsup_{\lambda\searrow 1}\frac{v(y)-v(\lambda y)}{\lambda-1} \le E\big[\hat{h}(y)I(\hat{h}(y))\big] \tag{3.15}$$

and

$$\liminf_{\lambda\searrow 1}\frac{v(y)-v(\lambda y)}{\lambda-1} \ge E\big[\hat{h}(y)I(\hat{h}(y))\big] \tag{3.16}$$

This will prove the validity of (3.14) with $v'(y)$ replaced by the right derivative $v'_r(y)$; using Lemma 3.7 we then can deduce the continuity of the function $y \to v'_r(y)$ which, by the convexity of v, implies the continuous differentiability of v, thus finishing the proof of the lemma. □

To show (3.15) we estimate

$$\begin{aligned}\limsup_{\lambda\searrow 1}\frac{v(y)-v(\lambda y)}{\lambda-1} &\le \limsup_{\lambda\searrow 1}\frac{1}{\lambda-1}E\left[V\left(\frac{1}{\lambda}\hat{h}(\lambda y)\right)-V\big(\hat{h}(\lambda y)\big)\right]\\ &\le \limsup_{\lambda\searrow 1}\frac{1}{\lambda-1}E\left[\left(\frac{1}{\lambda}-1\right)\hat{h}(\lambda y)V'\left(\frac{1}{\lambda}\hat{h}(\lambda y)\right)\right]\\ &= E\big[\hat{h}(\lambda y)I\big(\hat{h}(y)\big)\big],\end{aligned}$$

where in the last line we have used Remark 3.1.

To show (3.16) it suffices to apply the monotone convergence theorem,

$$\begin{aligned}\liminf_{\lambda\searrow 1}\frac{v(y)-v(\lambda y)}{\lambda-1} &\ge \liminf_{\lambda\searrow 1}\frac{1}{\lambda-1}E\big[V\big(\hat{h}(y)\big)-V\big(\lambda\hat{h}(y)\big)\big]\\ &\ge \liminf_{\lambda\searrow 1}\frac{1}{\lambda-1}E\big[(1-\lambda)\hat{h}(y)V'\big(\lambda\hat{h}(y)\big)\big]\\ &= E\big[\hat{h}(y)I\big(\hat{h}(y)\big)\big].\end{aligned}$$

Finally, v' is strictly increasing, because v is strictly convex by Theorem 3.1.

By (3.6) we have that u' is the inverse to $-v'$ and therefore, using Lemma 3.8, u' also is continuous and strictly decreasing.

LEMMA 3.9. *Under the assumptions of Theorem* 3.2, *suppose that the numbers x and y are related by $x = -v'(y)$. Then $\hat{g}(x) \triangleq I(\hat{h}(y))$ is the unique optimal solution to* (3.4).

PROOF. Let us first show that $\hat{g}(x) \triangleq I(\hat{h}(y))$ belongs to $\mathscr{C}(x)$. According to Proposition 3.1 it is sufficient to show that, for any $h \in \mathscr{D}(y)$,

$$E\big[hI\big(\hat{h}(y)\big)\big] \le xy = -yv'(y) = E\big[\hat{h}I\big(\hat{h}(y)\big)\big], \tag{3.17}$$

where the last equality follows from (3.14).

Let us fix $h \in \mathscr{D}(y)$ and denote

$$h_\delta = (1-\delta)\hat{h}(y) + \delta h, \qquad \delta \in (0,1).$$

From the inequality

$$\begin{aligned} 0 \le E\big[V(h_\delta)\big] - E\big[V\big(\hat{h}(y)\big)\big] &= E\bigg[\int_{h_\delta}^{\hat{h}(y)} I(z)\,dz\bigg] \\ &\le E\big[I(h_\delta)\big(\hat{h}(y) - h_\delta\big)\big], \end{aligned}$$

we deduce that

$$E\big[I\big((1-\delta)\hat{h}(y)\big)\hat{h}(y)\big] \ge E\big[I(h_\delta)h\big]. \tag{3.18}$$

Remark 3.1 implies that for δ close to 0,

$$E\big[I\big((1-\delta)\hat{h}(y)\big)\hat{h}(y)\big] < \infty.$$

The monotone convergence theorem and the Fatou lemma applied, respectively, to the left- and right-hand sides of (3.18), as $\delta \to 0$, now give us the desired inequality (3.17). Hence, $\hat{g}(x) \in \mathscr{C}(x)$.

For any $g \in \mathscr{C}(x)$ we have

$$\begin{aligned} E\big[g\hat{h}(y)\big] &\le xy, \\ U(g) &\le V\big(\hat{h}(y)\big) + g\hat{h}(y). \end{aligned}$$

It follows that

$$\begin{aligned} E[U(g)] \le v(y) + xy &= E\Big[V\big(\hat{h}(y)\big) + \hat{h}(y)I\big(\hat{h}(y)\big)\Big] \\ &= E\big[U\big(I\big(\hat{h}(y)\big)\big)\big] = E\big[U\big(\hat{g}(x)\big)\big], \end{aligned}$$

proving the optimality of $\hat{g}(x)$. The uniqueness of the optimal solution follows from the strict concavity of the function U. □

LEMMA 3.10. *Under the assumptions of Theorem* 3.2, *the asymptotic elasticity of u is less than or equal to the asymptotic elasticity of U*,

$$AE(u)_+ \le AE(U)_+ < 1,$$

where x_+ denotes $\max\{x, 0\}$.

PROOF. By passing from $U(x)$ to $U(x) + C$, if necessary, we may assume w.l.g. that $U(\infty) > 0$ (compare Lemma 6.1 below and the subsequent discussion). Fix $\gamma > \limsup_{x\to\infty}(xU'(x)/U(x))$; we infer from Lemma 6.3 that there is $x_0 > 0$, s.t.,

$$U(\lambda x) < \lambda^\gamma U(x) \quad \text{for } \lambda > 1, x > x_0. \tag{3.19}$$

We have to show that there is $x_1 > 0$, s.t.,

$$u(\lambda x) < \lambda^\gamma u(x) \quad \text{for } \lambda > 1, x > x_1. \tag{3.20}$$

First suppose that assertion (3.19) holds true for each $x > 0$ and $\lambda > 1$, which implies

$$\begin{aligned} u(\lambda x) &= E\big[U\big(\hat{g}(\lambda x)\big)\big] \\ &\leq E\bigg[\lambda^{\gamma} U\bigg(\frac{\hat{g}(\lambda x)}{\lambda}\bigg)\bigg] \\ &\leq \lambda^{\gamma} u(x). \end{aligned}$$

This gives the desired inequality (3.20) for all $x > 0$.

Now assume that (3.19) only holds true for $x \geq x_0$; replace U by the utility function $\tilde{U}$ which is defined by

$$\tilde{U}(x) = \begin{cases} c_1 \dfrac{x^{\gamma}}{\gamma}, & \text{for } x \leq x_0, \\ c_2 + U(x), & \text{for } x \geq x_0, \end{cases}$$

where the constants c_1, c_2 are such that we achieve smooth pasting at x_0: choose c_1 such that $c_1 x_0^{\gamma-1} = U'(x_0)$ and c_2 such that $c_1(x_0^{\gamma}/\gamma) = c_2 + U(x)$.

The utility function $\tilde{U}$ now satisfies (3.19) for all $x > 0$; hence we know that the corresponding value function $\tilde{u}$ satisfies (3.20), for all $x > 0$. Clearly there is a constant $K > 0$ such that

$$U(x) - K \leq \tilde{U}(x) \leq U(x + x_0) + K, \qquad x > 0;$$

hence we obtain for the corresponding value functions

$$u(x) - K \leq \tilde{u}(x) \leq u(x + x_0) + K,$$

and in particular there is a constant $C > 0$ and $x_2 > 0$ such that

$$u(x) - C \leq \tilde{u}(x) \leq u(x) + C \quad \text{for } x \geq x_2,$$

so that we may deduce from Lemma 6.4 that $AE(u) = AE(\tilde{u}) \leq \gamma$, which completes the proof. □

PROOF OF THEOREM 3.2. We have to check that the above lemmas imply all the assertions of Theorem 3.2.

As regards the assertions

$$u'(\infty) = \lim_{x\to\infty} u'(x) = 0 \quad \text{and} \quad -v'(0) = \lim_{x\to 0} -v'(y) = \infty,$$

they are equivalent as, by Theorem 3.1(i) and Lemma 3.8, $-v'(y)$ is the inverse function of $u'(x)$. Hence it suffices to prove the first one. We have established in Lemma 3.10 that $AE(u) < 1$, which implies in particular that $u'(\infty) = 0$.

To show the validity of the three assertions,

$$E\big[\hat{g}(x)\hat{h}(y)\big] = xy, \qquad u'(x) = E\bigg[\frac{\hat{g}(x)U'\big(\hat{g}(x)\big)}{x}\bigg],$$

$$v'(y) = E\bigg[\frac{\hat{h}(y)V'\big(\hat{h}(y)\big)}{y}\bigg],$$

we have established the third one in Lemma 3.8. The other two assertions are simply reformulations, when we use the relations $y = u'(x)$, $x = -v'(y)$, $\hat{g}(x) = -V'(\hat{h}(y))$ and $\hat{h}(y) = U'(\hat{g}(x))$.

The proof of Theorem 3.2 now is complete. □

We complete the section with the following proposition, which will be used in the proof of item (iv) of Theorem 2.2. Let $\tilde{\mathscr{D}}$ be a convex subset of $\mathscr{D}$ such that (1) for any $g \in \mathscr{C}$,

$$\sup_{h\in\tilde{\mathscr{D}}} E[gh] = \sup_{h\in\mathscr{D}} E[gh]. \tag{3.21}$$

(2) The set $\tilde{\mathscr{D}}$ is closed under countable convex combinations, that is, for any sequence $(h^n)_{n\geq 1}$ of elements of $\tilde{\mathscr{D}}$ and any sequence of positive numbers $(a^n)_{n\geq 1}$ such that $\sum_{n=1}^{\infty} a^n = 1$, the random variable $\sum_{n=1}^{\infty} a^n h^n$ belongs to $\tilde{\mathscr{D}}$.

PROPOSITION 3.2. *Assume that the assumptions of Theorem* 3.2 *hold true and that $\tilde{\mathscr{D}}$ satisfies the above assertions. The value function $v(y)$ defined in* (3.5) *equals*

$$v(y) = \inf_{h\in\tilde{\mathscr{D}}} E[V(yh)]. \tag{3.22}$$

PROOF. Let us fix $\varepsilon > 0$. For $n > 0$, we define

$$V^n(y) = \max_{0<x\leq n}\left[U(x) - xy\right], \qquad y > 0.$$

The function V^n is convex and $V^n \uparrow V$, $n \to \infty$. By Lemma 6.3 below for any random variable $h > 0$,

$$E[V(h)] < \infty \Rightarrow E[V(\lambda h)] < \infty \qquad \forall\, \lambda \in (0,1). \tag{3.23}$$

Hence, for any integer k we can find a number $n(k)$ such that

$$E\left[V^{n(k)}\left(\frac{1}{2^k}\hat{h}(y)\right)\right] \geq E\left[V\left(\frac{1}{2^k}\hat{h}(y)\right)\right] - \frac{\varepsilon}{2^k}, \tag{3.24}$$

where $\hat{h}(y)$ is the optimal solution to (3.5). Denote

$$W^0 = V^{n(0)}, \dots, W^k = V^{n(k+1)} - V^{n(k)}, \dots .$$

The functions W^k, $k \geq 1$, are convex and decreasing. Since $W^k \leq V - V^{n(k)}$, $k \geq 1$, we deduce from (3.24) that

$$E\left[W^k\left(\frac{\hat{h}(y)}{2^k}\right)\right] \leq \frac{\varepsilon}{2^k}, \qquad k \geq 1. \tag{3.25}$$

From (3.21) and the convexity of $\tilde{\mathscr{D}}$ we deduce, by applying the bipolar theorem [3], that $\mathscr{D}$ is the *smallest* convex, closed, solid subset of $L^0_+(\Omega, \mathscr{F}, P)$ containing $\tilde{\mathscr{D}}$. It follows that for any h in $\mathscr{D}$ one can find a sequence $(f_n)_{n\geq 1}$ in $\tilde{\mathscr{D}}$ such that $f = \lim_{n\to\infty} f_n$ exists almost surely and $f \geq h$. In particular such a sequence exists for $h = \hat{h}(y)$ and in this case we deduce from the maximality of $\hat{h}(y)$ that $h = f = \lim_{n\to\infty} f_n$ almost surely.

Since $V^k(y) = V(y)$, for $y \geq I(k)$, and $V^k(y)$ is bounded from above, we deduce from Lemma 3.2 that, for k fixed, the sequence $V^k(f^n)$, $n \geq 1$, is uniformly integrable and therefore $EV^k(f^n) \to EV^k(\hat{h}(y))$ as $n \to \infty$. We can construct the sequence $(f^n)_{n\geq 1}$ such that

$$EW^k\left(\frac{f^n}{2^k}\right) \leq EW^k\left(\frac{\hat{h}(y)}{2^k}\right) + \frac{\varepsilon}{2^k}, \qquad n \geq k,\, k \geq 0.$$

We now define

$$f = \sum_{k=1}^{\infty} \frac{1}{2^k} f^k.$$

We have $f \in \tilde{\mathscr{D}}$, because the set $\tilde{\mathscr{D}}$ is closed under countable convex combinations, and

$$\begin{aligned} EW^k(f) &\overset{(1)}{\leq} EW^k\left(\sum_{i=1}^{\infty} \frac{1}{2^{k+i}} f^{k+i}\right) \overset{(2)}{\leq} \sum_{i=1}^{\infty} \frac{1}{2^i} EW^k\left(\frac{f^{k+i}}{2^k}\right) \\ &\leq EW^k\left(\frac{\hat{h}(y)}{2^k}\right) + \frac{\varepsilon}{2^k}, \qquad k \geq 0, \end{aligned} \tag{3.26}$$

where in (1) and (2) we used the fact that the function W^k is decreasing and convex. Finally, we deduce from (3.25) and (3.26) that

$$\begin{aligned} EV(f) = \sum_{k=0}^{\infty} EW^k(f) &\leq \sum_{k=0}^{\infty} EW^k\left(\frac{\hat{h}(y)}{2^k}\right) + 2\varepsilon \\ &\leq EV\big(\hat{h}(y)\big) + 3\varepsilon = v(y) + 3\varepsilon. \end{aligned}$$

The proof now is complete. □

4. Proof of the main theorems. In order to make the link between Theorems 2.1 and 2.2 and their "abstract versions," 3.1 and 3.2, we still have to prove Proposition 3.1.

Let us first comment on the content of Proposition 3.1 and its relation to known results. First note that assertion (iii) as well as the convexity and solidity of $\mathscr{C}$ and $\mathscr{D}$ are rather obvious. The main content of Proposition 3.1 in the closedness of $\mathscr{C}$ and $\mathscr{D}$ (w.r.t. the topology of convergence in measure) and the bipolar relation (ii) between $\mathscr{C}$ and $\mathscr{D}$.

In order to deal with this bipolar relation in the proper generality recall that, for a nonempty set $C \subseteq L^0_+(\Omega, \mathscr{F}, P)$, we define its *polar* C^0 by

$$C^0 = \{h \in L^0_+(\Omega, \mathscr{F}, P)\colon E[gh] \leq 1, \text{ for all } g \in C\}.$$

Using this terminology, assertion (ii) of Proposition 3.1 states that $\mathscr{C} = \mathscr{D}^0$ and $\mathscr{D} = \mathscr{C}^0$.

Let us recall known results pertaining to the content of Proposition 3.1. It was shown by Delbaen and Schachermayer (see [7] for the case of a locally bounded semi-martingale S and [10], Theorems 4.1 and 5.5, for the general case) that assumption (2.2) implies that $\mathscr{C}$ is closed w.r.t. the topology of convergence in measure and that $g \in \mathscr{C}$ iff, for each $Q \in \mathscr{M}^e(S)$, we have

$$E_Q[g] = E\left[g\frac{dQ}{dP}\right] \leq 1. \tag{4.1}$$

Denoting by $\tilde{\mathscr{D}}$ the subset $\mathscr{D}$ consisting of the functions h of the form $h = dQ/dP$, for some $Q \in \mathscr{M}^e(S)$, and using the above terminology, assertion (4.1) may be phrased as

$$\mathscr{C} = \tilde{\mathscr{D}}^0. \tag{4.2}$$

On the other hand, it follows from the definition of $\mathscr{D}$ that, for $h \in \mathscr{D}$ and $g \in \mathscr{C}$, we have $E[gh] \leq 1$; in other words,

$$\mathscr{D} \subseteq \mathscr{C}^0 = \tilde{\mathscr{D}}^{00}. \tag{4.3}$$

It was shown in [3] that the following version of the *bipolar theorem* holds true: for a subset A of $L^0_+(\Omega, \mathscr{F}, P)$ the bipolar A^{00} of A is the smallest subset of $L^0_+(\Omega, \mathscr{F}, P)$ containing A, which is convex, solid and closed w.r.t. the topology of convergence in measure.

Hence, in order to complete the proof of Proposition 3.1 it will suffice to prove the following lemma.

LEMMA 4.1. *The set $\mathscr{D}$ is closed with respect to the topology of convergence in measure.*

In order to prove Lemma 4.1 we recall the concept of Fatou convergence in the setting of stochastic processes (see [13]).

DEFINITION 4.1. Let $(X^n)_{n\geq 1}$ be a sequence of stochastic processes defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ and τ be a dense subset of $\mathbb{R}_+$. The sequence $(X^n)_{n\geq 1}$ is *Fatou convergent on* τ to a process X, if $(X^n)_{n\geq 1}$ is uniformly bounded from below and

$$\begin{aligned} X_t &= \limsup_{s\downarrow t,\, s\in\tau}\ \limsup_{n\to\infty} X^n_s \\ &= \liminf_{s\downarrow t,\, s\in\tau}\ \liminf_{n\to\infty} X^n_s \end{aligned}$$

almost surely for all $t \geq 0$. If $\tau = \mathbb{R}_+$, then the sequence $(X^n)_{n\geq 1}$ is called simply *Fatou convergent*.

The following lemma on Fatou convergence was proved in [13].

LEMMA 4.2. *Let* $(X^n)_{n\geq 1}$ *be a sequence of supermartingales,* $X_0^n = 0$, $n \geq 1$, *which is uniformly bounded from below, and* τ *be a dense countable subset of* $\mathbb{R}_+$. *There is a sequence* $Y^n \in \mathrm{conv}(X^n, X^{n+1}, \ldots)$, $n \geq 1$, *and a supermartingale* Y, $Y_0 \leq 0$, *such that* $(X^n)_{n\geq 1}$ *is Fatou convergent on* τ *to* Y. $\square$

PROOF OF LEMMA 4.1. Let $(g^n)_{n\geq 1}$ be a sequence in $\mathscr{D}$, which converges almost surely to a function g, and $(Y^n)_{n\geq 1}$ be a sequence in $\mathscr{Y}$ such that $Y_T^n \geq g^n$. We have to show that g is in $\mathscr{D}$. Without restriction of generality we may suppose that these processes are constant on $[T, +\infty)$. By Lemma 4.2 there is a sequence $Z^n \in \mathrm{conv}(Y^n, Y^{n+1}, \ldots)$, $n \geq 1$, which is Fatou convergent to a process Z on the set of rational points. By the same lemma $(X_t Z_t)_{0\leq t\leq T}$ is a supermartingale, for each $X \in \mathscr{X}$ and $Z_0 \leq 1$. By passing from Z to Z/Z_0, if necessary, we may assume that $Z \in \mathscr{Y}$. The result now follows from the obvious inequality, $Z_T \geq g$. $\square$

PROOF OF PROPOSITION 3.1. Let us verify that Lemma 4.1 indeed implies Proposition 3.1: the set $\mathscr{D}$ contains $\tilde{\mathscr{D}}$ and clearly is convex and solid. By Lemma 4.1 it also is closed and therefore we may apply the bipolar Theorem to conclude that

$$\mathscr{D} \supseteq \tilde{\mathscr{D}}^{00}. \tag{4.4}$$

It follows that

$$\mathscr{D} = \tilde{\mathscr{D}}^{00} = \mathscr{D}^{00} \tag{4.5}$$

and therefore, using (4.2) and the fact that $\mathscr{C}^{00} = \mathscr{C}$,

$$\mathscr{D} = \mathscr{C}^0 \quad \text{and} \quad \mathscr{C} = \mathscr{D}^0 = \tilde{\mathscr{D}}^0,$$

which implies assertions (i) and (ii) of Proposition 3.1. As regards assertion (iii), it is obvious that $\mathscr{C}$ contains the constant function $\mathbb{1}$. The L^0-boundedness of $\mathscr{C}$, which by (ii) is equivalent to the existence of a strictly positive element $g \in D$, is implied by assumption (2.2). $\square$

If we combine Proposition 3.1 with Theorems 3.1 and 3.2, we obtain precisely Theorems 2.1 and 2.2, with the exception of item (iv) of Theorem 2.2, which now follows from the fact that $\mathscr{M}$ is closed under countable convex combinations and Proposition 3.2, observing that (3.29) is implied by (4.2) and (4.5).

The proof of Theorems 2.1 and 2.2 now is complete.

As regards Theorem 2.0 we still have to show the validity of the remaining assertions of Theorem 2.0 which are not directly implied by Theorem 2.1 (note that in Theorem 2.0 we did not make any assumption on the asymptotic elasticity of U so that Theorem 2.2 does not apply).

We start by observing that in the complete case, the definitions of $v(y)$ given in (2.8) and (2.9) indeed coincide.

LEMMA 4.3. *Assume that the family $\mathscr{M} = \mathscr{M}^e(S)$ of martingale measures consists of one element Q only. Then for the function $v(y)$ as defined in* (2.9) *we have*

$$v(y) = E\left[V\left(y\left(\frac{dQ}{dP}\right)\right)\right], \tag{2.10}$$

where dQ/dP is the Radon–Nikodym derivative of Q with respect to P on $(\Omega, \mathscr{F}_T)$.

PROOF. We denote by $Z = (Z_t)_{0\le t\le T}$ the density process of Q with respect to P. Let Y be an element of $\mathscr{Y}(1)$. We shall show that the set $A = \{Y_T > Z_T\}$ has measure zero, which will prove the lemma. Denoting by

$$a = Q(A),$$

we have to show $a = 0$, as the measures P and Q are equivalent.

Suppose that $a > 0$. The process

$$M_t = \frac{1}{Z_t} E[Z_T \mathbb{1}_A \mid \mathscr{F}_t]$$

is a martingale under Q with the initial value $M_0 = a$ and the terminal value $M_T = \mathbb{1}_A$. By our completeness assumption we may apply Jacod's theorem (see [19], page 338, Theorem 11.2) so that M can be represented as a stochastic integral with respect to S,

$$M_t = a + \int_0^t H_u \, dS_u.$$

Hence $M \in \mathscr{X}(a)$. However,

$$E[Y_T M_T] = E[Y_T \mathbb{1}_A] > E[Z_T \mathbb{1}_A] = a = Y_0 M_0,$$

which contradicts the supermartingale property of YM. $\square$

PROOF OF THEOREM 2.0. We first prove that

$$v'(y) = E\left[\frac{dQ}{dP} V'\left(y\frac{dQ}{dP}\right)\right], \tag{4.6}$$

for each $y > y_0$. Indeed, fix $y > y_0$ and $h > 0$; for almost each $\omega \in \Omega$ we have

$$V\left((y+h)\frac{dQ}{dP}(\omega)\right) - V\left(y\frac{dQ}{dP}(\omega)\right) = \int_y^{y+h} \frac{dQ}{dP}(\omega) V'\left(z\frac{dQ}{dP}(\omega)\right) dz;$$

hence

$$\begin{aligned} v(y+h) - v(y) &= E\left[V\left((y+h)\frac{dQ}{dP}\right) - V\left(y\frac{dQ}{dP}\right)\right] \\ &= E\left[\int_y^{y+h} \frac{dQ}{dP} V'\left(z\frac{dQ}{dP}\right) dz\right] \\ &= \int_y^{y+h} E\left[\frac{dQ}{dP} V'\left(z\frac{dQ}{dP}\right)\right] dz, \end{aligned}$$

where we are allowed to use Fubini's theorem above as the integrand (dQ/dP) $V'(z(dQ/dP))$ is negative on $\Omega \times [y, y+h]$. As the double integral is finite we obtain (4.6).

Using the definition of $\hat{X}(x)$ given in Theorem 2.0(ii) and the relations $y = u'(x)$, $x = -v'(y)$ for $0 < x < x_0$ and $y > y_0$, we obtain the formula

$$u'(x) = E\left[\frac{\hat{X}_T(x)U'(\hat{X}_T(x))}{x}\right], \qquad 0 < x < x_0 \tag{4.7}$$

and

$$E_Q[\hat{X}_T(x)] = E\left[I\left(y\frac{dQ}{dP}\right)\frac{dQ}{dP}\right] = -v'(y) = x, \tag{4.8}$$

thus proving items (ii) and (iii) of Theorem 2.0.

Formula (4.8) in conjunction with the martingale representation theorem shows in particular that $\hat{X}(x) \in \mathcal{X}(x)$. We still have to show that $\hat{X}(x)$ is the optimal solution of (2.3). To do so we follow the classical reasoning based on the fact that the marginal utility $U'(\hat{X}_T(x))$ is proportional to dQ/dP: let $X(x)$ be any element of $\mathcal{X}(x)$. As $E_Q[X_T(x)] \le x$ we obtain

$$\begin{aligned}
E\big[U(X_T(x))\big] &= E\big[U(\hat{X}_T(x)) + \big(U(X_T(x)) - U(\hat{X}_T(x))\big)\big] \\
&\le E\big[U\big(\hat{X}_T(x)\big)\big] + E\big[U'\big(\hat{X}_T(x)\big)\big(X_T(x) - \hat{X}_T(x)\big)\big] \\
&= E\big[U\big(\hat{X}_T(x)\big)\big] + E_Q\left[\frac{dP}{dQ}U'(\hat{X}_T(x))\big(X_T(x) - \hat{X}_T(x)\big)\right] \\
&= E\big[U\big(\hat{X}_T(x)\big)\big] + yE_Q\big[X_T(x) - \hat{X}_T(x)\big] \\
&\le E\big[U(\hat{X}_T(x))\big],
\end{aligned}$$

where, by the strict concavity of U, in the second line we have strict inequality if $X_T(x) \not\equiv \hat{X}_T(x)$. This readily shows that $\hat{X}(x)$ is the unique optimal solution of (2.3).

To prove item (i), note that it follows from (4.6) that v is continuously differentiable and strictly convex on (y_0, ∞), hence by general properties of the Legendre transform [31] we have that u is continuously differentiable and strictly concave on $(0, x_0)$. $\square$

5. Counterexamples. We start with an example of a continuous security market and a well-behaved utility function U for which the infimum in Theorem 2.2(iv) is not attained.

EXAMPLE 5.1. The construction of the financial market is exactly the same as in [9]. Let B and W be two independent Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, P)$, where the filtration $(\mathcal{F}_t)_{t\ge 0}$ is supposed to be generated by B and W. The process L defined as

$$L_t = \exp\left(B_t - \tfrac{1}{2}t\right), \qquad t \in \mathbb{R}_+,$$

is known to be a martingale but not a uniformly integrable martingale, because L_t tends to 0 almost surely as t tends to ∞. The stopping time τ is defined as

$$\tau = \inf\{t \geq 0\colon L_t = 1/2\}.$$

Clearly $\tau < \infty$ a.s. Similarly, we construct a martingale

$$M_t = \exp\left(W_t - \tfrac{1}{2}t\right).$$

The stopping time σ is defined as

$$\sigma = \inf\{t \geq 0\colon M_t = 2\}.$$

The stopped process $M^\sigma = (M_{t\wedge\sigma})_{t\geq 0}$ is a uniformly integrable martingale. In case M does not hit level 2, the stopping time σ equals ∞. Therefore we have that M_σ equals 2 or 0, each with probability 1/2.

We now define the security market model with the time horizon,

$$T = \tau \wedge \sigma,$$

and the (stock) price process,

$$S_t = \exp\left\{-B_t + \tfrac{1}{2}t\right\}.$$

The utility function U is defined as

$$U(x) = \ln x,$$

in which case $I(y) = -V'(y) = 1/y$ and $V(y) = -\ln y - 1$.

PROPOSITION 5.1. *The following assertions hold true*:

(i) *The process $L^T M^T = (L_{t\wedge T} M_{t\wedge T})_{t\geq 0}$ is the density process of an equivalent martingale measure and hence $\mathscr{M} \neq \emptyset$.*

(ii) *The process $L^T = (L_{t\wedge T})_{t\geq 0}$ is not a uniformly integrable martingale and hence is not the density process of an equivalent martingale measure.*

(iii) *The process L^T is the unique optimal solution of the optimization problem*,

$$v(1) = \inf_{Y\in\mathscr{Y}(1)} E[V(Y_T)] = -\sup_{Y\in\mathscr{Y}(1)} E[\ln Y_T + 1].$$

(iv) *The process S^T is in the unique optimal solution to the optimization problem*

$$u(1) = \sup_{X\in\mathscr{X}(1)} E[U(X_T)] = \sup_{X\in\mathscr{X}(1)} E[\ln(X_T)].$$

PROOF. The items (i) and (ii) were proved in [9]. Clearly, $L \in \mathscr{Y}(1)$. For any $Y \in \mathscr{Y}(1)$, the process $Y/L = YS$ is a supermartingale starting at $Y_0 S_0 = 1$. Hence, by Jensen's inequality,

$$E[\ln Y_T] = E\left[\ln \frac{Y_T}{L_T}\right] + E[\ln L_T] \le \ln\left(E\left[\frac{Y_T}{L_T}\right]\right) + E[\ln L_T] \le E[\ln L_T].$$

To complete the proof, it is sufficient to show that

$$v(1) = -E[\ln L_T] - 1 < \infty.$$

From the supermartingale property of the process

$$N_t = \sqrt{L_t}\exp\left(\frac{t}{8}\right) = \exp\left(\frac{B_t}{2} - \frac{t}{8}\right)$$

and the inequality $L_T \ge 1/2$, we deduce that

$$E\left[\exp\left(\frac{T}{8}\right)\right] \le \sqrt{2}.$$

It follows that B^T is a uniformly integrable martingale and

$$E[\ln L_T] = E\left[B_T - \tfrac{1}{2}T\right] = -\tfrac{1}{2}E[T] > -\infty.$$

Assertion (iv) now follows from Theorem 2.2(ii). □

We give one more example displaying a phenomenon similar to Example 5.1 above, that is, that the infimum in (2.2)(iv) is not attained.

Example 5.1′ below will not be a continuous process, which is a drawback in comparison to Example 5.1. On the other hand, Example 5.1′ has some other merits: it is a one-period process defined on a countable probability space Ω and it shows that the optimal solution $\hat{Y}(y)$ to (2.9) may fail to be a local martingale.

EXAMPLE 5.1′. Let $(p_n)_{n=0}^{\infty}$ be a sequence of strictly positive numbers, $\sum_{n=0}^{\infty} p_n = 1$, tending sufficiently fast to zero and $(x_n)_{n=0}^{\infty}$ a sequence of positive reals, $x_0 = 2$, decreasing also to zero [but less fast than $(p_n)_{n=0}^{\infty}$]. For example, $p_0 = 1 - \alpha,\ p_n = \alpha 2^{-n}$, for $n \ge 1$, and $x_0 = 2,\ x_n = 1/n$, for $n \ge 1$, will do, if $0 < \alpha < 1$ is small enough to satisfy $(1-\alpha)/2 + \alpha \sum_{n=1}^{\infty} 2^{-n}(-n+1) > 0$.

Now define $S \triangleq (S_0, S_1)$ by letting $S_0 \equiv 1$ and S_1 to take the values $(x_n)_{n=0}^{\infty}$ with probability p_n. As filtration we choose the natural filtration generated by S. Clearly, the process S satisfies $\mathscr{M}^e(S) \neq \emptyset$.

In this easy example we can explicitly calculate the family of processes $\mathcal{X}(1)$: it consists of all processes X with $X_0 = 1$ and such that X_1 equals the random variable $X^\lambda \triangleq 1 + \lambda(S_1 - S_0)$, for some $-1 \le \lambda \le 1$.

Using again $U(x) = \ln(x)$ as a utility function and writing $f(\lambda) = E[U(X^\lambda)]$, we obtain by an elementary calculation,

$$f'(\lambda) = \sum_{n=0}^{\infty} p_n \frac{x_n - 1}{1 + \lambda(x_n - 1)}$$

so that $f'(\lambda)$ is strictly positive for $-1 \le \lambda \le 1$ if $\alpha > 0$ satisfies the above assumption $f'(1) = (1-\alpha)\frac{1}{2} + \alpha \sum_{n=1}^{\infty} 2^{-n}(-n+1) > 0$. Hence $f(\lambda)$ attains its maximum on $[-1, 1]$ at $\lambda = 1$; in other words, the optimal investment process $\hat{X}(1)$ equals the process S.

We can also explicitly calculate $u(x)$ by

$$\begin{aligned} u(x) &= E[U(xS_1)] = \textstyle\sum_{n=0}^{\infty} p_n U(x x_n) \\ &= \sum_{n=0}^{\infty} p_n \bigl(\ln(x) + \ln(x_n)\bigr) = \ln(x) + \sum_{n=0}^{\infty} p_n \ln(x_n). \end{aligned}$$

In particular, $u'(1) = 1$ and by Theorem 2.2 we get $\hat{Y}(1) = U'\bigl(\hat{X}(1)\bigr) = (S_1)^{-1}$. Note that

$$E\bigl[S_1^{-1}\bigr] = \sum_{n=0}^{\infty} \frac{p_n}{x_n} = \frac{p_0}{2} + \sum_{n=1}^{\infty} n p_n$$

is strictly less than 1 by using again the condition $(1-\alpha)\frac{1}{2} + \alpha \sum_{n=1}^{\infty} 2^{-n}(-n+1) > 0$. In particular, the optimal element $\hat{Y}(1) \in \widehat{\mathscr{Y}}(1)$ is not a martingale (not even a local martingale) but only a supermartingale and $\hat{Y}_1(1)$ is not the density of a martingale measure for the process S. This finishes the presentation of Example 5.1$'$.

From this point on we will assume that the asymptotic elasticity of the utility function U equals 1. By Corollary 6.1(iii) below this is equivalent to the following property of the conjugate function V of U:

$$\begin{gathered} \text{For any } y_0 > 0,\ 0 < \mu < 1,\ C > 0, \text{ there is} \\ 0 < y < y_0 \text{ s.t. } V(\mu y) > C V(y). \end{gathered} \tag{5.1}$$

LEMMA 5.1. *Assume that the function V satisfies* (5.1). *Then there is a probability measure Q on $\mathbb{R}_+$ supported by a sequence $(x_k)_{k\ge 0}$ decreasing to* 0 *such that*:

(i) $\int_0^\infty V(x) Q(dx) < \infty$;
(ii) $\int_0^\infty x I(x) Q(dx) = -\int_0^\infty x V'(x) Q(dx) < \infty$;
(iii) $\int_0^\infty V(\gamma x) Q(dx) = \infty$ *for any* $\gamma < 1$.

PROOF. Without loss of generality, we may assume that $V > 0$. Since the function V satisfies (5.1), there is a decreasing sequence $(y_n)_{n\ge 1}$ of positive numbers converging to 0 such that, for any $0 < \gamma < 1$,

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n}} \frac{V(\gamma y_n)}{V(y_n)} = +\infty. \tag{5.2}$$

Denote

$$x_n = y_n \Big/ \left(1 - \frac{1}{2^n}\right),$$

$$p_n = \frac{K}{2^{2n} V(y_n)},$$

where the normalizing constant K is chosen s.t. $\sum_{n=1}^{\infty} p_n = 1$. We now are ready to define the measure Q, which is supported by the sequence $(x_n)_{n \geq 1}$,

$$Q(x_n) = p_n.$$

Let us check the assertions of our lemma. We have

$$\int_0^\infty V(x) Q(dx) = \sum_{n=1}^\infty p_n V(x_n) \leq \sum_{n=1}^\infty p_n V(y_n) = K \sum_{n=1}^\infty \frac{1}{2^{2n}} \frac{K}{3},$$

proving (i). As regards (ii), we use the inequality

$$xI(x) \leq \frac{1}{1-\gamma}\bigl(V(\gamma x) - V(x)\bigr) \leq \frac{1}{1-\gamma} V(\gamma x),$$

which is valid for any $\gamma < 1$ and $x > 0$, to get

$$x_n I(x_n) \leq 2^n V(y_n),$$

and hence

$$\begin{aligned}\int_0^\infty xI(x) Q(dx) &= \sum_{n=1}^\infty p_n x_n I(x_n) \leq \sum_{n=1}^\infty p_n 2^n V(y_n) \\ &= K \sum_{n=1}^\infty \frac{1}{2^n} = K.\end{aligned}$$

Finally, (5.2) implies (iii): for any $\gamma < 1$,

$$\int_0^\infty V(\gamma x) Q(dx) = \sum_{n=1}^\infty p_n V(\gamma x_n) = \infty.$$

The proof is complete. □

NOTE 5.1. The assertions (i)–(iii) of Lemma 5.1 are sensitive only to the behavior of Q near zero. For example, we can always choose Q in such a way that $\int_0^\infty xQ(dx) = 1$ or $Q((0,1)) = 1$.

We now construct an example of a *complete* continuous financial market such that the assertions (i), (ii) and (iii) of Theorem 2.2 fail to hold true as soon as $AE(U) = 1$. We start with an easy observation which shows the intimate relation between assertion (i) and (ii) of Theorem 2.2:

SCHOLIUM 5.1. *Under the hypotheses of Theorem* 2.1, *suppose that, for* $0 < x_1 < x_2$, *the optimal solutions* $\hat{X}(x_1) \in \mathcal{X}(x_1)$ *and* $\hat{X}(x_2) \in \mathcal{X}(x_2)$ *in* (2.3) *exist. Then*

$$u\left(\frac{x_1 + x_2}{2}\right) > \frac{u(x_1) + u(x_2)}{2}.$$

Hence, if $u'(x) \equiv 1$ *for* $x \geq a$, *there is at most one* $x \geq a$ *for which an optimal solution* $\hat{X}(x) \in \mathcal{X}(x)$ *to* (2.3) *can exist.*

PROOF. For $\hat{X}(x_1) \in \mathcal{X}(x_1)$ and $\hat{X}(x_2) \in \mathcal{X}(x_2)$, the convex combination $X = (\hat{X}(x_1)+\hat{X}(x_2))/2$ is an element of $\mathcal{X}((x_1 + x_2)/2)$. By the strict concavity of the utility function U we have

$$u\left(\frac{x_1 + x_2}{2}\right) \geq E[U(X)] > \frac{E\big[U\big(\hat{X}(x_1)\big)\big] + E\big[U\big(\hat{X}(x_2)\big)\big]}{2} = \frac{u(x_1) + u(x_2)}{2}.$$

The second assertion is an immediate consequence. □

After this preliminary result we give the construction of our example.

EXAMPLE 5.2. Let U be a utility function satisfying (2.4) and such that $AE(U) = 1$. Let W be a standard Brownian motion with $W_0 = 0$ defined on a filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$, where $0 < T < \infty$ is fixed and the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ is supposed to be generated by W. Let Q be a measure on $(0, \infty)$ for which the assertions (i)–(iii) of Lemma 5.1 hold true and such that (see Note 5.1)

$$\int_0^\infty x Q(dx) = 1. \tag{5.3}$$

Let

$$a = \int_0^\infty x I(x) Q(dx).$$

and η be a random variable on $(\Omega, \mathcal{F}_T)$, whose distribution under P coincides with the measure Q. Clearly, (5.3) implies that $E\eta = 1$. The process

$$Z_t = E[\eta \mid \mathcal{F}_t], \qquad t \geq 0.$$

is a strictly positive martingale with initial value $Z_0 = 1$. From the integral representation theorem we deduce the existence of a predictable process $\mu = (\mu)_{t \geq 0}$ such that

$$Z_t = 1 + \int_0^t \mu_s Z_s \, dW_s$$

or, equivalently,

$$Z_t = \exp\left(\int_0^t \mu_s \, dW_s - \tfrac{1}{2}\int_0^t \mu_s^2 \, ds\right).$$

The stock price process S is now defined as

$$S_t = 1 + \int_0^t S_u(-\mu_u \, du + dW_u). \tag{5.4}$$

The standard arguments based on the integral representation theorem and the Girsanov theorem imply that the family of martingale measures consists of exactly one element (i.e., the market is complete) and that the density process of the unique martingale measure is equal to Z.

PROPOSITION 5.2. *Let U be a utility function satisfying* (2.4) *and such that $AE(U) = 1$. Then for the security market model defined in* (5.4) *the following assertions hold true*:

(i) *For $x \le a$, the optimization problem* (2.3) *has a unique optimal solution $\hat{X}(x)$, while, for $x > a$, no optimal solution to* (2.3) *exists.*

(ii) *u is continuously differentiable*; *it is strictly concave on* $]0, a]$, *while $u'(x) = 1$, for $x \ge a$.*

(iii) *v is continuously differentiable and strictly convex on* $[1, \infty[$ *and the right derivative v'_r at $y = 1$ equals $v'_r(1) = -a$, while $v(y) = \infty$, for $y < 1$.*

PROOF. The equivalence of (ii) and (iii) follows from the fact that u and v are conjugate and from the following well-known relations from the theory of convex functions:

$$u(x) = \inf_{y>0}\,(v(y) + xy)\,, \qquad x > 0,$$

$$u'(s) = \inf\{t > 0\colon\ -v'(t) \le s\}, \qquad s \ge 0,$$

$$-v'(t) = \inf\{s > 0\colon u'(s) \le t\}, \qquad t \ge 0.$$

In order to prove (iii), note that

$$v(y) = E[V(yZ_T)] = \int_0^\infty V(yx)Q(dx), \qquad y > 0, \tag{5.5}$$

$$-v'(y) = E[Z_T I(yZ_T)] = \int_0^\infty xI(yx)Q(dx) \le a \quad \text{if } v(y) < \infty, \tag{5.6}$$

with equality holding in (5.6) for $y = 1$ [in which case $v'(y)$ has to be interpreted as the right derivative]. Indeed, equality (5.5) is the assertion of Lemma 4.3 and (5.6) follows from Theorem 2.0 and Lemma 5.1. The fact that $v'(y)$ is continuous on $[1, \infty[$ now follows from (5.6) by applying the monotone convergence theorem.

To show (i) note that, for $x \le a$, the random variable $\hat{X}(x) = I(y(dQ/dP))$ with $y = u'(x) \ge 1$ is the unique solution to the optimization problem (2.3).

Finally, it follows from Scholium 5.1, from (ii) and the fact that $\hat{X}(a)$ does exist that, for $x > a$ there cannot exist an optimal solution to (2.3).

NOTE 5.2. (a) The message of the above example is rather puzzling from an economic point of view (at least to the authors): consider an economic agent with utility function U satisfying (2.4) and $AE(U) = 1$, which is endowed with an initial capital x which is large enough such that $U'(x) < \varepsilon$, for a given small number $\varepsilon > 0$; in other words, by passing from the endowment x to $x + 1$, the utility $U(x)$ of the agent increases to $U(x + 1)$ by less than ε.

The situation changes drastically if the agent is allowed to invest in the complete market $S = (S_t)_{0 \le t \le T}$ and to maximize the expected utility of the resulting terminal wealth $X_T(x)$. In the above example, for $x \ge a$, the passage from x to $x + 1$ increases the maximal expected utility from $u(x)$ to $u(x + 1)$ by 1 [as $u'(z) \equiv 1$, for $z \ge a$]. How can this happen for such a "rich" agent, faced with small marginal utility $U'(z)$, if z is in the order of x?

We shall try to give an intuitive explanation of the phenomenon occuring in Example 5.2. What the agent does to choose an approximating sequence $X_n(x) \in \mathcal{X}(x)$ for the optimization problem (2.3) is the following: he or she uses the portion a of the initial endowment $x > a$ to finance the wealth $\hat{X}_T(a)$ at time T, which is the optimal investment for an agent endowed with initial capital a. With the remaining endowment $x - a$, he or she gambles in a very risky way: he or she bets it all on the event $B_n = \{Z_T = x_n\}$, for some large n. Noting that the random variable $\hat{X}(a)$ takes the value $\xi_n \triangleq I(x_n)$ on B_n, an easy calculation shows that the agent can increase the value of the investment at time $t = T$, contingent on B_n, from ξ_n to $(x - a)(x_n p_n)^{-1} + \xi_n$, by betting the amount $(x - a)$ at time $t = 0$ on the event $\{Z_T = x_n\}$. What is the increase $f_n(x - a)$ of expected utility? Clearly, we have

$$f_n(x - a) = p_n\big[U\big((x - a)(x_n p_n)^{-1} + \xi_n\big) - U(\xi_n)\big],$$

so that f_n is a strictly concave function of the variable $x - a \in \mathbb{R}_+$; another easy calculation reveals that $f_n'(0) = 1$ so that, "for small $x - a$" the gain in expected utility is approximately equal to (and slightly less then) $x - a$.

So far we have only followed the line of the usual infinitesimal Arrow–Debreu type arguments for the optimal investment $\hat{X}(a)$. The new ingredient is that, in the construction of Example 5.2, we have used the assumption $AE(U) = 1$ in order to choose the numbers x_n and p_n carefully, so that the functions $(f_n)_{n=1}^{\infty} = \big(f_n(x - a)\big)_{n=1}^{\infty}$ tend to the identity function uniformly on compact subsets of $\mathbb{R}_+$. Hence in Example 5.2 the above argument does not only hold for "small $x - a$" (in the sense of a first-order approximation); we now have that, *for any fixed* $(x - a) > 0$, the increase in expected utility $f_n(x - a)$ tends to $x - a$, as n tends to infinity.

This explanation of the phenomenon underlying Example 5.2 also indicates why, for $x > a$, there is no optimal solution $\hat{X}(x) \in \mathcal{X}(x)$, as in the above reasoning we obviously cannot "pass to the limit $n \to \infty$."

(b) We also note that Example 5.2 is in fact a very natural example: it may also be viewed, similarly to Examples 5.1 and 5.3 below, as an exponential Brownian motion with constant drift stopped at a stopping time T, which is finitely valued (but not bounded).

Indeed, fix Q as in Lemma 5.1 such that barycenter $(Q) = \int_0^\infty xQ(dx) = 1$ and such that for the decreasing sequence $(x_k)_{k\geq 0}$ supporting Q we have $x_0 > 1$ and $x_1 < 1$, which clearly is possible. Now let

$$R_t = \exp(W_t + t/2), \qquad t > 0.$$

By Girsanov's formula,

$$Z_t = \exp(-W_t - t/2), \qquad t > 0,$$

is the unique density process with $Z_0 = 1$ such that $R_t Z_t$ is a martingale.

We want to find a stopping time T such that the law of Z_T equals Q. Once we have done so, we may replace the definition of the stock price process S in (5.4) by

$$S_t = R_{t\wedge T} = \exp\bigl(W_{t\wedge T} + (t \wedge T)/2\bigr), \qquad t > 0 \tag{5.4$'$}$$

and deduce the conclusions of Proposition 5.2 for this stock price process in exactly the same way as above.

The existence of a stopping time T such that the law of Z_T equals Q is a variant of the well-known "Skorohod stopping problem." For the convenience of the reader we sketch a possible construction of T:

$$T = \inf\bigl\{t\colon Z_t = x_0 \text{ or } \bigl(Z_t = x_i \text{ and } t_{i-1} < t \leq t_i\bigr)\bigr\},$$

where the increasing sequence of deterministic times $(t_i)_{i=0}^\infty$ is defined inductively by $t_0 = 0$ and

$$t_i = \inf\bigl\{t\colon \mathbb{P}\bigl[Z_{t\wedge T_i} = x_i\bigr] = Q(x_i)\bigr\}.$$

The stopping times T_i are also inductively defined (after determining $t_0, \dots, t_{i-1}$) by

$$\begin{aligned} T_i = \inf\bigl\{t\colon Z_t = x_0 \text{ or } \bigl(Z_t = x_j \text{ and } t_{j-1} < t \leq t_j \text{ and } 1 \leq j < i\bigr)] \\ \text{or } \bigl(Z_t = x_i \text{ and } t_{i-1} < t\bigr)\bigr\}. \end{aligned}$$

Intuitively speaking, we start to define the stopping time T at time $t_0 = 0$ as the first moment when Z_t either hits $x_0 > 1$ or $x_1 < 1$ and continue to do so until the (deterministic) time t_1, when $\mathbb{P}[Z_{T\wedge t} = x_1]$ has reached the value $Q(x_1)$. Then we lower the stakes and define T to be the first moment when Z_t hits x_0 or x_2 and so on. It follows from the martingale property of Z_t and $\int_0^\infty Q(dx) = 1$ that T is finite almost surely and that the law of Z_T equals Q.

We close the section with an example of an (incomplete) continuous financial model such that assertion (iv) of Theorem 2.2 fails to hold true.

EXAMPLE 5.3. Let Q be a probability measure on $\mathbb{R}_+$ supported by a decreasing sequence $(x_k)_{k\geq 0}$: $1 > x_0 > x_1 > \cdots$ converging to 0, such that

$$\int_0^\infty V(x)Q(dx) < \infty,$$

$$\int_0^\infty V(\gamma x)Q(dx) = \infty \qquad \forall\, \gamma < 1.$$

The existence of such a measure follows from Lemma 5.1 and Note 5.1. Our construction will use a Brownian motion B and a sequence $(\varepsilon_n)_{n\geq 1}$ of independent (mutually as well as of B) random variables such that

$$\varepsilon_n = \begin{cases} 2^n, & \text{with probability } \dfrac{1}{2^{n+1}-1}, \\ \dfrac{1}{2}, & \text{with probability } 1 - \dfrac{1}{2^{n+1}-1}. \end{cases}$$

Note that $E\varepsilon_n = 1$.

The martingale L is defined as

$$L_t = \exp\bigl(B_t - \tfrac{1}{2}t\bigr).$$

Similarly to Note 5.2(b), we define the increasing sequence $0 = t_0 < t_1 < \cdots < t_k < \cdots$ in $\mathbb{R}_+$ in such a way that the deterministic function

$$\phi(t) = \sum_{k=0}^{\infty} x_k \mathbb{1}_{\{t_k \leq t < t_{k+1}\}}$$

has the property that the probability that the stopping time

$$\tau = \inf\{t \geq 0 \colon L_t = \phi(t)\}$$

belongs to the interval $[t_k, t_{k+1})$ is equal to $Q(x_k)$. In other words, the distribution of the random variable L_τ under P is equal to Q. Since $\sum_{k=0}^{\infty} Q(x_k) = 1$, the stopping time τ is finite a.s.

Using the sequence $(\varepsilon_n)_{n\geq 1}$, we construct the martingale

$$M_t = \prod_{i=1}^{[t]} \varepsilon_i,$$

where $[t]$ denotes the largest integer less then t. The stopping time σ is defined as

$$\sigma = \inf\{t \geq 0 \colon M_t = 2\}.$$

The stopped process $M^\sigma = (M_{t\wedge\sigma})_{t\geq 0}$ is a uniformly integrable martingale. In the case M does not hit level 2, the stopping time σ equals ∞. Therefore we have that M_σ equals 2 or 0, each with probability 1/2.

The final ingredient of our construction is the stopping time ψ defined as

$$\psi = \inf\{t \geq \sigma \colon L_t - L_\sigma \geq 1\}.$$

Note that L is a uniformly integrable martingale on $[\tau\wedge\sigma, \tau\wedge\psi]$, that is,

$$E\bigl[L_{\tau\wedge\psi} \mid \mathscr{F}_{\tau\wedge\sigma}\bigr] = L_{\tau\wedge\sigma}.$$

We now determine the security market model with the time horizon

$$T = \tau \wedge \psi \tag{5.7}$$

and the price process

$$S_t = \exp\bigl\{-B_t + \tfrac{1}{2}t\bigr\}, \qquad 0 \leq t \leq T = \tau\wedge\psi, \tag{5.8}$$

defined on a filtered probability space $(\Omega, \mathscr{F}, P)$, where the filtration is supposed to be generated by L^T and M^σ (note that M is stopped at time σ, which is less than or equal to T).

PROPOSITION 5.3. *Assume that the utility function U satisfies* (2.4) *and $AE(U) = 1$. Then for the financial model defined in* (5.7) *and* (5.8), *the following assertions hold true*:

(i) *The family of equivalent local martingale measures for the process S is not empty.*

(ii) *The process $L^T = L^{\tau\wedge\psi}$ is an element of $\mathscr{Y}(1)$ and*

$$E[V(L_T)] < \infty.$$

However L^T is not a uniformly integrable martingale and hence is not the density process of an equivalent martingale measure.

(iii) *If Y is an element of $\mathscr{Y}(1)$ and $Y \not\equiv L$, then $EV(Y_T) = \infty$. In particular,*

$$E\left[V\left(\frac{dQ}{dP}\right)\right] = \infty$$

for any martingale measure Q.

PROOF. (i) Let us show that the process $L^T M^\sigma$ is a uniformly integrable martingale and hence is the density process of a martingale measure. Indeed,

$$\begin{aligned} E[L_T M_\sigma] &\overset{(1)}{\leq} E[L_{\tau\wedge\sigma} M_\sigma] = 2E[L_{\tau\wedge\sigma} \mathbb{1}_{\{\sigma<\infty\}}] \\ &= \lim_{n\to\infty} 2E[L_{\tau\wedge\sigma\wedge n} \mathbb{1}_{\{\sigma\leq n\}}] = \lim_{n\to\infty} 2E[L_n \mathbb{1}_{\{\sigma\leq n\}}] \\ &= \lim_{n\to\infty} 2[L_n]P[\sigma \leq n] = \lim_{n\to\infty} 2P[\sigma \leq n] = 1, \end{aligned}$$

where in (i) we used the fact that L is a uniformly integrable martingale on $[\tau \wedge \sigma, T]$.

(ii) Since L is a martingale and $SL \equiv 1$, we have that L^T is an element of $\mathscr{Y}(1)$. From the equality

$$E[L_T \mathbb{1}_{\{\sigma<\infty\}}] = \tfrac{1}{2},$$

proved above, we deduce that

$$\begin{aligned} E[L_T] &= E[L_\tau \mathbb{1}_{\{\sigma=\infty\}}] + E[L_T \mathbb{1}_{\{\sigma<\infty\}}] \\ &= \tfrac{1}{2}(E[L_\tau] + 1) < \tfrac{1}{2}(x_0 + 1) < 1. \end{aligned}$$

Hence L^T is not a uniformly integrable martingale. Finally,

$$E[V(L_T)] \leq E[V(L_\tau)] = \int_0^\infty V(x) Q(dx) < \infty,$$

where the first inequality holds true, because $L_T \geq L_\tau$ and V is a decreasing function.

(iii) To avoid technicalities, we assume hereafter that $V > 0$. We start with two lemmas.

LEMMA 5.2. *Let χ be a stopping time. Then, for a set $A \in \mathscr{F}_\chi$, $P(A) > 0$, $A \subseteq \{\chi < \tau\}$ and $\gamma < 1$, we have*

$$E\big[V(\gamma L_\tau)\mathbb{1}_A\big] = \infty.$$

PROOF. The lemma can be equivalently reformulated as follows: for any stopping time χ and $\gamma < 1$,

$$E\big[V(\gamma L_\tau) \mid \mathscr{F}_\chi\big] = \infty \text{ on the set } \{\chi < \tau\}. \tag{5.9}$$

Let us denote by $k(\chi) = k(\chi)(\omega)$ the first index k such that $t_k > \chi$, where t_k is the number from our partition. Since

$$E\big[V(\gamma L_\tau) \mid \mathscr{F}_\chi\big] \geq \sum_{k \geq k(\chi)} V(\gamma x_k) P\big[(t_k \leq \tau < t_{k+1}) \mid \mathscr{F}_\chi\big],$$

(5.9) is satisfied if there exists a $\mathscr{F}_\chi$-measurable nonnegative function ξ such that $\{\chi < \tau\} \subseteq \{\xi > 0\}$ and

$$P\big[(t_k \leq \tau < t_{k+1}) \mid \mathscr{F}_\chi\big](\omega) \geq \xi Q(x_k) \qquad \forall\, k \geq k(\chi). \tag{5.10}$$

Let $\theta(y)$ denote the first passage time of the process L to the number $y < 1$,

$$\theta(y) = \inf\big(t \geq 0\colon L_t = y\big) = \inf\left(t \geq 0\colon B_t - \frac{t}{2} = \ln y\right).$$

The density of $\theta(y)$ equals (see, e.g., [23], Section 3.5.C)

$$f(t; y) \triangleq \frac{P(\theta(y) \in (t, t + dt))}{dt} = \sqrt{\frac{\ln^2 y}{2\pi t^3}} \exp\left[-\frac{(\ln y - t/2)^2}{2t}\right].$$

It follows that the random function ξ defined as

$$\xi = \operatorname{ess\,inf}_{t \geq k(\chi)} \frac{f(t - \chi; x_{k(\chi)}/L_\chi)}{f(t; x_{k(\chi)})} \mathbb{1}_{\{\chi < \tau\}}$$

is strictly positive on the set $\{\chi < \tau\}$.

Further, denoting by

$$g(t|x, s) \triangleq \frac{P(\tau \in (t, t + dt) \mid L_s = x, \tau > s)}{dt}$$

the density of τ conditioned to the event $\{L_s = x, \tau > s\}$ and using the strong Markov property for the process L, we deduce on the set $\{\chi < \tau\}$ and for

$k \geq k(\chi)$,

$$\begin{aligned} Q(x_k) &= P\big(t_k \leq \tau < t_{k+1}\big) = \int_{t_k}^{t_{k+1}} g(t \mid 1, 0)\, dt \\ &= \int_{t_k}^{t_{k+1}} \left(\int_{t_{k(\chi)}}^{t} g\big(t \mid x_{k(\chi)}, s\big) f\big(s; x_{k(\chi)}\big)\, ds \right) dt \\ &\leq \int_{t_k}^{t_{k+1}} \left(\int_{t_{k(\chi)}}^{t} g\big(t \mid x_{k(\chi)}, s\big) \frac{1}{\xi} f\left(s - \chi; \frac{x_{k(\chi)}}{L_\chi}\right) ds \right) dt \\ &= \frac{1}{\xi} \int_{t_k}^{t_{k+1}} g\Big(t \mid L_\chi, \chi\Big)\, dt = \frac{1}{\xi} P\big[\big(t_k \leq \tau < t_{k+1}\big) \mid \mathscr{F}_\chi bigr], \end{aligned}$$

proving (5.10). □

LEMMA 5.3. *Any process Y in $\mathscr{Y}(1)$ has the form*

$$Y = NL^T A, \tag{5.11}$$

where A is a decreasing, nonnegative, predictable process, $A_0 = 1$, and

$$N_t = \prod_{i=1}^{[t]} \big(1 + \alpha_i \mathbb{1}_{\{\sigma \wedge \tau \geq i\}} (\varepsilon_i - 1)\big),$$

is a purely discontinues local martingale, where α_i is an $\mathscr{F}_{i-}$-measurable random function such that $-1/(2^i - 1) \leq \alpha_i \leq 2$.

PROOF. The multiplicative decomposition of the positive supermartingale Y and the integral representation theorem imply that

$$Y = NKA,$$

where A and N are as in the lemma and K has the integral representation

$$K_t = 1 + \int_0^t K_{u-} \zeta_u \, dB_u,$$

for a predictable process ζ such that the stochastic integral above is well defined. Further, from (2.1) and (5.8) we deduce that any $X \in \mathfrak{X}(1)$ has the form

$$X_t = 1 + \int_0^t X_{u-} \big[\phi_u (du - dB_u)\big],$$

where ϕ is a predictable process. By Itô's formula,

$$XY = \text{“local martingale”} + \int_0^t X_{u-} Y_{u-} \left[\phi_u (1 - \zeta_u)\, du + \frac{dA_u}{A_{u-}} \mathbb{1}_{\{A_{u-} > 0\}} \right].$$

It follows that XY is a supermartingale for any X (hence for any integrable ϕ) if and only if $\zeta \equiv 1$ on the set $\{Y_- > 0\}$, that is, $K \equiv L$ on this set, which clearly implies the assertion of Lemma 5.3. □

Let us now continue proof of Proposition 5.3. By Lemma 5.3 any Y in the set $\mathscr{Y}(1)$ can be represented in the form given in (5.11). If $Y \not\equiv L^T$, that is, $N_T A_T \not\equiv 1$, then the supermartingale property of NA implies that $P[N_T A_T < 1] > 0$. Consequently, there exists a number $\gamma < 1$ such that the stopping time

$$\chi = \inf\{t \geq 0\colon N_t A_t \leq \gamma\}$$

is strictly less then T with probability greater than zero.

Let us denote by i_0 the first index i such that $P[\alpha_i < 0, \chi < i < T] > 0$. If $i_0 = \infty$, that is, the set $\{\alpha_i < 0, \chi < i < T\}$ is empty for any $i \geq 1$, then

$$\begin{aligned} EV(Y_T) &\geq EV(Y_\tau)\mathbb{1}_{\{\chi<\tau\}}\mathbb{1}_{\{\sigma=\infty\}} \overset{(1)}{\leq} EV(\gamma L_\tau)\mathbb{1}_{\{\chi<\tau\}}\mathbb{1}_{\{\sigma=\infty\}} \\ &\overset{(2)}{\leq} EV(\gamma L_\tau)\mathbb{1}_{\{\chi<\tau\}} P[\{\sigma=\infty\} \mid \mathscr{F}_\chi] = EV(\gamma L_\tau)\mathbb{1}_{\{\chi<\tau\}}\left[1 - \frac{1}{2^{[\chi]+1}}\right] \\ &\geq \frac{1}{2} EV(\gamma L_\tau)\mathbb{1}_{\{\chi<\tau\}}, \end{aligned}$$

where in (i) we used the inequality $N_\tau \leq N_\chi$, which holds true on the set $\{\chi < \tau, \sigma = \infty\}$ by our assumption that $\alpha_i \geq 0$ for $\chi < i < T$, and in (ii) the conditional independence of L_τ and σ on $\mathscr{F}_\chi$. The result now follows from Lemma 5.2.

On the other hand, if $i_0 < \infty$, then we similarly deduce that

$$\begin{aligned} EV(Y_T) &\geq EV(Y_\tau)\mathbb{1}_{\{\chi<i_0<\tau,\, \alpha_{i_0}<0\}}\mathbb{1}_{\{\sigma=i_0\}}\mathbb{1}_{\{\psi=\infty\}} \\ &\geq EV(\gamma L_\tau)\mathbb{1}_{\{\chi<i_0<\tau,\, \alpha_{i_0}<0\}}\mathbb{1}_{\{\sigma=i_0\}}\mathbb{1}_{\{\psi=\infty\}} \\ &= EV(\gamma L_\tau)\mathbb{1}_{\{\chi<i_0<\tau,\, \alpha_{i_0}<0\}}\mathbb{1}_{\{\sigma=i_0\}} P[\{\psi=\infty\} \mid \mathscr{F}_\tau] \\ &= EV(\gamma L_\tau)\mathbb{1}_{\{\chi<i_0<\tau,\alpha_{i_0}<0\}}\mathbb{1}_{\{\sigma=i_0\}}\left[1 - \frac{L_\tau}{1+L_\sigma}\right] \\ &\geq \frac{1}{1+x_0} EV(\gamma L_\tau)\mathbb{1}_{\{\chi<i_0<\tau,\, \alpha_{i_0}<0\}}\mathbb{1}_{\{\sigma=i_0\}} \\ &= \frac{1}{1+x_0} EV(\gamma L_\tau)\mathbb{1}_{\{\chi<i_0<\tau,\, \alpha_{i_0}<0\}} P[\{\sigma=i_0\} \mid \mathscr{F}_{i_0-}] \\ &= \frac{1}{(2^{i_0+1}-1)(1+x_0)} EV(\gamma L_\tau)\mathbb{1}_{\{\chi<i_0<\tau,\, \alpha_{i_0}<0\}} \end{aligned}$$

and the proof again follows from Lemma 5.2. $\square$

6. The asymptotic elasticity of a utility function. In this section we assemble some facts on the notion of asymptotic elasticity. We let $U(x)$ denote a strictly concave, increasing, real-valued function defined on $]0,\infty[$ satisfying (2.4). Recall that

$$\Psi(x) = \frac{xU'(x)}{U(x)}$$

denotes the *elasticity function* of U and

$$AE(U) = \limsup_{x\to\infty} \Psi(x) = \limsup_{x\to\infty} \frac{xU'(x)}{U(x)}$$

denotes the *asymptotic elasticity* of U.

LEMMA 6.1. *For a strictly concave, increasing, real-valued function U the asymptotic elasticity $AE(U)$ is well defined and, depending on $U(\infty) = \lim_{x\to\infty} U(x)$, takes its values in the following sets*:

(i) *For $U(\infty) = \infty$, we have $AE(U) \in [0, 1]$.*
(ii) *For $0 < U(\infty) < \infty$, we have $AE(U) = 0$.*
(iii) *For $-\infty < U(\infty) \le 0$, we have $AE(U) \in [-\infty, 0]$.*

PROOF. (i) Using the monotonicity and positivity of U', we may estimate, for $x \ge 1$,

$$\begin{aligned} 0 \le xU'(x) &= (x-1)U'(x) + U'(x) \\ &\le \big[U(x) - U(1)\big] + U'(1); \end{aligned}$$

hence, in the case $U(\infty) = \infty$,

$$0 \le \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} \le \limsup_{x\to\infty} \frac{U(x) - U(1) + U'(1)}{U(x)} = 1.$$

(ii) In the case $0 < U(\infty) < \infty$ we have to show that $\limsup_{x\to\infty} xU'(x) = 0$. So suppose to the contrary that $\limsup_{x\to\infty} xU'(x) = \alpha > 0$ and choose first x_0 such that $U(\infty) - U(x_0) < \alpha\, 2$ and then $x_1 > x_0$ such that $(x_1 - x_0)U'(x_1) > \alpha/2$ [note that $U(\infty) < \infty$ implies in particular $\lim_{x\to\infty} U'(x) = 0$]. We thus arrive at a contradiction, as

$$\frac{\alpha}{2} > U(x_1) - U(x_0) \ge (x_1 - x_0)U'(x_1) > \frac{\alpha}{2}.$$

(iii) By the strict concavity of U, we infer from $U(\infty) \le 0$ that $U(x) < 0$, for $x \in \mathbb{R}_+$, so that $\Psi(x) < 0$, for all $x \in \mathbb{R}_+$. □

What is the economic interpretation of the notion of the elasticity function $\Psi(x)$ and the asymptotic utility $AE(U)$ for a utility function U? First note that by passing from U to an affine transformation $\tilde{U}(x) = c_1 + c_2 U(x)$, with $c_1 \in \mathbb{R}$, $c_2 > 0$, the utility maximization problem treated in this paper obviously remains unchanged. On the other hand, the elasticities of the utility functions $\Psi(x)$ and $\tilde{\Psi}(x)$ are different if $c_1 \ne 0$. This seems to be bad news as a notion which is not invariant under affine transformations of utility functions does not seem to make sense, but the good news is that the notion of *asymptotic* elasticity does not change if we pass from U to an affine transformation, provided $U(\infty) > 0$ and $\tilde{U}(\infty) > 0$.

LEMMA 6.2. *Let $U(x)$ be a utility function satisfying* (2.4) *and $\tilde{U}(x) = c_1 + c_2U(x)$ an affine transformation, where $c_1 \in \mathbb{R}$, $c_2 > 0$. If $U(\infty) > 0$ and $\tilde{U}(\infty) > 0$, then*

$$AE(U) = AE(\tilde{U}) \in [0, 1].$$

We leave the easy verification of this lemma to the reader.

From now on we shall always assume that $U(\infty) > 0$ which, from an economic point of view, does not restrict the generality. Under this proviso we may interpret the asymptotic utility $AE(U)$ in economic terms as the ratio of the *marginal utility* $U'(x)$ to the *average utility* $U(x)/x$, for large $x > 0$ (in the sense of the lims superior).

EXAMPLES 6.1.

(i) *For $U(x) = \ln(x)$ we have $AE(U) = 0$.*
(ii) *For $\alpha < 1, \alpha \neq 0$ and $U(x) = x^\alpha/\alpha$, we have $AE(U) = \alpha$.*
(iii) *For a utility function $U(x)$ such that $U(x) = x/\ln(x)$, for $x > x_0$, we have $AE(U) = 1$.*

We now give the equivalent characterizations of $AE(U)$ in terms of conditions involving the functions U, V or the derivatives U', $V' = -I$, respectively.

LEMMA 6.3. *Let $U(x)$ be a utility function satisfying* (2.4) *and $U(\infty) > 0$. In each of the subsequent assertions, the infimum of $\gamma > 0$ for which these assertions hold true equals the asymptotic elasticity $AE(U)$.*

(i) *There is $x_0 > 0$ s.t.*,

$$U(\lambda x) < \lambda^\gamma U(x) \quad \text{for } \lambda > 1,\ x \geq x_0.$$

(ii) *There is $x_0 > 0$ s.t.*,

$$U'(x) < \gamma \frac{U(x)}{x} \quad \text{for } x \geq x_0.$$

(iii) *There is $y_0 > 0$ s.t.*,

$$V(\mu y) < \mu^{-\gamma/(1-\gamma)} V(y) \quad \text{for } 0 < \mu < 1,\ 0 < y \leq y_0.$$

(iv) *There is $y_0 > 0$ s.t.*,

$$-V'(y) < \left(\frac{\gamma}{1-\gamma}\right)\frac{V(y)}{y} \quad \text{for } 0 < y \leq y_0.$$

PROOF. It follows from the definition of the asymptotic elasticity that $AE(U)$ equals the infimum over all γ such that (ii) holds true. We shall show that for each of the above four conditions the inf of the γ's for which they hold true is the same.

(i) $\Leftrightarrow$ (ii) To show that (ii) $\Rightarrow$ (i), fix $x > 0$, $\gamma > 0$ and compare the two functions

$$F(\lambda) = U(\lambda x) \quad \text{and} \quad G(\lambda) = \lambda^{\gamma} U(x), \qquad \lambda > 1.$$

Here F and G are differentiable, $F(1) = G(1)$, and if (ii) holds true then, for $x > x_0$,

$$F'(1) = xU'(x) < \gamma U(x) = G'(1),$$

hence we have $F(\lambda) < G(\lambda)$ for $\lambda \in]1, 1+\varepsilon[$, for some $\varepsilon > 0$. To show that $F(\lambda) < G(\lambda)$ for all $\lambda > 1$, let $\widehat{\lambda} = \inf\{\lambda > 1\colon F(\lambda) = G(\lambda)\}$ and suppose that $\widehat{\lambda} < \infty$. Note that we must have $F'(\widehat{\lambda}) \geq G'(\widehat{\lambda})$, which leads to a contradiction as it follows from (ii) that

$$F'(\widehat{\lambda}) = xU'(\widehat{\lambda}x) < \frac{\gamma}{\widehat{\lambda}} U(\widehat{\lambda}x) = \frac{\gamma}{\widehat{\lambda}} F(\widehat{\lambda}) = \frac{\gamma}{\widehat{\lambda}} G(\widehat{\lambda}) = G'(\widehat{\lambda}).$$

The reverse implication (i) $\Rightarrow$ (ii) follows from

$$U'(x) = \frac{F'(1)}{x} \leq \frac{G'(1)}{x} = \gamma \frac{U(x)}{x}.$$

(ii) $\Leftrightarrow$ (iv) Let $y_0 = U'(x_0)$. Assuming (ii) we may estimate, for $y < y_0 \triangleq U'(x_0)$,

$$\begin{aligned} V(y) &= \sup_x [U(x) - xy] \\ &= U(-V'(y)) + yV'(y) \\ &> \frac{1}{\gamma}(-V'(y))U'\big(-V'(y)\big) + yV'(y) = \frac{1-\gamma}{\gamma} y(-V'(y)), \end{aligned}$$

which is precisely (iv). Conversely, assuming (iv) we get, for $x \geq x_0 \triangleq -V'(y_0)$,

$$\begin{aligned} U(x) &= \inf{}_y [V(y) + xy] \\ &= V(U'(x)) + xU'(x) \\ &> \frac{1-\gamma}{\gamma} U'(x)\big(-V'(U'(x))\big) + xU'(x) = \frac{1}{\gamma} xU'(x), \end{aligned}$$

which is precisely (ii).

(iii) $\Leftrightarrow$ (iv) Just as in the proof of (i) $\Leftrightarrow$ (ii) we compare, for $0 < y \leq y_0$ fixed, the functions

$$F(\mu) = V(\mu y) \quad \text{and} \quad G(\mu) = \mu^{-\gamma/(1-\gamma)} V(y), \qquad 0 < \mu < 1,$$

to obtain that (iv) is equivalent to $F(\mu) < G(\mu)$, for $0 < y \leq y_0$ and $0 < \mu < 1$. This easily implies the equivalence of (iii) and (iv). $\square$

Another way of describing the asymptotic elasticity is to pass to a logarithmic scaling of $\mathbb{R}_+$, that is, to pass from U to

$$\widehat{U}(z) = \ln(U(e^z)), \qquad z > z_0 \triangleq \ln(U^{-1}(0)).$$

One easily verifies that $AE(U) = \limsup_{z\to\infty} \widehat{U}'(z)$, and a similar characterization may be given in terms of

$$\widehat{V}(z) = \ln(V(e^z)), \qquad z \in \mathbb{R}.$$

We also indicate the connection of the condition $AE(U) < 1$ with the well-known Δ_2-condition in the theory of Orlicz spaces [26]. Obviously, we have $-V'(y) < (\gamma/1-\gamma)V(y)/y$, for $0 < y \le y_0$, iff we have for the function $\check{V}(z) = V(1/z)$ the inequality

$$V'(z) \le \frac{\gamma}{1-\gamma}\frac{\check{V}(z)}{z} \quad \text{for } z \ge z_0 \triangleq y_0^{-1},$$

that is, iff the function $\check{V}(z)$ satisfies the Δ_2 condition. [Note, however, that $\check{V}(z)$ is, in general, not a convex function of $z \in \mathbb{R}_+$.]

Finally, we note an easy and useful characterization of the condition $AE(U) < 1$, which immediately follows from Lemma 6.3.

COROLLARY 6.1. *Let $U(x)$ be a utility function satisfying* (2.4) *and $U(\infty) > 0$. The following assertions are equivalent*:

(i) *The asymptotic elasticity of U is less than* 1.

(ii) *There is $x_0 > 0$, $\lambda > 1$ and $c < 1$ s.t.*,

$$U(\lambda x) < c\lambda U(x) \quad \textit{for } x > x_0.$$

(ii′) *There is $x_0 > 0$ s.t., for every $\lambda > 1$ there is $c < 1$*,

$$U(\lambda x) < c\lambda U(x) \quad \textit{for } x > x_0.$$

(iii) *There is $y_0 > 0$, $\mu < 1$ and $C < \infty$ s.t.*,

$$V(\mu y) < CV(y) \quad \textit{for } y < y_0.$$

(iii′) *There is $y_0 > 0$ s.t., for every $0 < \mu < 1$, there is $C < \infty$ s.t.*,

$$V(\mu y) < CV(y) \quad \textit{for } y < y_0. \qquad \square$$

We now prove a technical result which was used in Section 3.

LEMMA 6.4. *Let u, w be two concave functions, defined on $\mathbb{R}_+$, verifying $u(\infty) > 0$, $w(\infty) > 0$ and such that there exist $x_0 > 0$ and $C > 0$, for which we have*

$$u(x) - C \le w(x) \le u(x) + C, \qquad x \ge x_0.$$

Then

$$AE(u) = \limsup_{x\to\infty} \frac{xu'(x)}{u(x)} = \limsup_{x\to\infty} \frac{xw'(x)}{w(x)} = AE(w).$$

PROOF. We may assume w.l.g. that $u(\infty) = w(\infty) = \infty$ [otherwise $AE(u) = AE(w) = 0$] as well as $u'(\infty) = w'(\infty) = 0$ [otherwise $AE(u) = AE(w) = 1$].

Suppose that $AE(u) = \gamma$ and $AE(w) > \gamma + \alpha$ for some $0 \le \gamma < 1$ and $\alpha > 0$; let us work towards a contradiction.

By Lemma 6.3 we may find arbitrarily large $x \in \mathbb{R}_+$ such that

$$w'(x) > (\gamma + \alpha)\frac{w(x)}{x}. \tag{6.1}$$

Let $h = h(x) = 8Cx/\alpha(\gamma+\alpha)u(x)$ and observe that $\lim_{x\to\infty} h(x)/x = 0$ so that in particular $x - h > 0$, for sufficiently large x. Fixing such an $x > 0$ satisfying also (6.1) we may estimate

$$\begin{aligned} hu'(x-h) + 2C &\ge u(x) - u(x-h) + 2C \\ &\ge w(x) - w(x-h) \\ &\ge hw'(x) \\ &\ge h(\gamma+\alpha)\frac{w(x)}{x} \ge h(\gamma+\alpha)\frac{u(x)-C}{x} \end{aligned}$$

so that

$$u'(x-h) \ge (\gamma+\alpha)\frac{u(x)-C}{x} - \frac{2C}{h}.$$

Using

$$\frac{2C}{h} = \frac{\alpha}{4}(\gamma+\alpha)\frac{u(x)}{x}$$

and the estimates

$$u(x) - C > \left(1 - \frac{\alpha}{4}\right)u(x), \qquad x - h > \frac{x}{1-\alpha/4}$$

which hold true for sufficiently large $x > 0$, we obtain

$$\begin{aligned} u'(x-h) &\ge (\gamma+\alpha)\left(1-\frac{\alpha}{4}\right)\frac{u(x)}{x} - \frac{\alpha}{4}(\gamma+\alpha)\frac{u(x)}{x} \\ &\ge (\gamma+\alpha)\left(1-\frac{\alpha}{2}\right)\frac{u(x-h)}{x-h}\left(1-\frac{\alpha}{4}\right) \ge \left(\gamma+\frac{\alpha}{4}\right)\frac{u(x-h)}{x-h}, \end{aligned}$$

so that Lemma 6.3 gives a contradiction to the assumption $AE(u) \le \gamma$. $\square$

We end this section by comparing the condition $AE(U) < 1$ with two other growth conditions [assertions (i) and (iii), respectively, in the subsequent lemma] which have been studied in [21], condition (4.8) and (5.4)].

LEMMA 6.5. *Let $U(x)$ be a utility function satisfying* (2.4) *and* $U(\infty) > 0$. *Consider the subsequent assertions*:

(i) *There is $x_0 > 0$, $\alpha < 1$ and $\beta > 1$ s.t.*,

$$U'(\beta x) < \alpha U'(x) \quad \text{for } x > x_0.$$

(ii) $AE(U) < 1$.

(iii) *There is $x_0 > 0$, $k_1 > 0$, $k_2 > 0$ and $\gamma < 1$ s.t.*

$$U(x) \le k_1 + k_2 x^{\gamma} \quad \text{for } x > x_0.$$

Then the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *hold true*, *while the reverse implications* (ii) $\Rightarrow$ (i) *and* (iii) $\Rightarrow$ (ii) *do not hold true*, *in general*.

PROOF. (i) $\Rightarrow$ (ii) Assume (i) and let $a = \alpha\beta$ and $b = 1/\alpha > 1$ and estimate, for $x > ax_0$,

$$\begin{aligned} U(bx) &= U(\beta x_0) + \int_{\beta x_0}^{bx} U'(t)\,dt \\ &= U(\beta x_0) + \beta \int_{x_0}^{x/a} U'(\beta t)\,dt \\ &\le U(\beta x_0) + \alpha\beta \int_{x_0}^{x/a} U'(t)\,dt \\ &= U(\beta x_0) + aU\left(\frac{x}{a}\right) - aU(x_0). \end{aligned}$$

It follows that criterion (ii) of Corollary 6.1 is satisfied; hence $AE(U) < 1$.

(ii) $\Rightarrow$ (iii) is immediate from assertion (i) of Lemma 6.3.

(ii) $\not\Rightarrow$ (i) For $n \in \mathbb{N}$, let $x_n = 2^{2^n}$ and define the function $U(x)$ by letting $U(x_n) = 1 - 1/n$ and to be linear on the intervals $[x_{n-1}, x_n]$ [for $0 < x \le x_1$ continue $U(x)$ in an arbitrary way, so that U satisfies (2.4)].

Clearly $U(x)$ fails (i) as for any $\beta > 1$ there are arbitrary large $x \in \mathbb{R}$ with $U'(\beta x) = U'(x)$. On the other hand, we have $U(\infty) = 1$ so that $AE(U) = 0$ by Lemma 6.1.

The attentive reader might object that $U(x)$ is neither strictly concave nor differentiable. But it is obvious that one can slightly change the function to "smooth out" the kinks and to "strictly concavify" the straight lines so that the above conclusion still holds true.

(iii) $\not\Rightarrow$ (ii) Let again $x_n = 2^{2^n}$ and consider the utility function $\tilde{U}(x) = x^{1/2}$. Define $U(x)$ by letting $U(x_n) = \tilde{U}(x_n)$, for $n = 0, 1, 2...$ and to be linear on the intervals $[x_n, x_{n+1}]$ [for $0 < x \le x_1$ again continue $U(x)$ in an arbitrary way, so that U satisfies (2.4)].

Clearly, $U(x)$ satisfies condition (iii) as U is dominated by $\tilde{U}(x) = x^{1/2}$.

To show that $AE(U) = 1$, let $x \in]x_{n-1}, x_n[$ and calculate the marginal utility U' at x,

$$U'(x) = \frac{U(x_n) - U(x_{n-1})}{x_n - x_{n-1}} = \frac{2^{2^{n-1}} - 2^{2^{n-2}}}{2^{2^n} - 2^{2^{n-1}}}$$
$$= \frac{2^{2^{n-1}}(1 - 2^{-2^{n-2}})}{2^{2^n}(1 - 2^{-2^{n-1}})} = 2^{-2^{n-1}}(1 + o(1)).$$

On the other hand we calculate the average utility at $x = x_n$,

$$\frac{U(x_n)}{x_n} = \frac{2^{2^{n-1}}}{2^{2^n}} = 2^{-2^{n-1}}.$$

Hence

$$AE(U) = \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} = 1.$$

As regards the lack of smoothness and strict concavity of U a similar remark applies as in (ii) $\not\Rightarrow$ (i) above. □

REFERENCES

[1] ANSEL, J. P. and STRICKER, C. (1994). Couverture des actifs contingents et prix maximum. *Ann. Inst. H. Poincaré* **30** 303–315.

[2] BISMUT, J. M. (1973). Conjugate convex functions in optimal stochastic control. *J. Math. Anal. Appl.* **44** 384–404.

[3] BRANNATH, W. and W. SCHACHERMAYER (1999). A bidual theorem for subsets of $L^0_+(\Omega, \mathscr{F}, \mathbb{P})$. *Séminaire de Probabilités*. To appear.

[4] COX, J. C. and HUANG, C. F. (1989). Optimal consumption and portfolio plicies when asset prices follow a diffusion process. *J. Econom. Theory* **49** 33–83.

[5] COX, J. C. and HUANG, C. F. (1991). A variational problem arising in financial economics. *J. Math. Econom.* **20** 465–487.

[6] DAVIS, M. H. A. (1997). Option pricing in incomplete markets. In *Mathematics of Derivative Securities* 216–226. Cambridge Univ. Press.

[7] DELBAEN, F. and SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300** 463–520.

[8] DELBAEN, F. and SCHACHERMAYER, W. (1995). The no-arbitrage property under a change of numéraire. *Stochastics Stochastics Rep.* **53** 213–226.

[9] DELBAEN, F. and SCHACHERMAYER, W. (1998). A simple counter-example to several problems in the theory of asset pricing, which arises in many incomplete markets. *Math. Finance*. To appear.

[10] DELBAEN F. and SCHACHERMAYER, W. (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann.* **312** 215–250.

[11] DIESTEL, J. (1975). *Geometry of Banach Spaces—Selected Topics*. *Lecture Notes in Math.* **485**. Springer, Berlin.

[12] FOLDES, L. P. (1990). Conditions for optimality in the infinite-horizon portfolio-cum-savings problem with semimartingale investments. *Stochastics Stochastics Rep.* **29** 133–171.

[13] FÖLLMER, H., YU and KABANOV, M. (1998). Optional decomposition and Lagrange multiplies. *Finance and Stochastics* **2** 69–81.

[14] FÖLLMER, H. and KRAMKOV, D. O. (1997). Optional decompositions under constraints. *Probab. Theory Related Fields* **109** 1–25.

[15] GROTHENDIECK, A. (1954). Espaces vectoriel topologiques. Sociedade de Matematica de São Paulo.

[16] HE, H. and PEARSON, N. D. (1991). Consumption and portfolio policies with incomplete markets and short-sale constraints: the infinite-dimensional case. *Math. Finance* **1** 1–10.

[17] HE, H. and PEARSON, N. D. (1991). Consumption and portfolio policies with incomplete markets and short-sale constraints: the infinite-dimensional case. *J. Econom. Theory* **54** 259–304.

[18] JACKA, S. D. (1992). A martingale representation result and an application to incomplete financial markets. *Math. Finance* **2** 239–250.

[19] JACOD, J. (1979). *Calcul stochastique et problémes de martingales. Lecture Notes in Math.* **714**. Springer, Berlin.

[20] KARATZAS, I. (1989). Optimization problems in the theory of continuous trading. *SIAM J. Control Optim.* **27** 1221–1259.

[21] KARATZAS, I., LEHOCZKY, J. P., SHREVE, S. E. and XU, G. L. (1991). Martingale and duality methods for utility maximisation in an incomplete market. *SIAM J. Control Optim.* **29** 702–730.

[22] KARATZAS, I., LEHOCZKY, J. P. and SHREVE, S. E. (1987). Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. *SIAM J. Control Optim.* **25** 1557–1586.

[23] KARATZAS, I. and SHREVE, S. (1991). *Brownian Motion and Stochastic Calculus.* Springer, New York.

[24] EL KAROUI, N. and QUENEZ, M.-C. (1995). Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim.* **33** 29–66.

[25] KRAMKOV, D. O. (1996). Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Probability Theory Related Fields* **105** 459–479.

[26] KRASNOSEL'SKII, M. A. and RUTICKII, YA. B. (1961). *Convex Functions and Orlicz Spaces.* Gordon and Breach, New York.

[27] MERTON, R. C. (1969). Lifetime portfolio selection under uncertainty: the continuous-time case. *Rev. Econom. Statist.* **51** 247–257.

[28] MERTON, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *J. Econom. Theory* **3** 373–413.

[29] MERTON, R. C. (1990). *Continuous-Time Finance.* Blackwell, Cambridge.

[30] PLISKA, S. R. (1986). A stochastic calculus model of continuous trading: optimal portfolio. *Math. Oper. Res.* **11** 371–382.

[31] ROCKAFELLAR, R. T. (1970). *Convex Analysis.* Princeton Univ. Press.

[32] SAMUELSON, P. (1969). Lifetime portfolio selection by the dynamic stochastic programming, *Rev. Econom. Stoch.* 239–246.

[33] STRASSER, H. (1985). *Mathematical Theory of Statistics: Statistical Experiments and Asymptotic. Decision Theory*. de Gruyter, Berlin.

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