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A HETEROPOLYMER NEAR A LINEAR INTERFACE

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We consider a quenched-disordered heteropolymer, consisting of hydrophobic and hydrophylic monomers, in the vicinity of an oil–water interface. The heteropolymer is modeled by a directed simple random walk $(i, S_i)_{i\in\mathbb{N}}$ on $\mathbb{N}\times\mathbb{Z}$ with an interaction given by the Hamiltonians $H_n^\omega(S) = \lambda\sum_{i=1}^n(\omega_i + h)\,\mathrm{sign}(S_i)$ $(n \in \mathbb{N})$. Here, λ and h are parameters and $(\omega_i)_{i\in\mathbb{N}}$ are i.i.d. ±1-valued random variables. The $\mathrm{sign}(S_i) = \pm1$ indicates whether the ith monomer is above or below the interface, the $\omega_i = \pm1$ indicates whether the ith monomer is hydrophobic or hydrophylic. It was shown by Bolthausen and den Hollander that the free energy exhibits a localization–delocalization phase transition at a curve in the (λ, h)-plane.

In the present paper we show that the free-energy localization concept is equivalent to pathwise localization. In particular, we prove that free-energy localization implies exponential tightness of the polymer excursions away from the interface, strictly positive density of intersections with the interface and convergence of ergodic averages along the polymer. We include an argument due to G. Giacomin, showing that free-energy delocalization implies that there is pathwise delocalization in a certain weak sense.

1. Introduction. Heteropolymers near an interface between two solvents are intriguing because of the possibility of a localization–delocalization phase transition. A typical example is a polymer consisting of hydrophobic and hydrophylic monomers in the presence of an oil–water interface.

In the bulk of a single solvent, the polymer is subject to thermal fluctuations and therefore is rough on all space scales. However, near the interface the polymer can benefit from the fact that part of its monomers prefer to be in one solvent and part in the other. The energy it may gain by placing as many monomers as possible in their preferred solvent can, at least for low temperatures, tame the entropy-driven fluctuations. Consequently, the polymer becomes captured by the interface and therefore is smooth on large space scales. The two regimes of characteristic behavior are separated by a phase transition.

1.1. *The model.* The polymer is modeled by a random walk path $(i, S_i)_{i\in\mathbb{L}}$, where $\mathbb{L} \subseteq \mathbb{Z}$ indexes the monomers, $S_i \in \mathbb{Z}$ and $S_i - S_{i-1} = \pm1$. The interface is the horizontal in $\mathbb{L} \times \mathbb{Z}$. We distinguish two cases:

1. The *singly infinite* polymer, where $\mathbb{L} = \mathbb{N}$ and $S_0 = 0$;
2. The *doubly infinite* polymer, where $\mathbb{L} = \mathbb{Z}$ and $S_0 \in 2\mathbb{Z}$.

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The heterogeneity within the polymer is represented by assigning a random variable $\omega_i = \pm 1$ to monomer i for each $i \in \mathbb{L}$, where $\omega_i = +1$ means that monomer i is hydrophobic and $\omega_i = -1$ that it is hydrophylic.

Let $F(\mathbb{L})$ be the set of all finite connected subsets of $\mathbb{L}$. In the simplest model, the thermodynamics of the heteropolymer is governed by the family $(H_\Lambda^{\omega,\lambda,h})_{\Lambda \in F(\mathbb{L})}$ of Hamiltonians

$$H_\Lambda^{\omega,\lambda,h}(S) = \lambda \sum_{i \in \Lambda} (\omega_i + h)\Delta_i(S) \tag{1.1}$$

w.r.t. the reference measure giving all paths $S = (S_i)_{i \in \mathbb{L}}$ equal probability, that is, the measure P for simple random walk (SRW). Here, λ and h are parameters, $\omega = (\omega_i)_{i \in \mathbb{L}}$ is the disorder configuration, and

$$\Delta_i(S) = \begin{cases} \operatorname{sign}(S_i), & \text{if } S_i \neq 0, \\ \operatorname{sign}(S_{i-1}), & \text{if } S_i = 0. \end{cases} \tag{1.2}$$

The role of the Hamiltonian is that (for $\lambda > 0$) it favors the combinations $S_i > 0$, $\omega_i = +1$ and $S_i < 0$, $\omega_i = -1$, so hydrophobic monomers in the oil above the interface ($\mathbb{L} \times \mathbb{Z}_+$) and hydrophylic monomers in the water below the interface ($\mathbb{L} \times \mathbb{Z}_-$). [Note that the definition of $\Delta_i(S)$ actually corresponds to a bond model.] The parameter λ plays the role of the inverse temperature, whereas h expresses the asymmetry between the affinities of the monomer species with the solvents.

The Hamiltonian is $(S, \omega, h) \to (-S, -\omega, -h)$ symmetric. In view of this, we shall henceforth take

$$\mathscr{I} = \{(\lambda, h) \colon \lambda > 0, h \geq 0\} \tag{1.3}$$

as our parameter space.

1.2. *The free energy and a phase transition.* The singly infinite quenched i.i.d. random model with Hamiltonian (1.1) and with a symmetric disorder distribution was recently analyzed in detail by Bolthausen and den Hollander (1997). For the reader's convenience we describe some of the results obtained in that paper.

The localization–delocalization phase transition is established by estimating the *free energy*

$$\phi(\lambda, h) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \log E\big(e^{H_{\Lambda_n}^{\omega,\lambda,h}}\big), \tag{1.4}$$

where $\Lambda_n = \{1, \dots, n\}$ and where E stands for the expectation w.r.t. SRW starting at 0. The limit is shown to exist and to be ω-independent a.s. by the subadditive ergodic theorem.

It was observed that $\phi(\lambda, h) \geq \lambda h$, with the lower bound attained for delocalized paths. Indeed, $P(S_i \geq 0 \;\forall\, 0 \leq i \leq n) \sim C/\sqrt{n}$ $(n \to \infty)$, and conditioned on this event,

$$\frac{1}{|\Lambda_n|} H_{\Lambda_n}^{\omega,\lambda,h} = \frac{1}{|\Lambda_n|} \lambda \sum_{i \in \Lambda_n} (\omega_i + h) = \lambda h(1 + o(1)), \quad \omega\text{-a.s.} \tag{1.5}$$

For this reason, it is natural to work with the *excess free energy*

$$\psi(\lambda, h) = \phi(\lambda, h) - \lambda h \tag{1.6}$$

and to put forward the following concept of a phase transition.

DEFINITION 1 [Bolthausen and den Hollander (1997)]. The polymer is said to be:

(a) Localized if $\psi > 0$;
(b) Delocalized if $\psi = 0$.

As indicated by (1.5), (b) is justified by noting that delocalized paths yield no contribution to ψ. Conversely, (a) is justified by noting that only those excursions that move below the interface can give a positive contribution to ψ. Nonetheless, Definition 1 makes no claims as to the actual path behavior. The present paper shows that, in fact, a bit of work is needed to obtain a path statement from (a) and (b).

Let us define

$$\mathscr{L} = \{\psi > 0\} \cap \mathscr{I}, \tag{1.7}$$

$$\mathscr{D} = \{\psi = 0\} \cap \mathscr{I} \tag{1.8}$$

as the sets of parameters for which the model is localized, respectively, delocalized in the sense of Definition 1. Neither of these sets is trivial, as shown by the following theorem.

THEOREM 1 [Bolthausen and den Hollander (1997)]. *There is a continuous nondecreasing function $h_c\colon (0, \infty) \to (0, 1)$ such that*

$$\mathscr{L} = \bigl\{(\lambda, h) \in \mathscr{I}\colon 0 \le h < h_c(\lambda)\bigr\}. \tag{1.9}$$

Moreover,

$$\lim_{\lambda\to\infty} h_c(\lambda) = 1 \quad \textit{and} \quad \lim_{\lambda\downarrow 0} \frac{h_c(\lambda)}{\lambda} = K_c, \tag{1.10}$$

where $0 < K_c < \infty$ is a number related to a Brownian version of the model.

Theorem 1 asserts that $\mathscr{L}$ and $\mathscr{D}$ are separated by a phase transition line $\lambda \to h_c(\lambda)$ (which extends over all temperatures). Although it is relatively easy to establish the existence and uniqueness of $h_c(\lambda)$ [essentially via the convexity of ϕ in (1.4)] and to evaluate the limit $\lambda \to \infty$ [through an appropriate lower bound on the expectation in (1.4)], the scaling law for $\lambda \downarrow 0$ is a rather involved problem. The intuitive reason why a Brownian constant should appear for $\lambda \downarrow 0$ is that for high temperatures the polymer excursions are large. Therefore, from a coarse-grained point of view, both the excursions and the disorder inside the excursions may be approximated by their Brownian counterparts. However, the details of this approximation are quite delicate.

1.3. *Earlier path results.* As already alluded to, Theorem 1 characterizes the phase transition in terms of the free energy rather than the path. One would like to prove that, indeed, $\mathscr{L}$ corresponds to a localized path and (the interior of) $\mathscr{D}$ to a delocalized path. Moreover, one would like to learn more about the path characteristics, for example, the length and the height of a typical excursion. Progress in this direction has been made by Sinai (1993), who proved pathwise localization in the symmetric case $h = 0$ for all $\lambda > 0$.

Sinai introduces a (Gibbsian) probability distribution $Q_n^{\omega,\lambda,0}$ in the volume $\Lambda_n = \{1, \dots, n\}$, defined by

$$\frac{dQ_n^{\omega,\lambda,0}}{dP_n}(S) = \frac{\exp(H_{\Lambda_n}^{\omega,\lambda,0}(S))}{Z_{\Lambda_n}^{\omega,\lambda,0}}, \tag{1.11}$$

where the reference measure P_n is the projection onto Λ_n of the SRW-measure P with $S_0 = 0$, and $Z_{\Lambda_n}^{\omega,\lambda,0}$ is the normalizing constant or position function. His result appears in the following theorem.

THEOREM 2 [Sinai (1993)]. *Let $h = 0$ and $\lambda > 0$. Then there exist a deterministic number $\zeta = \zeta(\lambda) > 0$ and two random variables $n(\omega) \in \mathbb{N}$, $m(\omega) \in \mathbb{N}$ such that for almost all ω,*

$$\sup_{0 \le i \le n} Q_n^{\omega,\lambda,0}\big(|S_i| > s\big) \le e^{-\zeta s}, \qquad n \ge n(\omega),\ s \ge m(\omega). \tag{1.12}$$

Theorem 2 states that the path measure is *exponentially tight* in the vertical direction. This result has been extended by Albeverio and Zhou (1996), who show that the length of the longest excursion in Λ_n is of order $\log n$ and so is the height of the highest excursion.

1.4. *Path results in the present paper—outline.* The goal of the present paper is to give a complete description of the path for all $(\lambda, h) \in \mathscr{L}$. We in fact adopt a more comprehensive attitude by discussing the entire Gibbsian structure associated with the Hamiltonian (1.1). (Theorems 1 and 2 are in this respect statements about the Gibbs measures generated by the free boundary condition for the singly infinite model.)

We begin by singling out a class of "regular" Gibbs measures (Section 2). For this, measurability and moderate growth of the boundary condition are the key concepts. Within this class we establish, for all $(\lambda, h) \in \mathscr{L}$, uniqueness of the Gibbs measure, exponential tightness of the path in the vertical direction and ergodicity in the horizontal direction (Sections 3 and 5). The proof requires three preparatory lemmas, leading up to positivity of the lower density of intersections with the interface, which is the key ingredient in the proof (Section 4). The paper is concluded by showing that for $(\lambda, h) \in \mathscr{D}$ the path is delocalized in a weak sense, namely, it spends a zero fraction of its time in any finite layer around the interface (Section 6).

The main results of the present paper are Theorem 3 (Section 3) and Theorem 4 (Section 6).

1.5. *Literature remarks.* The *annealed* model (i.e., the partition sum is averaged over ω) treated by Sinai and Spohn (1996) is exactly solvable when the ω_i's are i.i.d. or interact via an Ising Hamiltonian. It turns out that the annealed heteropolymer is delocalized despite the influence of the interface. To get localization, an additional binding potential at the interface has to be superimposed.

The *quenched* model (i.e., ω is kept frozen) is mathematically much harder. The *periodic* case (e.g., ω represents some periodic constraint within the polymer) has been successfully dealt with by using a transfer-matrix approach [Grosberg, Izrailev and Nechaev (1994)]. In that paper, the underlying random walk is three-dimensional, undirected and with Gaussian steps, while the interface is a two-dimensional plane. It turns out that the phase transition curve diverges at some finite value of λ. This may be attributed to the flexibility of the Gaussian random walk to keep its monomers in their preferred solvent (by making large steps when necessary).

Prior to Sinai (1993) and Bolthausen and den Hollander (1997), the *random* case (e.g., ω i.i.d.) had been analyzed by Garel, Huse, Leibler and Orland (1989) using the replica method. The latter study draws a conclusion qualitatively similar to that of Theorem 1.

The free-energy localization concept has proved to be useful also in the study of higher-dimensional generalizations of the present model (Bolthausen and Giacomin, in preparation). The latter authors consider a d-dimensional Gaussian surface, pinned at the interface outside a finite box and weighted by the same type of Hamiltonian as in (1.1). A localization–delocalization phase transition in the sense of Definition 1 is found, but the properties of the phase transition line are not yet fully understood and are possibly different from the ones in Theorem 1.

Whittington (1998a, b) and Orlandini, Tesi and Whittington (1998) consider the model where the heteropolymer is confined to a half-space above the interface and has an attractive interaction at the interface. Both for periodic and random quenched disorder they establish the existence of a localization–delocalization phase transition for the free energy.

2. Preliminaries.

2.1. *Gibbsian structure.* Let $(\omega_i)_{i\in\mathbb{L}}$ be an i.i.d. sequence of ±1-valued random variables defined on a probability space $(\Omega, \mathscr{B}, \mathbb{P})$. Here $\Omega = \{-1, 1\}^{\mathbb{L}}$, $\mathscr{B}$ is the σ-algebra generated by the cylinder sets and $\mathbb{P}$ is the i.i.d. measure with $\mathbb{P}(\omega_i = +1) = \mathbb{P}(\omega_i = -1) = 1/2$. Expectation w.r.t. $\mathbb{P}$ will be denoted by $\mathbb{E}$.

Let

$$\Sigma = \begin{cases} \{S = (S_i)_{i\in\mathbb{N}}\colon S_0 = 0, |S_i - S_{i-1}| = \pm 1\, \forall\, i \in \mathbb{N}\} \\ \{S = (S_i)_{i\in\mathbb{Z}}\colon S_0 \in 2\mathbb{Z}, |S_i - S_{i-1}| = \pm 1\, \forall\, i \in \mathbb{Z}\} \cup \{S \equiv \pm\infty\} \end{cases} \tag{2.1}$$

be the space of SRW-paths for the singly infinite ($\mathbb{L} = \mathbb{N}$) and the doubly infinite ($\mathbb{L} = \mathbb{Z}$) case, respectively.

Let $\mathscr{F}$ be the σ-algebra generated by the cylinder sets. For $\Lambda \subset \mathbb{L}$, let $\mathscr{F}_\Lambda$ be the projection of $\mathscr{F}$ onto Λ, and let $\mathscr{T} = \bigcap_{\Lambda \in F(\mathbb{L})} \mathscr{F}_{\Lambda^c}$ be the tail σ-field [remember that $F(\mathbb{L})$ denotes the set of all finite connected subsets of $\mathbb{L}$]. We use $\mathscr{P}(\Sigma, \mathscr{F})$ to denote the space of all probability measures on $(\Sigma, \mathscr{F})$. Note that $\mathscr{P}(\Sigma, \mathscr{F})$ is compact in the weak topology for both the singly infinite and the doubly infinite case. [This is why we added $\{S \equiv \pm\infty\}$ in (2.1) for the doubly infinite case. In Section 5 we shall see that when $(\lambda, h) \in \mathscr{L}$ the Gibbs measures assign zero probability to $\{S \equiv \pm\infty\}$.] Let P, E denote probability and expectation under SRW.

We define Gibbs measures by means of the *Gibbsian specification* [for details see Georgii (1988), Chapter 1]

$$\gamma_\Lambda^{\omega,\lambda,h}(S \mid \tilde{S}) = \frac{\exp(H_\Lambda^{\omega,\lambda,h}(S))}{Z_\Lambda^{\omega,\lambda,h}(\tilde{S})} P(S_\Lambda \mid \tilde{S}_{\Lambda^c}) 1_{\{S_{\Lambda^c} = \tilde{S}_{\Lambda^c}\}}, \qquad \Lambda \in F(\mathbb{L}). \tag{2.2}$$

This specification is a probability measure on infinite paths $S = S_\Lambda \vee \tilde{S}_{\Lambda^c} \in \Sigma$ (with $S_\Lambda = (S_i)_{i\in\Lambda}$ and $\tilde{S}_{\Lambda^c} = (\tilde{S}_i)_{i \in \Lambda^c}$), is absolutely continuous w.r.t. the conditional measure $P(S_\Lambda \mid \tilde{S}_{\Lambda^c})$ corresponding to the SRW-bridge, and is a measurable function of the boundary condition $\tilde{S} \in \Sigma$. The partition function $Z_\Lambda^{\omega,\lambda,h}(\tilde{S})$ is the normalizing constant [which actually only depends on $\tilde{S}_{\partial\Lambda} = (\tilde{S}_i)_{i\in\partial\Lambda}$, with $\partial\Lambda$ the outer boundary of Λ]. It is easy to verify that the specifications $(\gamma_\Lambda^{\omega,\lambda,h})_{\Lambda \in F(\mathbb{L})}$ form a consistent family.

Given $\omega \in \Omega$ and $(\lambda, h) \in \mathscr{I}$, the Gibbs measures are defined as follows:

$$\mathscr{G}_\omega^{\lambda,h} = \{\mu \in \mathscr{P}(\Sigma, \mathscr{F}) \colon \mu = \mu\gamma_\Lambda^{\omega,\lambda,h} \,\forall\, \Lambda \in F(\mathbb{L})\}, \tag{2.3}$$

that is, $\gamma_\Lambda^{\omega,\lambda,h}$ is the conditional expectation of μ in Λ given the boundary condition in Λ^c. By compactness of $\mathscr{P}(\Sigma, \mathscr{F})$, any weak (subsequential) limit of $\gamma_\Lambda^{\omega,\lambda,h}(\cdot \mid \tilde{S})$ as $\Lambda \to \mathbb{L}$, with a fixed boundary condition $\tilde{S}$, leads to a Gibbs measure (because the specifications are consistent). Hence $\mathscr{G}_\omega^{\lambda,h} \neq \varnothing$.

2.2. *Regular measures.* As is typical for Gibbs measures with unbounded single-component state spaces, an extreme boundary condition may overule the effect of the interaction itself. In our setting, for the singly infinite case and for any $(\lambda, h) \in \mathscr{I}$, there is a whole class of Gibbs measures (of at least countably infinite cardinality) for which delocalized behavior is enforced when $\tilde{S}_i$ grows linearly with i. Similarly for the doubly infinite case. This class we want to throw out.

One can analyze this situation by looking at the *lower excess free energy* $\psi_{\tilde{S}}$ corresponding to $\tilde{S}$, defined by

$$\phi_{\tilde{S}}(\lambda, h) = \liminf_{n\to\infty} \frac{1}{|\Lambda_n|} \mathbb{E}(\log Z_{\Lambda_n}^{\omega,\lambda,h}(\tilde{S})), \tag{2.4}$$

$$\psi_{\tilde{S}}(\lambda, h) = \phi_{\tilde{S}}(\lambda, h) - \lambda h. \tag{2.5}$$

LEMMA 1. (a) *Consider the singly infinite case. Let* $\lim_{i\to\infty} \tilde S_i/i = 0$. *Then* $\psi_{\tilde S}(\lambda, h) \ge \psi(\lambda, h)$ *and similarly for the doubly infinite case.*

PROOF. To find a lower bound on $\phi_{\tilde S}(\lambda, h)$, we take Λ_{2n} and restrict the summation in $Z^{\omega,\lambda,h}_{\Lambda_{2n}}(\tilde S)$ to paths that end by hitting the interface and subsequently moving at maximal speed. More precisely, if $c_n = \tilde S_{2n}/2n \ge 0$, then the path moves from height 0 at position $2n(1-c_n)$ to height $2nc_n$ at position $2n$. This gives

$$Z^{\omega,\lambda,h}_{\Lambda_{2n}}(\tilde S) \ge Z^{\omega,\lambda,h}_{\Lambda_{2n(1-c_n)}}(0)\exp\Biggl[\lambda \sum_{i=2n(1-c_n)+1}^{2n} (\omega_i + h)\Biggr] \frac{\binom{2n(1-c_n)}{n(1-c_n)}}{\binom{2n}{n(1-c_n)}}, \tag{2.6}$$

where $Z^{\omega,\lambda,h}_{\Lambda_{2m}}(0)$ denotes the partition sum with boundary condition $\tilde S_{2m} = 0$ ($m \in \mathbb{N}$). The binomial factors come from the fact that the path must match the boundary condition (recall that the partition sum is defined w.r.t. the SRW-bridge). Now, it was shown by Bolthausen and den Hollander (1997) that the ratio of $Z^{\omega,\lambda,h}_{\Lambda_{2n}}(0)$ and the partition function with free boundary condition, which was used to define $\phi(\lambda, h)$, is of linear order in n. Therefore, the claim follows after taking logarithms, dividing by $2n$, letting $n \to \infty$, using that $c_n \to 0$, and using the relation between ϕ and ψ in (1.6). The case $c_n \le 0$ and the doubly infinite case are completely analogous. □

Lemma 1 shows that any sublinear boundary condition cannot destroy localization in the sense of Definition 1. Thus, a natural distinction between sublinear and linear boundary conditions arises. This leads us to the following definition.

DEFINITION 2. Given $(\lambda, h) \in \mathscr{I}$, the *regular* Gibbs measures are those $\mu \in \mathscr{G}^{\lambda,h}_\omega$ for which $\lim_{i\to\pm\infty} S_i/i = 0$ μ-a.s. The set of regular Gibbs measures is denoted by $\mathscr{G}^{R,\lambda,h}_\omega$.

The theory of Gibbs measures guarantees that all regular Gibbs measures lie in the closed convex hull of all the weak limits generated by sublinear boundary conditions.

2.3. *Measurable Gibbsian sections.* As we noted earlier, $\mathscr{G}^{\lambda,h}_\omega \ne \varnothing$ for all ω by compactness. However, although $\mu_\omega \in \mathscr{G}^{\lambda,h}_\omega$ for different ω can be arranged into a measure-valued function of ω, it is not a priori clear that this can be done in a measurable way, because of possible nonuniqueness of the Gibbs measure. Formally, if we put $\mathscr{G}^{R,\lambda,h} = \bigcup_{\omega\in\Omega}\{(\omega,\mu): \mu \in \mathscr{G}^{R,\lambda,h}_\omega\}$, then the question is whether or not there are *measurable* sections $(\omega, \mu_\omega)_{\omega\in\Omega} \in \mathscr{G}^{R,\lambda,h}$. We shall answer this question affirmatively when $(\lambda, h) \in \mathscr{L}$. Measurability will be important later on because we shall want to integrate over ω.

Define

$$\hat{\mathscr{G}}^{R,\lambda,h} = \big\{\mu_{(\cdot)}\colon \Omega \to \mathscr{P}(\Sigma, \mathscr{F})\colon \mu_\omega \in \mathscr{G}_\omega^{R,\lambda,h}\ \forall\, \omega \in \Omega,\ \mu_{(\cdot)}(A)\ \mathscr{B}\text{-measurable}\ \forall\, A \in \mathscr{F}\big\} \tag{2.8}$$

to be the set of regular measurable Gibbsian sections. Observe that $\mu_{(\cdot)} \in \hat{\mathscr{G}}^{R,\lambda,h}$ implies that $\mu_{(\cdot)}$, when regarded as a measure-valued function on Ω, is measurable w.r.t. the Borel σ-algebra associated with the weak topology on $\mathscr{P}(\Sigma, \mathscr{F})$.

LEMMA 2. *Let $(\lambda, h) \in \mathscr{L}$. Then*:

(a) $\hat{\mathscr{G}}^{R,\lambda,h} \neq \varnothing$ *is nonempty both for the singly infinite and the doubly infinite case.*

(b) *For the doubly infinite case there is a $\mu_{(\cdot)}\colon \Omega \to \mathscr{P}(\Sigma, \mathscr{F})$ such that $\mu_{(\cdot)}(A)$ is $\mathscr{B}$-measurable for all $A \in \mathscr{F}$ and*

(i) $\mu_\omega(|S_0| < \infty) = 1$;

(ii) μ_ω *is regular Gibbsian, that is*, $\mu_\omega \in \mathscr{G}_\omega^{R,\lambda,h}$;

(iii) $\mu_{\sigma\omega}(\sigma A) = \mu_\omega(A)$ *for all* $A \in \mathscr{F}$

hold for $\mathbb{P}$-almost all ω. Here σ denotes the left-shift by two (!) *lattice sites, acting on path and disorder.*

The proof of Lemma 2 is given in Section 5 and requires a large deviation estimate on the partition function appearing in (2.2), which is derived in Lemma 3 (Section 4). The main point here is to rule out that mass escapes to infinity under the doubly infinite measure μ_ω (i.e., $\mu_\omega(\{S \equiv \pm\infty\}) = 0$)). This is in fact likely to happen when $(\lambda, h) \in \mathscr{D}$, but here we are only considering $(\lambda, h) \in \mathscr{L}$.

3. Uniqueness and positive density in the localization regime. It is intuitively clear that $\psi > 0$ implies recurrence, that is, the path hits the interface infinitely often. Indeed, if $\psi > 0$, then by Lemma 7 for any regular boundary condition $\tilde{S}$ we have $Z_\Lambda^{\omega,\lambda,h}(\tilde{S})e^{-\lambda h|\Lambda|} = \exp(|\Lambda|\psi_{\tilde{S}}(\lambda, h) + o(|\Lambda|)) \to \infty$ as $\Lambda \to \mathbb{L}$, which implies that the set $\{S \in \Sigma\colon S_i > 0\,\forall\, i \geq n\}$ has zero probability for all n [recall (1.1) and (2.1)]. Below we shall in fact prove more, namely, that all regular Gibbs measures are *positively recurrent*, that is, the path hits the interface with a certain positive frequency.

For $a \in \mathbb{Z}$, let

$$\varrho_a^-(S) = \liminf_{\Lambda \to \mathbb{L}} \frac{2}{|\Lambda|} \sum_{i \in \Lambda} 1_{\{S_i = a\}}, \tag{3.1}$$

where the factor 2 takes care of the parity of SRW. We shall say that $(\omega, \mu_\omega)_{\omega\in\Omega} \in \mathscr{G}^{R,\lambda,h}$ is *localized* if $\mathbb{E}\mu_\omega(\varrho_0^- > 0) = 1$, that is, if $\mu_\omega(\varrho_0^- > 0) = 1$ for $\mathbb{P}$-almost all ω. Now we are ready to state the main theorem of our paper.

THEOREM 3. *Let* $(\lambda, h) \in \mathscr{L}$. *Then*:

(a) $\hat{\mathscr{G}}^{R,\lambda,h}$ *is a singleton both for the singly infinite and the doubly infinite case.*

(b) *The unique doubly infinite Gibbsian section* $(\omega, \mu_\omega)_{\omega\in\Omega}$ *is localized and is jointly translation invariant* (*i.e.*, $\mu_{\sigma\omega}(\sigma A) = \mu_\omega(A)$ *for* $\mathbb{P}$-*almost all* ω *and all* $A \in \mathscr{F}$).

(c) *The unique singly infinite Gibbsian section* $(\omega, \nu_\omega)_{\omega\in\Omega}$ *is localized and is asymptotically equal to* $(\omega, \mu_\omega)_{\omega\in\Omega}$:

$$\lim_{n\to\infty} \sup_{A\in\mathscr{F}} \left|\nu_\omega(\sigma^n A) - \mu_\omega(\sigma^n A)\right| = 0 \quad \textit{for } \mathbb{P}\textit{-almost all } \omega. \tag{3.2}$$

(d) *Both Gibbsian sections have* a.s. *constant densities*, *that is*, *for* $\mathbb{P}$-*almost all* ω *and all* $A \in \mathscr{F}$,

$$\lim_{\Lambda\to\mathbb{L}} \frac{1}{|\Lambda|} \sum_{i\in\Lambda} 1_{\sigma^i A}(S) = \mathbb{E}\mu_\omega(A) \quad \textit{for } \mu_\omega\textit{-almost all } S \in \Sigma, \tag{3.3}$$

and similarly for ν_ω.

(e) *Both Gibbsian sections are exponentially tight*: *for any* $s \in \mathbb{Z}$ *and* $\varepsilon > 0$ *there exists a random number* $n_0(s, \varepsilon, \omega)$ *such that*

$$\nu_\omega(S_n = s) \le \mathscr{O}(1)\exp(-(\zeta_s - \varepsilon)|2s|), \qquad n \ge n_0(s, \varepsilon, \omega), \tag{3.4}$$

with $\zeta_s = \psi(\lambda, h)$ *when* $s > 0$ *and* $\zeta_s = \psi(\lambda, h) + \lambda h$ *when* $s < 0$.

Assertions (a) and (b) establish uniqueness within the class of regular Gibbsian sections [for $(\lambda, h) \in \mathscr{L}$]. The measures ν_ω, respectively, μ_ω can be viewed as describing the behavior of the polymer near the endpoint, respectively, away from the endpoints. Assertion (c) claims that these two blend into each other at infinity. Assertion (d) corresponds to ergodicity along the polymer. (Note that the probabilities $\mu_\omega(\sigma^i A)$ typically vary a great deal with i according to the local disorder.) Assertion (e) provides an extension of Sinai's result cited in Theorem 2, with explicit bounds on the decay rate.

4. Three preparatory lemmas. In order to prove Lemma 2 and Theorem 3, we first have to state a couple of technical lemmas that establish exponential growth of the partition function (Lemma 3), exponential tightness of the interarrival times to the interface (Lemma 4), and, most importantly, a.s. positivity of the lower density of intersections with the interface under both μ_ω and ν_ω (Lemma 5). To avoid confusion, we emphasize that the proof of Lemma 2 requires only the result of Lemma 3, hence there is no problem with the assumption of measurability in Lemmas 4 and 5. Throughout the sequel we assume $(\lambda, h) \in \mathscr{L}$ and suppress these parameters from the notation.

4.1. *Large deviations for the partition sum.* The assertion of Lemma 3 is a large deviation estimate for the partition sum that will be needed later on, in particular, in conjunction with a Borel–Cantelli argument.

LEMMA 3. *Let* $Z_{2n}^{\omega} = Z_{\Lambda_{2n}}^{\omega}(0)$ *be the partition function for the boundary condition* $\tilde{S}_0 = \tilde{S}_{2n} = 0$. *Then for each* $\varepsilon \in (0, \psi)$ *there is a* $\delta_\varepsilon > 0$ *such that*

$$\mathbb{P}\left(\frac{1}{2n}\log Z_{2n}^{\omega} < \psi + \lambda h - \varepsilon\right) \le \mathscr{O}(1)\exp(-\delta_\varepsilon 2n), \qquad n \to \infty. \tag{4.1}$$

PROOF. Given $\varepsilon > 0$, there is an m large enough such that

$$\frac{1}{2m}\mathbb{E}(\log Z_{2m}^{\omega}) \ge \psi + \lambda h - \varepsilon/2. \tag{4.2}$$

This follows from the fact that a sublinear boundary condition does not lower the free energy (see Lemma 1). Pick any such an m and put $k = \lfloor n/m \rfloor$. Then, by restricting the path to return to 0 at positions $2m, 4m, \ldots, 2km$ $(\le 2n)$, we obtain

$$Z_{2n}^{\omega} \ge \frac{\binom{2m}{m}^k \binom{2(n-km)}{n-km}}{\binom{2n}{n}} \left[\prod_{j=0}^{k-1} Z_{2m}^{\sigma^{jm}\omega}\right] Z_{2n-2km}^{\sigma^{km}\omega}. \tag{4.3}$$

Here the binomial factor reflects the fact that the partition sum is defined w.r.t. the SRW-bridge. After taking logarithms and dividing by $2n$ we get

$$\begin{aligned} &\mathbb{P}\left(\frac{1}{2n}\log Z_{2n}^{\omega} < \psi + \lambda h - \varepsilon\right) \\ &\qquad \le \mathbb{P}\left(\frac{1}{k}\sum_{j=0}^{k-1}\frac{1}{2m}\log Z_{2m}^{\sigma^{jm}\omega} < \psi + \lambda h - \frac{3\varepsilon}{4}\right), \end{aligned} \tag{4.4}$$

where we have assumed n so large that the factors outside the square brackets in (4.3) give rise to a correction less than $\varepsilon/4$. Now, $(1/2m)\log Z_{2m}^{\sigma^{jm}\omega}$ $(j = 0, \ldots, k-1)$ are i.i.d. bounded random variables. Therefore a standard large deviation estimate gives that the r.h.s. of (4.4) is bounded by $\mathscr{O}(1)\exp(-\delta'_\varepsilon 2k)$ for some $\delta'_\varepsilon > 0$. From this the claim easily follows by choosing $\delta_\varepsilon = \delta'_\varepsilon/m$ [we neglect the additional correction coming from rounding off $\lfloor n/m \rfloor$, which is absorbed into the $\mathscr{O}(1)$-term]. □

4.2. *Exponential tightness of the interarrival times.* Let us introduce the notion of *arrival times*, defined as the positions where the path hits the interface, that is,

$$\cdots < N_{-1} < N_0 \le 0 < N_1 < N_2 < \cdots \tag{4.5}$$

specified by $S_{2N_k} = 0$ $(k \in \mathbb{Z})$ and $S_{2r} \ne 0$ if $r \notin (N_k)$. Let $\xi_k = N_{k+1} - N_k$ $(k \in \mathbb{Z})$ be the *interarrival times*. (Both sequences end when no further arrivals occur.) Note that only the even sites are counted in the excursions.

LEMMA 4. *If* $(\omega,\mu_\omega)_{\omega\in\Omega}\in\mathscr{G}^{R,\lambda,\mathbb{R}}$ *is a measurable Gibbsian section, then there is a* $\kappa>0$ *such that for any* $i\in\mathbb{Z}$, $K\in\mathbb{N}$, $L\in\mathbb{Z}$, *and any* $m_{i+j}\in\mathbb{N}$ $(j=0,\dots,K-1)$,

$$\mathbb{E}\big[\mu_\omega\big(\xi_{i+j}=m_{i+j}\,\forall\, j=0,\dots,K-1\,|\,N_i=L\big)\big] \le \mathscr{O}(1)\prod_{j=0}^{K-1}\exp(-\kappa m_{i+j}). \tag{4.6}$$

PROOF. Fix i,K,L. The event

$$A=\big\{\xi_{i+j}=m_{i+j}\,\forall\, j=0,\dots,K-1\big\}, \tag{4.7}$$

if conditioned on $\{N_i=L\}$, means that $S_{2k_j}=0$ for $k_j=L+\sum_{l=0}^{j-1}m_{i+l}$ $(j=0,\dots,K)$ and $S_{2r}\neq 0$ for $k_j<r<k_{j+1}$ $(j=0,\dots,K-1)$. Since μ_ω is Gibbsian, we can apply conditioning to write [recall (1.1)]

$$\mu_\omega(A\,|\,N_i=L)=\left[\prod_{j=0}^{K-1}\frac{1+\exp(-2\lambda(\Omega_{I_j}+h|I_j|))}{2Z^\omega_{I_j}\exp(-\lambda(\Omega_{I_j}+h|I_j|))}P_{I_j}\right] \times\mu_\omega\big(S_{2k_{j+1}}=0\,\forall\, j=0,\dots,K-1\,|\,N_i=L\big), \tag{4.8}$$

where $I_j=(2k_j,2k_{j+1}]\cap\mathbb{Z}$, $\Omega_{I_j}=\sum_{l\in I_j}\omega_l$, and P_{I_j} is the probability that SRW conditioned on $S_{2k_j}=0=S_{2k_{j+1}}$ never touches the interface in between. By neglecting the last factor, we obtain

$$\mu_\omega(A\,|\,N_i=L)\le\prod_{j=0}^{K-1}\frac{1+\exp(-2\lambda(\Omega_{I_j}+h|I_j|))}{2Z^\omega_{I_j}\exp(-\lambda(\Omega_{I_j}+h|I_j|))}P_{I_j}. \tag{4.9}$$

Pick $\varepsilon>0$. By Lemma 3, there exists a $\delta_\varepsilon>0$ such that

$$\mathbb{P}\big(Z^\omega_{I_j}\exp(-\lambda h|I_j|)<\exp((\psi-\varepsilon)|I_j|)\big) \le\mathscr{O}(1)\exp(-\delta_\varepsilon|I_j|)\quad\text{for all } j. \tag{4.10}$$

Moreover, a standard large deviation estimate gives that there exists a $\delta'_\varepsilon>0$ such that

$$\mathbb{P}\bigg(|\Omega_{I_j}|>\frac{\varepsilon}{\lambda}|I_j|\bigg)\le\mathscr{O}(1)\exp(-\delta'_\varepsilon|I_j|)\quad\text{for all } j. \tag{4.11}$$

Hence, by using (4.11) to estimate the numerator of the fractions in (4.9), and (4.10) to estimate the denominator of the fractions in (4.9) on the complement of the event in (4.11), we get

$$\mathbb{E}\big[\mu_\omega(A\,|\,N_i=L)\big] \le\mathscr{O}(1)\prod_{j=0}^{K-1}\big[\exp(-\delta_\varepsilon|I_j|)+\exp(-\delta'_\varepsilon|I_j|)+\exp(-(\psi-2\varepsilon)|I_j|)\big]. \tag{4.12}$$

[Here we also used that each factor in the r.h.s. of (4.9) is less than or equal to 1.] The desired estimate (4.6) is now obtained by setting $\kappa=2\sup_{\varepsilon>0}\min\{\delta_\varepsilon,\delta'_\varepsilon,\psi-2\varepsilon\}$. Since $\psi>0$, we obviously have $\kappa>0$. □

4.3. *Positive lower density of intersections.* Now comes the most important lemma, which establishes a.s. positivity of ϱ_0^- [recall (3.1)]. As explained in Section 5, this result will make accessible certain coupling techniques that will be used to prove Theorem 3.

LEMMA 5. *There is a $\hat{\varrho} > 0$ such that for any measurable Gibbsian section $(\omega, \mu_\omega)_{\omega\in\Omega} \in \hat{\mathscr{G}}^{R,\lambda,h}$: $\mu_\omega(\varrho_0^- \geq \hat{\varrho}) = 1$ for $\mathbb{P}$-almost all ω.*

PROOF. Let us concentrate on the doubly infinite case. (The singly infinite case can be handled analogously.) Let

$$A_{n;k} = \left\{ \sum_{j=-n}^{n} 1_{\{S_{2j}=0\}} \leq k \right\}. \tag{4.13}$$

Let $-2(n+n_-)$ label the last arrival before $-2n$ and $2(n+n_+)$ the first arrival after $2n$. Since Lemma 4 provides an estimate for interarrival times in a row, we have for $0 \leq k \leq n$,

$$\begin{aligned} \mathbb{E}\mu_\omega(A_{n;k}) &\leq \mathscr{O}(1)\left[\sum_{l=0}^{k}\binom{2n+1}{l}\right]\sum_{n_+,n_-=1}^{\infty}\exp(-\kappa(2n+n_-+n_++1)) \\ &\leq \mathscr{O}(n)\binom{2n+1}{k}\exp(-\kappa n), \end{aligned} \tag{4.14}$$

where the binomial factor accounts for all possible positions of the k arrivals within $[-2n, 2n]$.

Pick $0 < \hat{\varrho} < 1/2$ and pick $k = k(n) = \lfloor (2n+1)\hat{\varrho} \rfloor$. Then, using Stirling's formula, we obtain

$$\mathbb{E}\mu_\omega(A_{n;k(n)}) \leq \mathscr{O}(n)\left[\exp(-\kappa/2)\hat{\varrho}^{-\hat{\varrho}}(1-\hat{\varrho})^{-(1-\hat{\varrho})}\right]^{2n}. \tag{4.15}$$

So if $\hat{\varrho}$ satisfies $\hat{\varrho}\log\hat{\varrho} + (1-\hat{\varrho})\log(1-\hat{\varrho}) + \kappa/2 > 0$, which is the case for $\hat{\rho}$ small enough because $\kappa > 0$, then the r.h.s. is summable on n, and hence

$$\mathbb{E}\mu_\omega\big(A_{n;k(n)}\text{ i.o.}\big) = 0 \tag{4.16}$$

by the Borel–Cantelli lemma. Consequently,

$$\sum_{j=-n}^{n} 1_{\{S_{2j}=0\}} > \lfloor (2n+1)\hat{\varrho} \rfloor \tag{4.17}$$

eventually under $\mathbb{E}\mu_\omega$, and hence under μ_ω for $\mathbb{P}$-almost all ω. Therefore the claim follows [recall (3.1)]. □

5. Proofs.

5.1. *Proof of Lemma 2.* Fix $(\lambda, h) \in \mathscr{L}$ and suppress these parameters from the notation. We shall consider the doubly infinite case and construct a measure-valued function $\omega \to \mu_\omega$ with the desired properties. The existence proof in the singly infinite case is analogous.

We start the construction by defining a finite-volume jointly translation-invariant Gibbs measure on SRW-paths and disorder configurations and then identifying a thermodynamic limit thereof. The construction guarantees that the limit fulfils the requirements stated in Lemma 2(b). Lemma 4 will be used to show that no mass escapes to infinity.

Consider a finite string $\Lambda \subset \mathbb{Z}$, with $|\Lambda|$ even and $\Lambda \ni 0$. Let $\tilde{\gamma}_\Lambda^\omega$ be the Gibbsian specification in Λ defined by

$$\tilde{\gamma}_\Lambda^\omega(S_\Lambda) = \frac{1}{2^{|\Lambda|}} \frac{\exp(H_\Lambda^\omega(S_\Lambda))}{\tilde{Z}_\Lambda^\omega} 1_{\{\exists (2i)\in\Lambda:\ S_{2i}=0\}} 1_{\{S_{\max\Lambda} - S_{\min\Lambda} = \pm 1\}}. \tag{5.1}$$

Note that here we force the path to intersect the interface somewhere and that we impose periodic boundary conditions. Clearly, $\tilde{\gamma}_\Lambda^{\sigma\omega}(\sigma A) = \tilde{\gamma}_\Lambda^\omega(A)$ for any $A \in \mathscr{F}_\Lambda$ (where σ acts cyclically).

Pick a sequence (Λ_{2n}) of such intervals with $|\Lambda_{2n}| = 2n$. Now define $\mu_B^{(n)}(A)$ by

$$\mu_B^{(n)}(A) = \int_\Omega \mathbb{P}(d\omega) 1_B(\omega) \tilde{\gamma}_{\Lambda_{2n}}^\omega(A), \qquad A \in \mathscr{F}_{\Lambda_{2n}}, B \in \mathscr{B}_{\Lambda_{2n}}. \tag{5.2}$$

By compactness, we have $\mu_B^{(n_k)}(A) \to \mu_B(A)$ for some $\mu_B(A)$ along a subsequence (n_k) for all $A \in \bigcup_n \mathscr{F}_{\Lambda_{2n}}$, all $B \in \bigcup_n \mathscr{B}_{\Lambda_{2n}}$. Since $\mu_B(A)$ is σ-additive on $\bigcup_n \mathscr{B}_{\Lambda_{2n}} \times \bigcup_n \mathscr{F}_{\Lambda_{2n}}$, it has a unique extension $\bar{\mu}_B(A)$ to $\mathscr{B} \times \mathscr{F}$ by the Caratheodory theorem.

Before we extract μ_ω from $\bar{\mu}_\Omega$, we first verify that $\bar{\mu}_\Omega$ assigns zero probability to $\{S \equiv \pm\infty\}$, which is Lemma 2(b)(i). This will follow if $\bar{\mu}_\Omega(|S_0| \geq a) \to 0$ for $a \to \infty$. Indeed, since $\tilde{\gamma}_\Lambda^\omega$ is Gibbsian, we may estimate with the aid of (4.6)

$$\begin{aligned} \mu_\Omega^{(n_k)}\big(|S_0| \geq 2a\big) &\leq \mathbb{E}\tilde{\gamma}_{\Lambda_{2n_k}}^\omega(N_0 \leq -a,\ N_1 \geq a) \\ &\leq \mathscr{O}(1) \sum_{i_1, i_2 = a}^{\infty} \exp(-\kappa(i_1 + i_2 + 1)) \to 0, \qquad a \to \infty \end{aligned} \tag{5.3}$$

uniformly in n_k. Moreover, $\mu_B(A) \leq \mathbb{P}(B)$ implies $\bar{\mu}_B(A) \leq \mathbb{P}(B)$, so by the Radon–Nikodym theorem there exists a unique μ_ω such that

$$\bar{\mu}_B(A) = \int_\Omega \mathbb{P}(d\omega) 1_B(\omega) \mu_\omega(A), \qquad A \in \mathscr{F}, B \in \mathscr{B}. \tag{5.4}$$

Clearly, by (5.3), $\mu_\omega(|S_0| < \infty) = 1$ for $\mathbb{P}$-almost all ω, so Lemma 2(b)(i) is established.

The uniqueness of the representation in (5.4) implies that μ_ω is a σ-additive probability measure and that $\mu_\omega \in \mathscr{G}_\omega^{R,\lambda,h}$ for $\mathbb{P}$-almost all ω. To prove the latter property, which is Lemma 2(b)(ii), pick $\Lambda \subset \Delta \in F(\mathbb{Z})$, $D \in \mathscr{F}$, $\tilde{\omega} \in \Omega$, take $B = \{\omega \in \Omega\colon \omega|_\Delta = \tilde{\omega}|_\Delta\}$, and subtract (5.2) with $A = D$ from (5.2) with

A replaced by the function $\gamma_\Lambda^{\tilde\omega}(D\,|\,\cdot)$ [i.e., the specification defined in (2.2)], to get

$$\int_\Omega \mathbb{P}(d\omega)1_{\{\omega\in\Omega:\ \omega|_\Delta=\tilde\omega|_\Delta\}}\big[\mu_\omega(D)-E_{\mu_\omega}\gamma_\Lambda^{\tilde\omega}(D\,|\,\cdot)\big]=0, \tag{5.5}$$

where E_{μ_ω} stands for expectation w.r.t. μ_ω. Here we have been able to use the Gibbsianness of $\tilde\gamma^\omega_{\Lambda_{2n}}$ (for n such that $\Delta\subset\Lambda_{2n}$) even under integration over ω, because ω is equal to $\tilde\omega$ inside Λ. Now proceed by letting $\Delta\to\mathbb{Z}$ to show that $\mu_{\tilde\omega}(D)=E_{\mu_{\tilde\omega}}\gamma_\Lambda^{\tilde\omega}(D\,|\,\cdot)$ for $\mathbb{P}$-almost all $\tilde\omega$. Since Λ and D are arbitrary, it follows that μ_ω is a Gibbs measure for the Hamiltonian in (1.1). It follows from (5.3) and a straightforward Borel–Cantelli argument that $\bar\mu_\Omega$ is regular and, consequently, μ_ω is regular $\mathbb{P}$-almost surely, which completes the proof of Lemma 2(b)(ii).

Finally, Lemma 2(b)(iii) follows from the periodicity of the specification $\tilde\gamma^\omega_\Lambda$, which is trivially jointly translation-invariant. □

5.2. *Proof of Theorem 3.* The proof will come in four steps. Step 1, which is the most technical, shows that for two arbitrary Gibbsian sections corresponding to the *same* medium, the paths intersect infinitely often. Via a coupling argument in Step 2 this will prove items (a)–(c) of Theorem 3. Items (d) and (e) are established in Steps 3 and 4, respectively.

STEP 1. Let $(\omega,\mu_\omega)_{\omega\in\Omega}\in\mathscr{G}^{R,\lambda,h}$ be the doubly infinite measurable Gibbsian section whose existence was established in Lemma 2. Let $(\omega,\nu_\omega)_{\omega\in\Omega}\in\hat{\mathscr{G}}^{R,\lambda,h}$, either singly infinite or doubly infinite. In order to make coupling possible, we have to show that paths intersect infinitely often under a joint measure. The result of Lemma 5 allows us to choose the product measure.

Label the paths under μ_ω by 1, the paths under ν_ω by 2. Let

$$C_\infty=\big\{(S^1,S^2)\colon S^1_n=S^2_n \text{ i.o.}\big\} \tag{5.6}$$

be the set of pairs of paths that intersect infinitely often. We shall show that $(\mu_\omega\times\nu_\omega)(C_\infty)=1$ for $\mathbb{P}$-almost all ω. The proof goes as follows.

As was shown in Lemma 5, both measures have a positive lower density of intersections with the interface. Hence the function $f_M=1_{\{N_1-N_0\geq M\}}$, which is the indicator of the event that 0 belongs to an excursion larger than M [recall (4.5)], is well defined on a set of full measure. Since $f_M\in L^1(\Sigma,\mathscr{F},\mathbb{E}\mu_\omega)$, we have by the ergodic theorem (recall that $\mathbb{E}\mu_\omega$ is σ-invariant) that there exists an $\bar f_M$ such that

$$\lim_{n\to\infty}\frac{1}{2n+1}\sum_{j=-n}^{n}\sigma^j f_M=\bar f_M,\qquad \mathbb{E}\mu_\omega\text{-a.s.}, \tag{5.7}$$

where σ acts on Σ, and $\mathbb{E}E_{\mu_\omega}(\bar f_M)=\mathbb{E}E_{\mu_\omega}(f_M)$. Moreover, $\sigma\bar f_M=\bar f_M$. Hence, for every $a>0$, $\mu_\omega(\bar f_M>a)$ is constant $\mathbb{P}$-a.s. (by ergodicity w.r.t. the disorder) and

$$\mu_\omega(\bar f_M>a)=\mathbb{E}\mu_\omega(\bar f_M>a)\leq\frac{\mathbb{E}E_{\mu_\omega}(\bar f_M)}{a}=\frac{\mathbb{E}E_{\mu_\omega}(f_M)}{a}. \tag{5.8}$$

The r.h.s. can be further estimated with the help of Lemma 4, namely,

$$\mathbb{E}E_{\mu_\omega}(f_M) \le \mathscr{O}(1)\sum_{n=M}^{\infty} n\, e^{-\kappa n} = \mathscr{O}(M)e^{-\kappa M}, \qquad M \to \infty, \tag{5.9}$$

where we use that $\mathbb{E}\mu_\omega(f_M = 1)$ is bounded by the sum over $n \ge M$ of the l.h.s. of (4.6) with $i = 0$, $K = 1$, and L running from $-n+1$ to 0. Therefore, combining (5.8) and (5.9), we have

$$\mu_\omega(\bar f_M > a) \le \frac{\mathscr{O}(M)}{a} e^{-\kappa M} \quad \text{for } \mathbb{P}\text{-almost all } \omega. \tag{5.10}$$

Now, on $\{\bar f_M \le a\}$ the fraction of sites of $2\mathbb{Z}$ covered by excursions of length $\ge M$ is at most a. Hence, on $\{\bar f_M < \hat\varrho/2\} \times \Sigma$ at least half of the arrivals of S^2 occur within the S^1-excursions of length $< M$ [recall from Lemma 5 that $\hat\varrho$ is a lower bound for ϱ_0^- defined in (3.1)]. If the two paths (S^1, S^2) are to avoid each other, then the first has to stay either above or below the other during *all* of these (infinitely many) excursions.

To show that the probability of the latter event is zero we introduce some definitions. Let

$$p(n,\omega) = \frac{\gamma_{\Lambda_{2n}}(S_i > 0\,\forall\, 1 \le i < 2n \,|\, 0)}{\gamma_{\Lambda_{2n}}(S_i \ne 0\,\forall\, 1 \le i < 2n \,|\, 0)} \tag{5.11}$$

and put $p_M = \max_{n<M}\max_{\omega\in\Omega}\max\{p(n,\omega), 1 - p(n,\omega)\}$. An easy computation shows that $p_M = (1 + \exp(-2\lambda(1+h)(M-1)))^{-1} < 1$. This is the least price to pay (when conditioning upon the arrivals) to avoid that the path S^1 be swapped to $-S^1$ during an excursion of length less than M.

Next, define the remotest intersection time as

$$\tau = \begin{cases} \max\{k\colon S^1_{2k} = S^2_{2k} \text{ or } S^1_{-2k} = S^2_{-2k}\}, & (S^1, S^2) \notin C_\infty, \\ \infty, & (S^1, S^2) \in C_\infty. \end{cases} \tag{5.12}$$

Also define $\mathscr{N}_{k,M}(n) = \#\{i\colon k \le i \le n+k,\ S^2_{2i} = 0,\ f_M(\sigma^i S^1) = 0\}$ and $A_M = \{\bar f_M < \hat\varrho/2\} \times \Sigma$. On A_M we have

$$\liminf_{n\to\infty} \frac{\mathscr{N}_{k,M}(n)}{n} \ge \hat\varrho/2 > 0, \qquad \mathbb{E}\nu_\omega\text{-a.s.} \tag{5.13}$$

as follows from Lemma 5 and the reasoning below (5.10). Therefore we get

$$\begin{aligned} &(\mu_\omega \times \nu_\omega)\big(A_M \cap [C_\infty]^c\big) \\ &\quad \le \sum_{k=1}^{\infty}\Big[2(\mu_\omega \times \nu_\omega)\Big(\Big[\lim_{n\to\infty} p_M^{\mathscr{N}_{k,M}(n)}\Big]1_{A_M}1_{\{\tau=k\}}\Big)\Big] = 0. \end{aligned} \tag{5.14}$$

Here we decompose according to the values of τ, condition upon the arrivals of both S^1 and S^2 in $[\tau, \tau+n]$, then bound by p_M the interarrival probabilities of S^1 for excursions of length $< M$ containing at least one arrival of S^2, and bound by 1 otherwise, and finally use that $1_{A_M}\, p_M^{\mathscr{N}_{k,M}(n)} \le \exp(n(\hat\varrho/4)\log p_M)$ for n large enough, as follows from (5.13). The factor 2 reflects whether S^1

stays above S^2 from τ onward or vice versa. Note that there is no problem with 1_{A_M} in the conditioning, because A_M is a tail event.

The conclusion of (5.14) is that $(\mu_\omega \times \nu_\omega)(A_M \cap [C_\infty]^c) = 0$ for all M. On the other hand,

$$\mu_\omega\left(\bigcup_{M=1}^{\infty} A_M\right) = 1 \tag{5.15}$$

by the Borel–Cantelli lemma and (5.10). Hence, combining (5.14) and (5.15), we find that $(\mu_\omega \times \nu_\omega)([C_\infty]^c) = 0$, that is, the paths S^1 and S^2 intersect infinitely often $(\mu_\omega \times \nu_\omega)$-almost surely.

STEP 2. We show by a coupling inequality that μ_ω and ν_ω have to agree on the tail σ-field $\mathscr{T}$. Besides other things, this implies uniqueness of the Gibbs measure. The proof is done for ν_ω singly infinite, the doubly infinite case requiring only formal alterations.

Let $k \in \mathbb{N}$ and $A \in \mathscr{F}_{\Lambda_k^c}$ (A should be thought of as approximating a tail event). Define

$$\tau = \inf\{n \geq 0 \colon S_n^1 = S_n^2\}. \tag{5.16}$$

Let E_ω denote the expectation w.r.t. the product measure $\mu_\omega \times \nu_\omega$. Then we can write

$$\begin{aligned} \left|\mu_\omega(A) - \nu_\omega(A)\right| &= \left|E_\omega(A \times \Sigma) - E_\omega(\Sigma \times A)\right| \\ &\leq \left|E_\omega(1_{\{\tau > k\}} 1_{A\times\Sigma}) - E_\omega(1_{\{\tau>k\}} 1_{\Sigma\times A})\right| \\ &\leq E_\omega(1_{\{\tau>k\}}), \end{aligned} \tag{5.17}$$

where we use that $E_\omega(1_{\{\tau\leq k\}} 1_{A\times\Sigma}) = E_\omega(1_{\{\tau\leq k\}} 1_{\Sigma\times A})$ because μ_ω and ν_ω have the same conditional probabilities. Hence

$$\sup_{A\in\mathscr{F}_{\Lambda_k^c}} \left|\mu_\omega(A) - \nu_\omega(A)\right| \leq E_\omega(1_{\{\tau>k\}}). \tag{5.18}$$

By Step 1 the r.h.s. tends to 0 as $k \to \infty$. Consequently, μ_ω and ν_ω agree on the tail σ-field $\mathscr{T}$. In particular, we get (3.2):

$$\lim_{k\to\infty} \sup_{A\in\mathscr{F}} \left|\mu_\omega(\sigma^k A) - \nu_\omega(\sigma^k A)\right| = 0 \quad \text{for } \mathbb{P}\text{-almost all } \omega. \tag{5.19}$$

STEP 3. The a.s. convergence of ergodic averages under ν_ω can be proved through a comparison with the a.s. convergence under $\mathbb{E}\mu_\omega$, which is translation invariant. Namely, given a set $A \in \mathscr{F}$, let

$$A_> = \left\{\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{\sigma^k A} > \mathbb{E}\mu_\omega(A)\right\}. \tag{5.20}$$

Clearly, $A_>$ is a tail event, and $\mathbb{E}\mu_\omega(A_>) = 0$ by the translation invariance of $\mathbb{E}\mu_\omega$. However, this implies $\mathbb{E}\nu_\omega(A_>) = 0$, since $\mathbb{E}\nu_\omega$ coincides with $\mathbb{E}\mu_\omega$

on $\mathscr{T}$. So

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1_{\sigma^k A} \le \mathbb{E}\mu_\omega(A) \quad \nu_\omega\text{-a.s. for } \mathbb{P}\text{-almost all } \omega. \tag{5.21}$$

The same argument works for the limes inferior, so the limit in (3.3) is established.

STEP 4. The last property to prove is that μ_ω and ν_ω are exponentially tight. Since we know by (5.19) that $|\mu_\omega(\sigma^n A) - \nu_\omega(\sigma^n A)| \to 0$ as $n \to \infty$, it suffices to study the tail of μ_ω. To that end, pick $s \in \mathbb{Z}$, $s > 0$. We have from Gibbsianness

$$\begin{aligned}\mu_\omega(S_0 = 2s) = \sum_{n_+, n_- = s}^{\infty} & \frac{P_{n_+, n_-}(S_0 = 2s)}{Z^\omega_{I_{n_+, n_-}} \exp(-\lambda(\Omega_{I_{n_+, n_-}} + h|I_{n_+, n_-}|))} \\ & \times \mu_\omega(S_{-2n_-} = S_{2n_+} = 0),\end{aligned} \tag{5.22}$$

where $I_{n_+, n_-} = (-2n_-, 2n_+] \cap \mathbb{Z}$, and $P_{n_+, n_-}(S_0 = 2s)$ is the probability that SRW, conditioned on hitting the interface at $-2n_-$ and $2n_+$, climbes to height $2s$ at 0 without ever touching the interface in between. By using Lemma 3 [and using the Borel–Cantelli lemma to get rid of $\mathbb{E}$ as in (5.10)], we have for any $\varepsilon > 0$,

$$\left(Z^\omega_{I_{n_+, n_-}} \exp(-\lambda h |I_{n_+, n_-}|)\right)^{-1} \le \mathscr{O}(1) \exp(-|I_{n_+, n_-}|(\psi - \varepsilon)), \tag{5.23}$$

so the r.h.s. of (5.22) is $\mathbb{P}$-a.s. absolutely summable and of order $\exp(-4s \cdot (\psi - \varepsilon))$ as $s \to \infty$ [note that $\exp(-\lambda\Omega_I) = o(\exp(\varepsilon|I|))$ for each $\varepsilon > 0$ as $|I| \to \infty$]. After letting $\varepsilon \downarrow 0$, we obtain that the tail property in (3.4) is proved for $s > 0$, with $\zeta_s = \psi$. For $s \in \mathbb{Z}$, $s < 0$ there is an additional factor

$$\exp\left[-2\lambda \sum_{l \in I_{n_-, n_+}} (\omega_l + h)\right] \tag{5.24}$$

in the numerator of each summand. This raises ζ_s by λh. □

6. Zero density in the delocalization regime. In this section we consider the singly infinite case and present an argument due to G. Giacomin (private communication), showing that in the interior of the delocalization regime the path is delocalized in the following sense:

THEOREM 4. *Let $(\lambda, h) \in \mathrm{int}(\mathscr{D})$ and let $\nu_\omega \in \mathscr{G}^{R, \lambda, h}_\omega$ be an arbitrary singly infinite regular Gibbs measure. Then for $\mathbb{P}$-almost all ω,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} 1_{\{S_i = a\}} = 0 \quad \textit{in } \nu_\omega\textit{-probability for all } a \in \mathbb{Z}. \tag{6.1}$$

REMARK. Note that Theorem 4 makes a claim about *all* regular Gibbs measures under a typical disorder. Since we do not have Lemma 2 for $(\lambda, h) \in \mathscr{D}$, the notion of a measurable Gibbsian section is not available.

PROOF. Fix $a \in 2\mathbb{Z}$ (without loss of generality). For $k, l \in \mathbb{N}$, define

$$A^a_{k,l} = \left\{ \sum_{i=0}^{l} 1_{\{S_{2i}=a\}} \geq k+1 \right\}. \tag{6.2}$$

We shall show that for any boundary condition $\tilde{S}$ and any $\varepsilon > 0$ the event $A^a_{\lfloor \varepsilon n \rfloor, n}$ has a probability decaying to zero under the finite-volume specification $\gamma^\omega_{\Lambda_{2n}}(\cdot \mid \tilde{S})$ in the limit as $n \to \infty$. The key ingredient is the well-known entropy inequality

$$\gamma^\omega_{\Lambda_{2n}}\big(A^a_{\lfloor \varepsilon n \rfloor, n} \,\big|\, \tilde{S}\big) \leq \frac{\log 2 + \mathscr{H}^\omega_{2n}}{\log\big(1/P_{2n}(A^a_{\lfloor \varepsilon n \rfloor, n} | \tilde{S})\big)}, \tag{6.3}$$

where $P_{2n}(\cdot \mid \tilde{S})$ is the SRW-bridge probability measure between 0 and $\tilde{S}_{2n}$, and

$$\mathscr{H}^\omega_{2n} = \mathscr{H}\big(\gamma^\omega_{\Lambda_{2n}}(\cdot | \tilde{S}) \,\big|\, P_{2n}(\cdot | \tilde{S})\big) \tag{6.4}$$

denotes the relative entropy of the probability measure $\gamma^\omega_{\Lambda_{2n}}(\cdot \mid \tilde{S})$ w.r.t. $P_{2n}(\cdot \mid \tilde{S})$.

We first note that for $(\lambda, h) \in \operatorname{int}(D)$ the specific relative entropy $\mathscr{H}^\omega_{2n}/2n$ vanishes in the thermodynamic limit:

$$\lim_{n \to \infty} \frac{\mathscr{H}^\omega_{2n}}{2n} = -\phi(\lambda, h) + \lambda \frac{\partial \phi}{\partial \lambda}(\lambda, h) = 0 \quad \text{for } \mathbb{P}\text{-almost all } \omega. \tag{6.5}$$

Indeed, by (1.1) and (2.2),

$$\begin{aligned}
\mathscr{H}^\omega_{2n} &= \sum_{S_{\Lambda_{2n}}} \gamma^\omega_{\Lambda_{2n}}(S_{\Lambda_{2n}} \mid \tilde{S}) \log \frac{\gamma^\omega_{\Lambda_{2n}}(S_{\Lambda_{2n}} \mid \tilde{S})}{P_{2n}(S_{\Lambda_{2n}} \mid \tilde{S})} \\
&= -\log Z^\omega_{\Lambda_{2n}}(\tilde{S}) + \frac{1}{Z^\omega_{\Lambda_{2n}}(\tilde{S})} \sum_{S_{\Lambda_{2n}}} H^\omega_{\Lambda_{2n}}(S_{\Lambda_{2n}} \vee S_{\Lambda^c_{2n}}) \\
&\qquad\qquad \times \exp(H^\omega_{\Lambda_{2n}}(S_{\Lambda_{2n}} \vee S_{\Lambda^c_{2n}})) P_{2n}(S_{\Lambda_{2n}} \mid \tilde{S}) \\
&= -\log Z^\omega_{\Lambda_{2n}}(\tilde{S}) + \lambda \frac{\partial}{\partial \lambda} \log Z^\omega_{\Lambda_{2n}}(\tilde{S}).
\end{aligned} \tag{6.6}$$

Hence, the first equality in (6.5) follows after letting $n \to \infty$ and interchanging the limit with $\partial/\partial\lambda$ [which is allowed because of the convexity and regularity of ϕ in $\operatorname{int}(\mathscr{D})$], while the second equality in (6.5) holds because $\phi(\lambda, h) = \lambda h$ on $\mathscr{D}$. Thus, after we show that

$$\limsup_{n \to \infty} \frac{1}{2n} \log P_{2n}\big(A^a_{\lfloor \varepsilon n \rfloor, n} \mid \tilde{S}\big) < 0 \quad \text{for all } \varepsilon > 0, \tag{6.7}$$

it will follow from (6.3) and (6.5) that $\lim_{n\to\infty} \gamma^\omega_{\Lambda_{2n}}(A^a_{\lfloor \varepsilon n \rfloor, n} \mid \tilde{S}) = 0$. Conditioning then implies the same for any (regular) Gibbs measure ν_ω.

Pick ν_ω and define τ_1 (τ_2) to be the leftmost (rightmost) site i with $0 \le i \le 2n$ such that $S_i = a$. If no such sites occur, then (6.1) is trivially satisfied. Hence

$$\begin{aligned} &P_{2n}(A^a_{\lfloor \varepsilon n \rfloor, n} \,|\, \tilde S) \\ &\quad = \sum_{0 \le l_1 \le l_2 \le n} P_{2n}\big(\tau_1 = 2l_1, \tau_2 = 2l_2 \mid \tilde S\big) P_{2(l_2-l_1)}\big(A^0_{\lfloor \varepsilon n \rfloor, l_2-l_1} \,|\, 0\big), \end{aligned} \tag{6.8}$$

where the last factor can be further estimated by the corresponding number for the free SRW, namely,

$$\begin{aligned} P_{2(l_2-l_1)}\big(A^0_{\lfloor \varepsilon n \rfloor, l_2-l_1} \,|\, 0\big) &\le \frac{P\big(A^0_{\lfloor \varepsilon n \rfloor, l_2-l_1} \cap \{S_{2(l_2-l_1)} = 0\}\big)}{P\big(S_{2(l_2-l_1)} = 0\big)} \\ &\le \mathscr{O}(\sqrt{n}) P\big(A^0_{\lfloor \varepsilon n \rfloor, n}\big), \end{aligned} \tag{6.9}$$

where we used that $l_2 - l_1 \le n$ and $P(S_{2n} = 0) \sim C/\sqrt{n}$. Thus

$$P_{2n}\big(A^a_{\lfloor \varepsilon n \rfloor, n} \,|\, \tilde S\big) \le \mathscr{O}(\sqrt{n}) P\big(A^0_{\lfloor \varepsilon n \rfloor, n}\big), \tag{6.10}$$

so we need only consider the case $a = 0$.

Next, similarily as in the proof of Theorem 3, let us define the interarrival time ξ_i as the duration between the ith and the $(i+1)$st intersection with the interface. Then we may write

$$A^0_{\lfloor \varepsilon n \rfloor, n} = \left\{ \sum_{i=1}^{\lfloor \varepsilon n \rfloor} \xi_i \le n \right\}. \tag{6.11}$$

Now, under P the ξ_i are i.i.d. with distribution function satisfying

$$\sum_{l=1} P(\xi_1 = l) z^l = 1 - \sqrt{1 - z^2} \quad \text{for all } 0 \le z < 1. \tag{6.12}$$

By the exponential Chebyshev inequality we therefore have

$$P\left(\sum_{i=1}^{\lfloor \varepsilon n \rfloor} \xi_i \le n \right) \le z^{-n} \big(1 - \sqrt{1 - z^2}\big)^{\lfloor \varepsilon n \rfloor} \quad \text{for all } 0 < z < 1. \tag{6.13}$$

The r.h.s. attains its minimum at z such that $z^2 = (1 - 2\varepsilon'_n)(1 - \varepsilon'_n)^{-2}$ with $\varepsilon'_n = \lfloor \varepsilon n \rfloor / n \le \varepsilon$. Consequently, using (6.10), (6.11) and (6.13) we get the bound

$$\begin{aligned} &\limsup_{n \to \infty} \frac{1}{2n} \log P_{2n}\big(A^a_{\lfloor \varepsilon n \rfloor, n} \,|\, \tilde S\big) \\ &\qquad \le (1 - \varepsilon) \log(1 - \varepsilon) - \frac{1}{2}(1 - 2\varepsilon) \log(1 - 2\varepsilon), \end{aligned} \tag{6.14}$$

when ε is small enough. The r.h.s. is $\sim -\varepsilon^2$ as $\varepsilon \downarrow 0$. Hence (6.7) holds for all $\varepsilon > 0$ and the proof of (6.1) is complete. $\square$

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