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VOLATILITY MISSPECIFICATION, OPTION PRICING AND SUPERREPLICATION VIA COUPLING

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Consider the performance of an options writer who misspecifies the dynamics of the price process of the underlying asset by overestimating asset price volatility. When does he overprice the option? If he follows the hedging strategy suggested by his model, when does the terminal value of his strategy dominate the option payout?

We show that both these events happen if the option payoff is a convex function of the price of the underlying at maturity. The proofs involve the simple, powerful and intuitive techniques of coupling.

1. Robust hedging and superreplication. The standard approach in mathematical finance, and particularly in the pricing of derivative securities, is to begin by writing down a stochastic model, which is assumed, without comment, to correctly and precisely specify the dynamics of the underlying asset. Arbitrage theory, backed by change of measure techniques and martingale representation theorems, then allows options to be priced and hedged. The fairness of the price and the success of the hedge depend crucially on the truth of the underlying model.

The purpose of this article is to consider the implications of a misspecification of the dynamics of the asset price process. In particular, if the options writer uses an incorrect model, when does he overcharge for the option? Further, if he attempts to hedge using this incorrect model, can he still replicate (or rather superreplicate) the option payoff? (A superreplicating strategy is a dynamic hedging strategy which generates a terminal wealth which stochastically dominates the option payout.)

It is well known that the price of a call option in the Black–Scholes model is an increasing function of the volatility parameter and that this monotonicity property extends to all European options with convex payoff profiles. We generalize this result to prove a price comparison theorem for pairs of models with stochastic volatilities. We have in mind the scenario where one model is used by the options writer to price and hedge options, and the second model represents the (unknown) truth.

The substantive results of this paper are not new and can be found in [7]. Theorem 3.1 can be traced back to [4]; see also [19]. Instead, the aim of this paper is to give proofs of the main results which are shorter and which in the opinion of the author, are both more natural and more intuitive. This improved clarity leads to an extension of the main results to nondiffusion

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processes (see Remark 2.3) and to an understanding of the counterexamples in [7] (see Remark 2.4). A second justification for this paper is that the results are extensions of those proved in [4] and [7] in the sense that they require weaker conditions. For example the bounded derivative condition (see Remark 3.3) is not needed for the proof of the convexity theorem, Theorem 3.1, and the price process need only be an H^1 martingale rather than L^2. See Section 3.

The key tool we use here is that of the coupling of stochastic processes; those unaware of the joys of coupling are referred to [17]. This contrasts with the use of stochastic flows in [7] or the analysis of the partial differential pricing equation in [4]. As a first example of coupling (taken from the introduction to [17]) let X be a continuous Markov process and h an increasing function. Let a superscript denote the initial value. Given independent realizations X^x and X^y with $x < y$, define $\tau = \inf_u\{X^y_t \le X^x_t\}$. Define a new process $\tilde{X}^x_s$ via

$$\tilde{X}^x_s \equiv \begin{cases} X^x_s, & 0 \le s \le T \wedge \tau, \\ X^y_s, & T \wedge \tau \le s \le T. \end{cases}$$

Then $\tilde{X}^x$ has the same law as X^x and moreover, for all s,

$$\tilde{X}^x_s \le X^y_s \tag{1.1}$$

by construction. Hence $\mathbb{E}(h(X^x_T)) \equiv \mathbb{E}(h(\tilde{X}^x_T)) \le \mathbb{E}(h(X^y_T))$. See Remark 3.2 for a financial interpretation of this simple monotonicity result.

Throughout this article we will consider a continuous-time model for the economy with a finite horizon T. Markets are frictionless with no transaction costs or taxes, assets are infinitely divisible and their prices are semimartingales, and the prices of options are given by the (discounted) expected payoff of the option under a risk-neutral probability measure. This will be true if the market is complete, which in turn will follow, given our other assumptions if the filtration is generated by the asset price process. The fundamental problem is to price a derivative security which at time T has nonnegative payoff profile h.

For the sections on European options we will further assume that interest rates are zero. This has the advantage of minimizing the amount of notation required and thus facilitates maximum insight. All the results extend easily to the case of deterministic interest rates. Indeed, provided that the interest rates are adapted to the filtration generated by the asset price process and the market is complete, nondeterministic interest rates can also be accommodated. In the consideration of American options, one of the motivations for early exercise is to capture profits which might otherwise depreciate over time. Hence it would be restrictive to limit attention to the case of zero interest rates; however, for economy of notation we assume a constant rate of interest. Again the extension to deterministic interest rates is easy.

There are two main results in this article which we call option price monotonicity and the superreplication property. Suppose that there are two candidate models for the behavior of the underlying asset; P_0 is given and either the price process solves

$$dP_s = P_s \hat{\sigma}(P_s, s)\, dB_s \tag{1.2}$$

or

$$dP_s = P_s \tilde{\sigma}(P_s, s)\, dB_s. \tag{1.3}$$

(For the purposes of this introduction we implicitly assume that the price process is a diffusion, and that the market is complete; we comment later on when these assumptions can be weakened.)

Option price monotonicity. For options with a convex payoff profile $h(P_T)$, if for all p and s we have $\hat{\sigma}(p, s) \geq \tilde{\sigma}(p, s)$, then the option price is greater under the model (1.2) than under the model (1.3).

Superreplication property. Suppose h is convex and as before $\hat{\sigma}(p, s) \geq \tilde{\sigma}(p, s)$. Then if an options writer uses the model (1.2) to price and hedge options, but in reality the price process solves (1.3), then the sum of the initial option price and the gains from the (incorrect) hedging strategy dominate the option payout.

The superreplication property implies option price monotonicity.

The superreplication property is of great importance since it means that the option writer's hedging strategy is robust to model misspecification of the dynamics of the underlying asset. The acknowledgement of the issue of model risk has led to several new modelling developments and strategies. Dupire [5, 6] has argued that by considering calls as liquid trading instruments it is possible to price exotic options in a model-free fashion. See also [13] for a related approach to robust hedging. The special case where the model risk is restricted to a misspecification of volatility has also been the subject of much attention. Stochastic volatility models fall into this area; see [24] for a recent review. Frey and Stremme [11] show how a nonconstant volatility can arise in a model in which hedging demand affects the price dynamics of the underlying asset. Avellaneda, Levy and Parás [2] and Lyons [18] consider the situation where volatility is unconstrained except to lie in an interval. Lyons gives an example of a barrier option which illustrates that if the option is nonconvex then the option price may not increase monotonically with volatility. Frey and Sin [10] find bounds on European option prices in an incomplete market with stochastic volatility. All the papers cited in this paragraph are predicated upon a superreplication philosophy; for an alternative utility based approach to pricing and hedging with misspecified volatility, see [1].

This paper is organized as follows. In Section 2 we provide a direct proof of the monotonicity of option prices with respect to volatility using a coupling argument. (This direct proof is original and shows that option price monotonicity does not require a diffusion assumption.) In Section 3 we prove the superreplication property and hence derive a second proof of option price monotonicity. The key step is to prove that for a diffusion model the option value at intermediate times is a convex function of the asset price. Again, the proof is based on a coupling story. Finally, Section 4 contains an extension of the main results to American options; yet again there is a natural intuitive proof involving a coupling-style argument.

2. Option price monotonicity via coupling. Suppose that the price of an asset is a diffusion process and is given by the solution to the stochastic differential equation

$$dP_s = P_s \sigma_s \, dB_s \tag{2.1}$$

for a fixed initial value $P_0 > 0$, and for an adapted volatility process σ_s. Suppose that there are two candidate models for σ, namely $\sigma = \hat{\sigma}$ or $\sigma = \tilde{\sigma}$ and suppose that there is an ordering on these candidate volatilities in the sense that $\hat{\sigma}(x, s) \geq \tilde{\sigma}(x, s)$ for all asset values x and times s.

THEOREM 2.1. *In a complete market the* (*non-pathdependent*) *option with convex payoff h is more expensive under the model with volatility $\hat{\sigma}$ than under the model with volatility $\tilde{\sigma}$.*

Notation and assumptions. When we define quantities τ, A and P with respect to σ, this should be taken as implicitly defining pairs of quantities, the elements of which are denoted in a natural fashion with the accents $\hat{}$ and $\tilde{}$. As a first example, for a Brownian motion W with $W_0 = P_0$, define τ to be the solution of the ordinary differential equation

$$\frac{d\tau_t}{dt} = \frac{1}{W_t^2 \sigma(W_t, \tau_t)^2}. \tag{2.2}$$

Assume that, almost surely, this ordinary differential equation has a unique strictly increasing solution which explodes when W first reaches zero.

PROOF. The main idea is that in each model P_t is a continuous local martingale so we can hope to write P as a time-change of Brownian motion. By constructing both models from the same Brownian motion, we can hope to deduce comparison theorems for option prices from comparison theorems for the time changes.

Begin with a Brownian motion W started at P_0. (Note that since W is not defined with respect to σ we are thinking of a single Brownian motion rather than a pair.) Use this Brownian motion to define a (pair of) increasing functional(s) τ. Denote the inverse to τ by A_t. Now define a process P via $P_t \equiv W(A_t)$; then P is a local martingale with quadratic variation

$$\frac{dA_t}{dt} \equiv W(A_t)^2 \sigma(W(A_t), \tau(A_t))^2 \equiv P_t^2 \sigma(P_t, t)^2.$$

In particular, we can represent P as a (weak) solution of the SDE $dP_t = P_t \sigma(P_t, t) \, dB_t$.

If $\hat{P}_t = \tilde{P}_t$ then $d\hat{A}_t \geq d\tilde{A}_t$ and this is certainly true at $t = 0$. Moreover if $\hat{A}_t = \tilde{A}_t$ then by construction $\hat{P}_t \equiv \tilde{P}_t$ (since both processes have been constructed as a time change of the Brownian motion W) and hence $\hat{A}_t \geq \tilde{A}_t$ for all t.

Finally, $h(\hat{P}_T) \equiv h(W(\hat{A}_T)) \equiv h(W(\tilde{A}_T) + (W(\hat{A}_T) - W(\tilde{A}_T))$, so that the claimed option price monotonicity follows by a conditional version of Jensen's inequality. □

REMARK 2.1. This argument proves price monotonicity, but it does not seem possible to extend this coupling argument to show that the options trader using the wrong model will superreplicate. This is the subject of Section 3. However the above provides a general philosophy for showing the price monotonicity property: construct solutions $\hat{P}_t = W(\hat{A}_t)$ and $\tilde{P}_t = W(\tilde{A}_t)$ with respect to the same Brownian motion W, in such a way that $\hat{A}_T \geq \tilde{A}_T$ almost surely, and then apply Jensen's inequality. Thus, for example, if both $\hat{\sigma}$ and $\tilde{\sigma}$ are functions of time alone then a simple sufficient condition for option price monotonicity is that

$$\int_0^T \hat{\sigma}(t)^2 \, dt \geq \int_0^T \tilde{\sigma}(t)^2 \, dt. \tag{2.3}$$

REMARK 2.2. Completeness of the market is used to justify the assumption that the option price is the expected payoff of the option under the risk-neutral probability measure $\mathbb{P}$. More generally under any measure $\mathbb{P}$ for which B, and by construction W, are Brownian motions, the expected option payout is larger for the model with volatility $\hat{\sigma}$ than the model with volatility $\tilde{\sigma}$. Thus, provided that a suitable pricing measure has been fixed, option price monotonicity follows. This situation may arise in an incomplete market, for example with the selection of a minimal martingale measure (in the sense of [8]) from a family of equivalent martingale measures, and in this sense option price monotonicity is sometimes found in incomplete markets. However, the natural setting for the arguments of this article is a diffusion model which is complete, at least if the volatility is correctly specified.

REMARK 2.3. In fact, it is not necessary to restrict to diffusion models so that $\hat{\sigma}$, $\tilde{\sigma}$ can be functions of the whole price history provided that τ and its inverse can be defined and that for all p and for all pairs of paths $\{\hat{p}_u\}_{0\leq u\leq s}$, $\{\tilde{p}_u\}_{0\leq u\leq s}$ with $\hat{p}_s \equiv p \equiv \tilde{p}_s$ we have that $\hat{\sigma}(\{\hat{p}_u\}_{0\leq u\leq s}, s) \geq \tilde{\sigma}(\{\tilde{p}_u\}_{0\leq u\leq s}, s)$.

REMARK 2.4. An example in [7] (due in part to Marc Yor) at first sight appears to contradict the claim in Remark 2.3. In the example, two volatility processes are given for which there is an ordering of the volatilities but such that the prices of call options decrease with apparently increasing volatility. The point to note is that the Yor construction defines the price in each model as the strong solution of the SDE (2.1) with respect to the same Brownian motion B. Upon time change, this means that the processes must be defined relative to different Brownian motions W and the coupling argument cannot be applied. For financial modelling and derivative pricing, it is the law of the price process that is important and not the relationship with a particular driving Brownian motion. Hence option price monotonicity does hold in the sense of Remark 2.3. If volatility is path dependent, then the "correct" notion of a volatility comparison condition is that given in Remark 2.3.

3. Superreplication of European contingent claims. The purpose of this section is to provide a further proof that for European options with convex payoffs the option price is increasing in the volatility. Moreover, an options trader who prices and hedges according to the model with the higher volatility will superreplicate the option payout if the true underlying model is the alternative regime.

The key result in this section, which is of independent interest, is that, for diffusion models of the price of the underlying asset, the prices of options with convex payoffs are a convex function of the underlying. This is Theorem 3.1 below, and again there is a simple and intuitive coupling proof of this result. The superreplication result (Theorem 3.2) then follows.

THEOREM 3.1. *For a* (*complete*) *diffusion model of the asset price*, *and a European option with convex payoff* h, *the option value at intermediate times is convex in the price of the underlying*.

ASSUMPTIONS. Suppose the model is $dP_s = P_s\sigma(P_s, s)\,dB_s$; then in a complete market the option price is given by $v(P_t, t) = \mathbb{E}(h(P_T)|P_t)$, at least under the assumption that $\mathbb{E}(|h(P_T)|) < \infty$. We assume that σ has sufficient continuity properties to ensure that the solution to the SDE is unique in law (for example, a Lipschitz condition on $p\sigma(p, s)$; see [23], Remark V.16.4), and that P is a true martingale. A simple sufficient condition for this second property is that $\sigma(p, s)$ is bounded; more generally, the problem translates into one of verifying that the Doléans exponential is a true martingale; see [22], Section 3.5.

PROOF. *The set-up*. Suppose $0 < z < y < x$ and for independent Brownian motions α, β, γ define X, Y, Z via

$$dX_s = X_s\sigma(X_s, s)\,d\alpha_s, \quad X_0 = x,$$

$$dY_s = Y_s\sigma(Y_s, s)\,d\beta_s, \quad Y_0 = y,$$

$$dZ_s = Z_s\sigma(Z_s, s)\,d\gamma_s, \quad Z_0 = z.$$

The "*coupling*." Let $H_x \equiv \inf_u\{X_u = Y_u\}$, and similarly $H_z \equiv \inf_u\{Y_u = Z_u\}$. Now define $\tau = H_x \wedge H_z \wedge T$. Then, by symmetry, on $\tau = H_x$,

$$(X_T - Z_T)h(Y_T) \overset{\mathcal{D}}{=} (Y_T - Z_T)h(X_T)$$

so that

$$\mathbb{E}[(X_T - Z_T)h(Y_T); \tau = H_x] = \mathbb{E}[(Y_T - Z_T)h(X_T); \tau = H_x]$$

Also $\mathbb{E}[(X_T - Y_T)h(Z_T); \tau = H_x] = 0$. Similarly

$$\mathbb{E}[(X_T - Z_T)h(Y_T); \tau = H_z] = \mathbb{E}[(X_T - Y_T)h(Z_T); \tau = H_z] + \mathbb{E}[(Y_T - Z_T)h(X_T); \tau = H_z].$$

Finally, on $\tau = T$, we have that $Z_T < Y_T < X_T$, and by convexity of h,

$$(X_T - Z_T)h(Y_T) \le (X_T - Y_T)h(Z_T) + (Y_T - Z_T)h(X_T).$$

Taking expectations and adding gives

$$\mathbb{E}[(X_T - Z_T)h(Y_T)] \le \mathbb{E}[(X_T - Y_T)h(Z_T)] + \mathbb{E}[(Y_T - Z_T)h(X_T)],$$

so that by the independence of X, Y and Z,

$$(x - z)\mathbb{E}[h(Y_T)] \le (x - y)\mathbb{E}[h(Z_T)] + (y - z)\mathbb{E}[h(X_T)]. \qquad \square$$

REMARK 3.1. It is clear that this proof cannot be extended to nondiffusion models because then there can be no identity in law between, for example, X_T and Y_T on $\tau = H_x$.

REMARK 3.2. The argument given in the introduction shows that, if h is increasing, then the option price function v must also be increasing. This simple result, Theorem 1 in [4], was described as "the intuitive link between a diffusion process and properties of options prices." As the referee observed, the stochastic comparison $X_s^x \le X_s^y$, for all s [see (1.1)], and hence monotonicity of the option price function, holds more generally than just for diffusion models (see, e.g., [21], Theorem 54, page 268).

REMARK 3.3. If h has bounded derivative on $[0, \infty]$, and bound C say, then it is clear that $h(0) - CP_t \le v(P_t, t) \le h(0) + CP_t$. Given also that v is convex, and hence that the space derivative is increasing, it must follow that

$$|v'(P_t, t)| \le C. \tag{3.1}$$

One of the motivations for this paper was to provide a simple, direct and general proof of Theorem 3.1 using probabilistic methods. The result in Theorem 3.1 has appeared elsewhere (see [4], [7] and [9]), though the proofs in these papers use different methods and require more restrictive technical conditions. The authors in [4] and [9] differentiate the partial differential equation for the option price v, with respect to space, to deduce a partial differential equation for v'. This requires the diffusion coefficient $p^2\sigma(p, s)^2$ to be differentiable. In order to use a method of stochastic flows, [7] again requires continuous differentiability. Both methods require the payoff function h to have a bounded derivative. In each case, the proof uses the Feynman–Kac formula to deduce a representation for the derivative of the option payoff in the form $v'(P_t, t) = \mathbb{E}^*(h(P_T)|P_t)$. Here the superscript * denotes the fact that P solves a different SDE from (2.1) in this representation.

Theorem 3.1 was used by [7] as a key step in the proof of the main result of this paper, namely that an options trader who uses a model which systematically overestimates volatility is following a robust pricing and hedging strategy. Here we reproduce the essence of their argument.

Suppose that the options trader uses a diffusion model

$$dP_s = P_s \sigma(P_s, s)\, dB_s$$

under which options prices are given by

$$v(P_s, s) = \mathbb{E}(h(P_T)|P_s), \qquad v(P_T, T) = h(P_T).$$

Suppose further that the true volatility is given by $\tilde{\sigma}(p, s)$ and that the model volatility dominates the true volatility in the sense that $\sigma(p, s) \geq \tilde{\sigma}(p, s)$ for all asset values p and intermediate times s.

THEOREM 3.2. *If the model volatility dominates the true volatility, and if the option payoff function h is convex, then the options trader who prices and hedges according to the model will superreplicate the option payout.*

ASSUMPTIONS. We need to apply Itô's formula to the option price function $v(P_s, s)$ in the proof below. Sufficient conditions for $v \in C^{2,1}((0, \infty) \times [0, T))$ are that σ is positive and Hölder continuous. For further details see [7], Hypothesis 6.1, [16] Section 5.7 or [12].

PROOF. Under the option writer's model, since the market is complete, the payout can be decomposed into the initial price and the gains from trade:

$$h(P_T) = v(P_0, 0) + \int_0^T v'(P_t, t)\, dP_t.$$

Moreover, since $v(P_t, t)$ is a martingale, we must have that v solves the partial differential equation

$$\tfrac{1}{2} p^2 \sigma^2(p, s) v'' + \dot{v} = 0. \tag{3.2}$$

Consider the effect of following the strategy specified by $\theta_s = \theta_s(p, s) = v'(p, s)$ in the real (or tilded) world. Start with initial fortune $v(P_0, 0)$. By time T this has increased to

$$v(P_0, 0) + \int_0^T \theta_s\, d\tilde{P}_s = v(P_0, 0) + \int_0^T v'(\tilde{P}_s, s)\, d\tilde{P}_s, \tag{3.3}$$

and by assumption this stochastic integral is well defined. For suitable paths α, we have $v(\alpha_T, T) - v(\alpha_0, 0) = \int_0^T dv(\alpha_s, s)$ so that if $\alpha_0 = P_0 = \tilde{P}_0$ and $\alpha_s = \tilde{P}_s$, then

$$h(\tilde{P}_T) = v(\tilde{P}_0, 0) + \int_0^T dv(\tilde{P}_s, s). \tag{3.4}$$

Substituting $v(P_0, 0)$ for $v(\tilde{P}_0, 0)$, and combining (3.3) and (3.4) we obtain

$$v(P_0, 0) + \int_0^T \theta_s \, d\tilde{P}_s = h(\tilde{P}_T) + \int_0^T v'(\tilde{P}_s, s) \, d\tilde{P}_s - \int_0^T dv(\tilde{P}_s, s).$$

Moreover, by Itô's formula, and then using (3.2),

$$\begin{aligned} dv(\tilde{P}_s, s) &= v'(\tilde{P}_s, s) \, d\tilde{P}_s + \tfrac{1}{2} v''(\tilde{P}_s, s)(d\tilde{P}_s)^2 + \dot{v}(\tilde{P}_s, s) \, ds \\ &= v'(\tilde{P}_s, s) \, d\tilde{P}_s + \left[\tfrac{1}{2} \tilde{P}_s^2 v''(\tilde{P}_s, s) \tilde{\sigma}(\tilde{P}_s, s)^2 + \dot{v}(\tilde{P}_s, s)\right] ds \\ &= v'(\tilde{P}_s, s) \, d\tilde{P}_s + \left[\tfrac{1}{2} \tilde{P}_s^2 v''(\tilde{P}_s, s)\{\tilde{\sigma}(\tilde{P}_s, s)^2 - \sigma(\tilde{P}_s, s)^2\}\right] ds. \end{aligned}$$

Thus

$$\begin{aligned} &v(P_0, 0) + \int_0^T \theta_s \, d\tilde{P}_s \\ &\quad = h(\tilde{P}_T) + \int_0^T \tfrac{1}{2} \tilde{P}_s^2 v''(\tilde{P}_s, s)\{\sigma(\tilde{P}_s, s)^2 - \tilde{\sigma}(\tilde{P}_s, s)^2\} \, ds \end{aligned} \tag{3.5}$$

and under our volatility comparison assumption

$$v(P_0, 0) + \int_0^T \theta_s \, d\tilde{P}_s \geq h(\tilde{P}_T). \tag{3.6}$$ □

The quantity $e_t = \int_0^t \frac{1}{2} \tilde{P}_s^2 v''(\tilde{P}_s, s)\{\sigma(\tilde{P}_s, s)^2 - \tilde{\sigma}(\tilde{P}_s, s)^2\} \, ds$ was labelled the tracking error [7]. Superreplication is equivalent to $e_T \geq 0$, almost surely.

COROLLARY 3.3. *The model option price is greater than the fair price in the true world.*

ASSUMPTIONS. We need that the local martingale $\int_0^t \theta_s \, d\tilde{P}_s$ is a true martingale. If h has bounded derivative then by Remark 3.3 we may restrict attention to the case of bounded integrands. Assume that

$$\mathbb{E}\left[\left(\int_0^T \tilde{P}_s^2 \tilde{\sigma}(\tilde{P}_s, s)^2 \, ds\right)^{1/2}\right] < \infty,$$

so that $\tilde{P}$ is an element of the Hardy space H^1 of martingales, then by the Burkholder–Davis–Gundy inequalities ([23], Theorem IV.42.1), the stochastic integral is a true martingale. Exercise 4.22 in [22] shows that at this level of generality we cannot hope to do better.

The proof follows on taking expectations in (3.6).

REMARK 3.4. Clearly the proof can be applied in reverse in the sense that if the option has been bought for a price based on a model with volatility σ which is dominated by the true volatility, then by using the model hedge, the option purchaser can guarantee that his income (the option payoff), dominates his obligations (the sum of the initial premium and the losses from his model hedging strategy).

REMARK 3.5. Although it is crucial that the option writer's model be a diffusion process so that we can apply the convexity result, there is no requirement that the true process is a diffusion, merely that at all times s, and for all price histories $\{p_u\}_{0\le u\le s}$ we have that $\sigma(p_s, s) \ge \tilde{\sigma}(\{p_u\}_{0\le u\le s}, s)$.

REMARK 3.6. In Remark 2.1 we gave a simple condition (2.3) for option price monotonicity. Note that even the the stronger condition

$$\int_0^s \sigma(t)^2\,dt \ge \int_0^s \tilde{\sigma}(t)^2\,dt \quad \forall s \in [0, T]$$

is not sufficient to guarantee the superreplication property.

4. American options via coupling. We next turn our attention to American contingent claims. We assume that the claim is an entitlement to receive $h(P_t)$ at time t for some time-independent payoff profile h where the (stopping) time t is at the discretion of the option holder.

With American contingent claims, since the timing of payments can vary, it is not possible to completely remove discount factors from the analysis by a change of numeraire. For notational simplicity, we assume a fixed and constant rate of interest $r \ge 0$, though the extension to deterministic interest rates is easy.

Suppose that the market is complete and that (under the risk-neutral measure) the price process solves the SDE $dP_s = P_s\sigma_P(P_s, s)\,dB_s + rP_s\,ds$. Let Q_s denote the discounted price process $Q_s = e^{-rs}P_s$; then Q solves the SDE

$$dQ_s = Q_s\sigma(Q_s, s)\,dB_s, \tag{4.1}$$

where $\sigma(Q_s, s) \equiv \sigma_P(e^{rs}Q_s, s)$. Discounted into time 0 prices, the payoff of the option, if exercised at time t, is $e^{-rt}h(P_t) \equiv e^{-rt}h(e^{rt}Q_t)$.

THEOREM 4.1. *For a* (*complete*) *diffusion model of the asset price and an American option with convex payoff h, the option value at intermediate times is convex in the price of the underlying.*

ASSUMPTIONS. We assume that σ_P has sufficient continuity properties to ensure that the solution to the SDE for the price process [or equivalently (4.1)] is unique in law, that Q is a true martingale, and that for all stopping times ρ, $\mathbb{E}(|h(P_\rho)|) < \infty$. By analogy with the assumptions of Theorem 3.1, sufficient conditions are that σ_P is bounded, $p\sigma_P(p, s)$ is Lipschitz and h has bounded derivative.

PROOF. Bensoussan [3], Karatzas [14, 15] and Myneni [20] have shown that the price v of an American option is given by a Snell envelope:

$$v(q, s) = \operatorname*{ess\,sup}_{\rho\in\Gamma(s)} e^{-r\rho}\mathbb{E}[h(e^{r\rho}Q_\rho)|Q_s = q],$$

where $\Gamma(s)$ is the set of all stopping times which take values in the interval $[s, T]$. The optimal stopping time or exercise time is $E = \inf\{u\colon v(Q_s, s) =$

$e^{-rs}h(e^{rs}Q_s)\}$. In general $v(Q_s, s)$ is a supermartingale; it is a martingale on the continuation region $v(q, s) > e^{-rs}h(e^{rs}q)$.

The set-up. As in the proof of Theorem 3.1, let $0 < z < y < x$ and for independent Brownian motions α, β, γ define discounted price processes X, Y, Z via

$$dX_s = X_s\sigma(X_s, s)\,d\alpha_s, \qquad X_0 = x,$$

$$dY_s = Y_s\sigma(Y_s, s)\,d\beta_s, \qquad Y_0 = y,$$

$$dZ_s = Z_s\sigma(Z_s, s)\,d\gamma_s, \qquad Z_0 = z.$$

The "coupling." Let $E \equiv E(y)$ be the optimal exercise time for the process started at y. Define $\tau = H_x \wedge H_z \wedge E$ where as before $H_x \equiv \inf_u\{X_u = Y_u\}$ and $H_z \equiv \inf_u\{Y_u = Z_u\}$.

On $\tau = E$, we have that $e^{-r\tau}h(e^{r\tau}Y_\tau) \equiv v(Y_\tau, \tau)$ and, by convexity,

$$\begin{aligned}(X_\tau - Z_\tau)v(Y_\tau, \tau) &\le (Y_\tau - Z_\tau)e^{-r\tau}h(e^{r\tau}X_\tau) + (X_\tau - Y_\tau)e^{-r\tau}h(e^{r\tau}Z_\tau)\\ &\le (Y_\tau - Z_\tau)v(X_\tau, \tau) + (X_\tau - Y_\tau)v(Z_\tau, \tau).\end{aligned}$$

By symmetry, on $\tau = H_x$,

$$(X_\tau - Z_\tau)v(Y_\tau, \tau) = (Y_\tau - Z_\tau)v(X_\tau, \tau)$$

and $0 = (X_\tau - Y_\tau)v(Z_\tau, \tau)$. By playing a similar trick on the set $\tau = H_z$, we get that always

$$(X_\tau - Z_\tau)v(Y_\tau, \tau) \le (Y_\tau - Z_\tau)v(X_\tau, \tau) + (X_\tau - Y_\tau)v(Z_\tau, \tau).$$

Now taking expectations, and using independence,

$$(x - z)\mathbb{E}[v(Y_\tau, \tau)] \le (y - z)\mathbb{E}[v(X_\tau, \tau)] + (x - y)\mathbb{E}[v(Z_\tau, \tau)].$$

But $v(Y_t, t)$ is a martingale on $t \le \tau$, so $\mathbb{E}[v(Y_\tau, \tau)] = v(y, 0)$ and $v(X_t, t)$ and $v(Z_t, t)$ are supermartingales so that, for example, $\mathbb{E}[v(X_\tau, \tau)] \le v(x, 0)$. The convexity property for $v(\cdot, 0)$ now follows. □

COROLLARY 4.2. *Option price monotonicity and the superreplication property hold for American options.*

PROOF. Subject to minor modifications, the proof follows Theorem 3.2. The main change involves considering the gains from trade-up to an intermediate time t. We have that

$$e^{-rt}h(P_t) \le v(Q_t, t) = v(Q_0, 0) + \int_0^t v'(Q_s, s)\,dQ_s.$$

Moreover, assuming that Itô's formula applies, we have by the supermartingale property that $\frac{1}{2}q^2\sigma^2 v'' + \dot{v} \le 0$. In conclusion,

$$v(Q_0, 0) + \int_0^t \theta_s \, d\tilde{Q}_s \ge v(\tilde{Q}_t, t) + \tfrac{1}{2}\int_0^t \tilde{Q}_s^2 v''(\tilde{Q}_s, s)[\sigma(\tilde{Q}_s, s)^2 - \tilde{\sigma}(\tilde{Q}_s, s)^2]\, ds$$
$$\ge e^{-rt} h(\tilde{P}_t),$$

provided that $\sigma(q, s) \ge \tilde{\sigma}(q, s)$ everywhere. In order to deduce option price monotonicity we need that $\int_0^t \theta_s \, d\tilde{Q}_s$ is a true martingale; again if h has bounded derivative, then θ is bounded and a sufficient condition is $\mathbb{E}[(\int_0^T \tilde{Q}_s^2 \tilde{\sigma}(\tilde{Q}_s, s)^2)^{1/2}] < \infty$. □

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