Convergence almost everywhere of certain singular integrals and multiple Fourier series

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Introduction

We shall here study certain several variable analogues of the operator M^* defined by

$$M^*f(x) = \sup_n \left| \int_{-\pi}^{\pi} \frac{e^{-int}f(t)}{x-t}\,dt \right|, \quad |x| \leq \pi \,,$$

and the Fourier series maximal operator treated by Carleson [4] and Hunt [7].

Let $\mathbf{R}^s$ be the Euclidean space of dimension s and let $T_s = \{x = (x_1, \ldots, x_s) \in \mathbf{R}^s;\ 0 \leq x_i \leq 2\pi,\ i = 1, 2, \ldots, s\}$. If $x = (x_1, \ldots, x_s)$ and $\xi = (\xi_1, \ldots, \xi_s)$ belong to $\mathbf{R}^s$ we set $x \cdot \xi = \sum_{i=1}^{s} x_i\xi_i$ and $|x| = \left(\sum_{i=1}^{s} x_i^2\right)^{1/2}$.

L. Hörmander has observed that the first part of the proof in [4] can be generalized to yield the following (unpublished) result.

If k is a C^∞ Calderón — Zygmund kernel defined in $\mathbf{R}^s$, $s \geq 2$, and if $\int_{T_s} |f(x)|\,(\log^+ |f(x)|)^{1+\delta}\,dx < \infty$ for some $\delta > 0$, then $\left|\int_{T_s} k(x-t)e^{-i\xi\cdot t}f(t)dt\right| = o(\log\log|\xi|)$, $|\xi| \to \infty$, for almost every x in T_s.

In Sections 1 to 3 in this paper we prove among other things the following theorem, which generalizes the L^p estimate of the operator M^* in [7].

THEOREM. *Assume that k is a Calderón — Zygmund kernel defined in $\mathbf{R}^s$, $s \geq 2$, which has continuous derivatives of order $\leq s+1$ outside the origin. Let the operator M be defined by*

$$Mf(x) = \sup_{\xi\in\mathbf{R}^s} \left| \int_{T_s} k(x-t)\, e^{-i\xi\cdot t} f(t)dt \right|, \quad x \in T_s \,.$$

Then $\|Mf\|_p \leq C_p\|f\|_p$, $1 < p < \infty$, *where the norms are* L^p *norms taken with respect to* T_s *and* C_p *is a constant depending only on* s, k *and* p.

To prove the above theorem the second part of Carleson's proof in the version of Hunt is used. Most of the steps in the proof can easily be carried over to the case of several variables, but we need a new method to get the estimate required for the analogue of the »change of pairs» in [4] and [7]. This method is described in Section 1 and the details are carried out in Section 3.

In Section 4 we prove that the above theorem holds also in the case when k is odd but without smoothness.

Section 5 contains extensions and applications of the results mentioned above. In Sections 5 and 6 the following convergence result is proved.

THEOREM. *Let* μ *be a bounded Borel measure in* $\mathbf{R}^s$ *and assume that* μ *has no point mass at the origin. Let* k *be a Calderón — Zygmund kernel with the property* that $k \log^+ |k|$ *is integrable over the unit sphere and let its Fourier transform* $\hat{k}$ *satisfy* $\int_{\mathbf{R}^s} \hat{k}(x)\, d\mu(x) = 1$. *Define* K *and* K_R *by*

$$K(x) = \hat{\mu}(x)\, k(x) \quad \textit{and} \quad K_R(x) = R^s K(Rx)\ , \quad R > 0\ .$$

Then $\lim_{R\to\infty} \int_{\mathbf{R}^s} K_R(x-t)\, f(t) dt = f(x)$ *for almost every* x *in* $\mathbf{R}^s$ *if* $f \in L^p(\mathbf{R}^s)$ *for some* p *with* $1 < p < \infty$. *(Here as always the integral is taken in the principal value sense.)*

In Section 6 we consider the square partial sums $S_n f(x) = \sum_{|k_i| \leq n} c_k e^{ik\cdot x}$ of the Fourier series of a function $f \in L^1(T_s)$ with Fourier coefficients c_k, $k \in Z^s$. We prove that if $f \in L^p(T_s)$ for some $p > 1$, then $\lim_{n\to\infty} S_n f(x) = f(x)$ for almost all x. This result has been obtained simultaneously and independently by C. Fefferman [5], who uses a method different from ours. In the case $s = 2$, $p = 2$, the convergence has also been proved by N. R. Tevzadze [16]. Fefferman's proof can unlike ours be modified to handle other types of convergence than the one just described. Our method, however, gives a stronger result than Fefferman's when we extend the above convergence result to classes of functions close to $L^1(T_s)$. More precisely we can prove that $f \in L\,(\log L)^s \log\log L$ is a sufficient condition for the convergence almost everywhere of the square partial sums.

In Section 7 we estimate the rectangular partial sums $S_{mn} f(x\,,y) = \sum_{-m}^{m}\sum_{-n}^{n} c_{kl} e^{i(kx+ly)}$ of the Fourier series of a function $f \in L^1(T_2)$. We prove that if $f \in L^p(T_2)$ for some $p > 1$, then $S_{mn} f(x\,,y) = o(\log \min\,(m\,,n))$, $m\,,n \to \infty$, almost everywhere. Fefferman [6] has constructed a counter-example, which shows that this result is best possible in the sense that $\log \min\,(m\,,n)$ can not be replaced by

$\varepsilon_{mn} \log \max (m, n)$ for any double sequence $\{\varepsilon_{mn}\}$ with $\lim\limits_{m,n\to\infty} \varepsilon_{mn} = 0$. The above estimate is used to prove that if the coefficients of a function f satisfy $\sum\limits_{m,n} |c_{mn}|^2 (\log \min (|m| + 2, |n| + 2))^2 < \infty$, then $\lim\limits_{m,n\to\infty} S_{mn} f(x, y) = f(x, y)$ for almost all (x, y). This result improves a theorem of Kaczmarz [9].

In Section 8 finally we point out a connection between the operator M^* and Bochner — Riesz summability of critical index of multiple Fourier series in odd dimensions.

1. A theorem on maximal singular integrals

Let s be an integer not less than 2 and let $\mathscr{A}$ denote the class of all complex-valued functions k defined in $\mathbf{R}^s \setminus \{0\}$, which satisfy the following three conditions k is positively homogeneous of degree $-s$, i.e. $k(\lambda x) = \lambda^{-s} k(x)$ for

$$\lambda > 0, \quad x \in \mathbf{R}^s \setminus \{0\} \tag{1.1}$$

$\int\limits_S k(x)\, d\sigma(x) = 0$, where S is the unit sphere in $\mathbf{R}^s$ and $d\sigma$ is the surface element on S (1.2)

$$k \in C^{s+1}(\mathbf{R}^s \setminus \{0\}) \tag{1.3}$$

Operators defined by convolution with kernels of this type are a subclass of the operators studied by Calderón and Zygmund in [1]. We notice that the kernels in $\mathscr{A}$ can be written in the form $k(x) = \Omega(x)|x|^{-s}$, $x \neq 0$, where Ω is homogeneous of degree zero.

For a fixed kernel k in $\mathscr{A}$ and $f \in L^1(T_s)$ define the operator M by

$$Mf(x) = \sup_{\xi \in \mathbf{R}^s} \left| \int\limits_{T_s} k(x - t)\, e^{-i\xi\cdot t} f(t) dt \right|, \quad x \in T_s, \tag{1.4}$$

where the integral is taken in the principal value sense. It is not difficult to prove that for almost every $x \in T_s$

$$\lim_{\varepsilon \to 0} \int\limits_{T_s \setminus \{t\,;\, |x-t| < \varepsilon\}} k(x - t)\, e^{-i\xi\cdot t} f(t) dt$$

exists for every $\xi \in \mathbf{R}^s$ and by first taking the sup in (1.4) over a countable set of ξ:s we can also prove that Mf is measurable. For M we will give the following estimates, in which $\| \|_p$ denotes the norm in $L^p(T_s)$ and m denotes s-dimensional Lebesgue measure.

THEOREM 1.1.

(A) *If* $\int_{T_s} |f(x)| \log^+ |f(x)| \log^+ \log^+ |f(x)| dx < \infty$, *then* $Mf(x)$ *is finite a.e. in* T_s.

(B) $\|Mf\|_1 \leq \text{Const.} \int_{T_s} |f(x)| (\log^+ |f(x)|)^2 \, dx + \text{Const.}$

(C) $\|Mf\|_p \leq C_p \|f\|_p$, $1 < p < \infty$.

(D) $m\{x \in T_s ; Mf(x) > y\} \leq \text{Const.} \exp(-\text{Const.}\, y / \|f\|_\infty)$, $y > 0$.

Here and in Sections 2—5 by Const. we mean a number depending only on the dimension s and the kernel k and C_p denotes a number depending only on s, k and p.

In Section 3 will be given the proof of the following basic result.

LEMMA 1.2. *If* χ_F *is the characteristic function of a measurable set* $F \subset T_s$, *then*

$$m\{x \in T_s ; M\chi_F(x) > y\} \leq \begin{cases} \text{Const.}\, \frac{1}{y} \log \frac{1}{y}\, mF\,, & \text{if } 0 < y \leq \frac{1}{2} \\ \text{Const.} \exp\{-\text{Const.}\, y\}\, mF\,, & \text{if } y > \frac{1}{2} \end{cases}$$

(A), (B) and (D) follow from Lemma 1.2 in essentially the same way as in the proof of (A), (B) and (D) in [11]. (See [11], pp. 551—552 and 563—570.) That M is a bounded operator on $L^p(T_s)$, $1 < p < \infty$, is most easily proved by use of the interpolation theorem of Stein and Weiss. (See [15], p. 264.)

The proof of Lemma 1.2 is modelled on the proof of the estimate of $m\{M^*\chi_F(x) > y\}$ in [7]. The proof in the case of several variables is different from the one dimensional proof at some points. The greatest difficulty in our case lies in the proof of the inequality needed for the change of pairs (Lemma 3.3), in which we estimate an expression of the form

$$\int_\omega k(x-t)\,(1 - e^{i\xi\cdot(x-t)})\, h(t)dt\,,$$

where ω is a cube in $\mathbf{R}^s$, $x \in \omega$ and $h \in L^1(\omega)$. In the one variable case $k(t) = t^{-1}$ and $k(t)(1 - e^{i\xi\cdot t}) = t^{-1}(1 - e^{i\xi t})$ is a C^∞ function, which makes the estimate of the above integral easy. (See [7], p. 252.) In the case when k is a Calderón — Zygmund kernel the function $k(t)(1 - e^{i\xi\cdot t})$ has a singularity at the origin and we need a new technique to get the desired estimate. For simplicity assume that $\omega = T_s$. We proceed in the following way. Let φ be a non-negative function in $C^\infty(\mathbf{R}^s)$ with compact support and let $\varphi(t) = 1$ for $|t| \leq 2\pi\sqrt{s}$. Define K by $K(t) = k(t)(1 - e^{i\xi\cdot t})\,\varphi(t)$, $t \in \mathbf{R}^s$. Let H be the function in $L^1(R^s)$ with Fourier transform $\hat{H}(u) = (1 - |u|^2)^{(s+1)/2}$, $|u| \leq 1$, and $\hat{H}(u) = 0$, $|u| > 1$. Define H_R by $H_R(t) = R^s H(Rt)$. The above integral equals the sum of

$\int_{T_s} K*H_R(x-t)\,h(t)dt$ and $\int_{T_s} (K(x-t) - K*H_R(x-t))\,h(t)dt$. We estimate the first integral by use of the Fourier coefficients of h and the second in terms of the Hardy — Littlewood maximal function of h. We then choose R so that the majorants of both the integrals are small. For further details see Lemma 3.3.

2. Notation

Let Z denote the integers, Z_+ the non-negative integers and let B be the set of all cubes $\omega = \{x \in \mathbf{R}^s\,; r_i 2\pi \cdot 2^{-\nu} < x_i < (r_i+1)2\pi \cdot 2^{-\nu},\ i = 1, 2, \ldots, s\}$, for which r_i and $\nu \geq 0$ are integers and $\omega \subset (-4\pi, 4\pi)^s = \{x \in \mathbf{R}^s\,; |x_i| < 4\pi,\ i = 1, 2, \ldots, s\}$. We define B^* to be the set of all cubes which can be written as above, but for which we replace the condition that r_i shall be an integer with the condition that $2r_i$ is an integer. We let $\delta(\omega)$ denote the side length of a cube ω and let b_k denote the number 2^{-k}, $k = 0, 1, 2, \ldots$ For $\alpha \in \mathbf{R}^s$, $\omega \in B$, $\delta(\omega) = 2\pi \cdot 2^{-\nu}$, and $f \in L^1(\omega)$ set $c_\alpha(\omega) = c_\alpha(\omega, f) = (m\omega)^{-1} \int_\omega f(x) \exp(-i2^\nu \alpha \cdot x)dx$ and $C_\alpha(\omega) = C_\alpha(\omega, f) = C' \sum_{\mu \in Z^s} |c_{\alpha+\mu/3}(\omega)|(1+|\mu|)^{-s-1}$, where $C' = \{\sum_{\mu \in Z^s} (1+|\mu|)^{-s-1}\}^{-1}$.

If $\omega^* \in B^*$, $n \in Z^s$ and $f \in L^1(\omega^*)$ we define $C_n^*(\omega^*) = \max_{\omega'} C_n(\omega')$, where ω' ranges over the 4^s subcubes of ω^* with $4\delta(\omega') = \delta(\omega^*)$. If ξ is a non-negative real number, $\xi = \sum_{i=-\infty}^{N} \varepsilon_i 2^i$ (where ε_i equals 0 or 1 and we choose ε_i so that $\varliminf_{i \to -\infty} \varepsilon_i = 0$), and $\omega \in B$ with $\delta(\omega) = 2\pi \cdot 2^{-\nu}$, we define $\xi[\omega] = \sum_{i \geq \nu}^{N} \varepsilon_i 2^{i-\nu}$. If $\xi = (\xi_1, \ldots, \xi_s) \in \mathbf{R}_+^s$, i.e. $\xi \in \mathbf{R}^s$ and its components are non-negative, we define $\xi[\omega] = (\xi_1[\omega], \ldots, \xi_s[\omega])$. For $\omega^* \in B^*$ by an abuse of notation we also set $\xi[\omega^*] = \xi[\omega']$, where $4\delta(\omega') = \delta(\omega^*)$. If $\delta(\omega^*) = 4 \cdot 2\pi \cdot 2^{-\nu}$ let $Z_+^s(\omega^*) = \{n \in Z_+^s;\ n[\omega^*] = 2^{-\nu}n\}$. If ω is a cube and $\lambda > 0$ let $\lambda\omega$ denote the cube with the same center as ω and side length $\delta(\lambda\omega) = \lambda\delta(\omega)$.

If K is a function defined in $\mathbf{R}^s$ and R a positive number define K_R by $K_R(t) = R^s K(Rt)$, $t \in \mathbf{R}^s$. Finally let

$$S_\xi(x, \omega^*) = S_\xi(x, \omega^*, f) = \int_{\omega^*} k(x-t)e^{-i\xi \cdot t} f(t)dt\,,\ \ f \in L^1(\omega^*)\,,$$

where the integral is taken in the principal value sense.

3. Proof of the basic result

To shorten the proof of Lemma 1.2 we will assume that the reader is familiar with [7] and mention only the points where there are differences between the two proofs. For instance in the definitions of the polynomials $P_k(x, \omega)$ and the partitions $\Omega((n[\omega^*], \omega^*), k)$ we split cubes into 2^s subcubes instead of splitting intervals into two subintervals. In the definition of $\tilde{G}_k$ $|\omega|$ is replaced by $\delta(\omega)$ and $|n - \lambda[\omega]|$ by $\max_{1 \leq i \leq s} |n_i - \lambda_i[\omega]|$. f equals the characteristic function χ_F of a measurable set F included in T_s.

The following lemma is needed.

LEMMA 3.1. *Let* $\omega \in B$, $\delta(\omega) = 2\pi \cdot 2^{-\nu}$, *and let* $\varphi \in C^{s+1}(\overline{\omega})$. *Then there is a representation of* φ

$$\varphi(t) = \sum_{\mu} \gamma_\mu e^{-i3^{-1}2^\nu \mu \cdot t}, \quad t \in \omega,$$

with

$$|\gamma_\mu| \leq \text{Const.}\ (1 + |\mu|)^{-s-1} \sum_{\|\alpha\| \leq s+1} \sup_{\omega} |D^\alpha \varphi| 2^{-\nu\|\alpha\|}, \quad \mu \in Z^s, \tag{3.1}$$

where $\|\alpha\|$ *denotes* $\sum_1^s \alpha_i$.

Proof. φ can be extended to a function in $3\overline{\omega}$ with Fourier coefficients satisfying (3.1). (For the extension see e.g. [14], ch. VI, p. 18.)

We now describe how the remainder terms can be estimated. Assume that the pair $p^* = (n[\omega^*], \omega^*)$ satisfies the condition

$$\Omega(k): p^* \in G^*_{kL}, \quad C^*(p^*) < b_{k-1}y \quad \text{and} \quad n \in Z^s_+(\omega^*)$$

and that the partition $\Omega(p^*, k)$ has been constructed. For $x \in \frac{1}{2}\omega^*$ $\omega^*(x)$ is defined to be one of the cubes in the set $\{\tilde{\omega} \in B^*; x \in \frac{1}{2}\tilde{\omega}$ and there exists $\omega \in \Omega(p^*, k)$ such that $\omega \subset \tilde{\omega}$ and $\delta(\tilde{\omega}) = 2\delta(\omega)\}$ which has maximal side length. We need the following lemma, which is essentially due to Hörmander.

LEMMA 3.2. *There exists a set* $T^*(p^*) \subset \omega^*$ *with*
$mT^*(p^*) \leq \text{Const.} \exp\{-\text{Const.}\ Lk\} m\omega^*$, *such that*
$x \in \frac{1}{2}\omega^*$, $x \notin T^*(p^*)$ *implies*
$|S_\xi(x, \omega_0^*) - S_\xi(x, \omega^*(x))| \leq \text{Const.}\ Lkb_{k-1}y$
for all $\xi \in \mathbf{R}^s_+$ *with* $\xi[\omega^*] = n[\omega^*]$ *and all* $\omega_0^* \in B^*$ *such that* $x \in \frac{1}{2}\omega_0^*$ *and* $\omega^*(x) \subset \omega_0^* \subset \omega^*$.

Proof. For $t \in \omega \in \Omega(p^*, k)$, $\delta(\omega) = 2\pi \cdot 2^{-\nu}$, set $E_\xi(t) = c_{2-\nu_\xi}(\omega)$ and $M_\xi(t) = C_{2-\nu_\xi}(\omega)$. From the construction of $\Omega(p^*, k)$ it follows that for $t \in \omega$ and $\xi[\omega^*] = n[\omega^*]$ we have

$$|E_\xi(t)| \leq \text{Const.}\ M_\xi(t) \leq \text{Const.}\ C_{n[\omega]}(\omega) \leq \text{Const.}\ b_{k-1}y.$$

Set

$$S_\xi(x, \omega_0^*) - S_\xi(x, \omega^*(x)) = \int_{\omega_0^* \setminus \omega^*(x)} k(x-t)e^{-i\xi\cdot t}f(t)dt = A(x) + B(x) + C(x),$$

where

$$A(x) = \int_{\omega_0^* \setminus \omega^*(x)} k(x-t)e^{-i(\xi-n)\cdot t}E_n(t)dt,$$

$$B(x) = \int_{\omega_0^* \setminus \omega^*(x)} k(x-t)\,(E_\xi(t) - e^{-i(\xi-n)\cdot t}E_n(t))dt \quad \text{and}$$

$$C(x) = \int_{\omega_0^* \setminus \omega^*(x)} k(x-t)\,(e^{-i\xi\cdot t}f(t) - E_\xi(t))dt.$$

First consider $A(x)$. From the estimate

$$|k(x-t)(e^{i(\xi-n)\cdot(x-t)} - 1)| \leq \text{Const. } \delta(\omega^*)^{-1}|x-t|^{-s+1}, \quad \xi[\omega^*] = n[\omega^*],$$

it follows that

$$|e^{i(\xi-n)\cdot x}A(x) - \int_{\omega_0^* \setminus \omega^*(x)} k(x-t)E_n(t)| \leq \text{Const. } E_n^*(x),$$

where E_n^* is the Hardy — Littlewood maximal function of E_n. If $\tilde{g}$ is defined for $g \in L^1(\mathbf{R}^s)$ by

$$\tilde{g}(x) = \sup_{\varepsilon>0} \left| \int_{|x-t|>\varepsilon} k(x-t)g(t)dt \right|, \quad x \in \mathbf{R}^s,$$

the above estimate yields $|A(x)| \leq \text{Const. } (\tilde{E}_n(x) + E_n^*(x))$.

To estimate $B(x)$ we observe that for $t \in \omega \in \Omega(p^*, k)$

$$E_\xi(t) - e^{-i(\xi-n)\cdot t}E_n(t) = e^{-i(\xi-n)\cdot t}(m\omega)^{-1}\int_\omega e^{-in\cdot u}f(u)\,(e^{i(\xi-n)\cdot(t-u)} - 1)du.$$

Applying Lemma 3.1 to the function $\varphi(u) = e^{i(\xi-n)\cdot(t-u)} - 1$, $u \in \omega$, we get $|E_\xi(t) - e^{-i(\xi-n)\cdot t}E_n(t)| \leq \text{Const. } \delta(\omega)\delta(\omega^*)^{-1}M_n(t)$ if $t \in \omega$ and $\xi[\omega^*] = n[\omega^*]$. From this estimate it follows that

$$\left|\int_\omega k(x-t)\,(E_\xi(t) - e^{-i(\xi-n)\cdot t}E_n(t))dt\right| \leq \text{Const. } \delta(\omega^*)^{-1}\int_\omega |x-t|^{-s+1}M_n(t)dt$$

and we conclude that $|B(x)| \leq \text{Const. } M_n^*(x)$.

It remains to estimate $C(x)$. Suppose $\omega \in \Omega(p^*, k)$ and that ω is not contained in $\omega^*(x)$. For $u \in \mathbf{R}^s$ set $L(u) = (1 + |u|)^{-s-1}$. Denoting the center of ω by t_0 we use Lemma 3.1 to prove that

$$k(x-t) - k(x-t_0) = \sum_{\mu} \gamma_\mu e^{-i3^{-1}2^\nu \mu \cdot t}, \quad t \in \omega,$$

where $|\gamma_\mu| \leq \text{Const.}\ (1 + |\mu|)^{-s-1} L_{\delta(\omega)^{-1}}(x - t_0)$. Using this representation we obtain

$$\left| \int_\omega k(x-t)\,(e^{-i\xi\cdot t} f(t) - E_\xi(t))dt \right| \leq$$

$$\leq \left| \int_\omega (k(x-t) - k(x-t_0))e^{-i\xi\cdot t} f(t)dt \right| + \left| \int_\omega (k(x-t) - k(x-t_0))E_\xi(t)dt \right| \leq$$

$$\leq \text{Const.} \int_\omega L_{\delta(\omega)^{-1}}\,(x-t) M_n(t)dt\,.$$

Defining $\delta(t)$ by $\delta(t) = \delta(\omega)$, $t \in \omega \in \Omega(p^*, k)$, and setting

$$\bar{M}_n(x) = \int_{\omega^*} L_{\delta(t)^{-1}}\,(x-t) M_n(t)dt \quad \text{we have} \quad |C(x)| \leq \text{Const.}\ \bar{M}_n(x)\,.$$

Collecting the estimates of A, B and C we get

$$\left| \int_{\omega_0^* \setminus \omega^*(x)} k(x-t) e^{-i\xi\cdot t} f(t)dt \right| \leq \text{Const.}\ (\tilde{E}_n(x) + M_n^*(x) + \bar{M}_n(x))$$

for all ξ and ω_0^* with the properties in the statement of the lemma.

$\bar{M}_n$ can be estimated by use of the fact that the adjoint of the operator $g \to \bar{g}$ can be majorized by the Hardy — Littlewood maximal function. (See e.g. [17], pp. 253—255.) For $\tilde{E}_n$ and M_n^* well-known estimates hold and we obtain

$$m\{x \in \omega^*\,; \tilde{E}_n(x) + M_n^*(x) + \bar{M}_n(x) > \lambda\} \leq \text{Const.}\ \exp\{-\text{Const.}\ \lambda/\|M_n\|_\infty\} m\omega^*\,, \quad \lambda > 0\,.$$

The lemma now follows if we choose $\lambda = \text{Const.}\ Lk\, b_{k-1} y$.

The reason for proving the above lemma for all ξ with $\xi[\omega^*] = n[\omega^*]$ is that the several variable analogue of the Lemma 3.4 in [7] fails.

We will now describe how the estimate necessary for the change of pairs (cf. [7], Lemma (10.2)) can be obtained. Also in this case it is the lack of an analogue of Lemma 3.4 in [7] that makes the proof more complicated. What is needed is the following lemma.

LEMMA 3.3. *Suppose that* $\omega_0^* \in B^*$, $x \in \frac{1}{2}\omega_0^*$, ξ_0 *and* $\xi \in \mathbf{R}_+^s$ *and* $|\xi_0 - \xi| \leq A\delta(\omega_0^*)^{-1}$ *where* A *is a constant. Assume that there exist a complex number* ϱ *and* $\tilde{n} \in Z^s$ *such that* $C_\alpha^*(\omega_0^*, f - \varrho e^{i\tilde{n}\cdot t}) \leq \varepsilon$ *if* $|\xi_0[\omega_0^*] - \alpha| \leq C$, *for some constants* ε *and* C. *Then for all* R *with* $0 < R \leq C$

$$\big|\,|S_{\xi_0}(x, \omega_0^*)| - |S_\xi(x, \omega_0^*)|\,\big| \leq \text{Const.}\ \{|\varrho| + AR^s\varepsilon + AR^{-1/2}(f^*(x) + |\varrho|)\}\,.$$

We use the above lemma in the proof of the analogue of Lemma (10.2) in [7]. We then have (for the definitions of $\tilde{p}_0^*$ and $\tilde{n}$ see [7])

$$A = \text{Const.}\, b_k^{-2}\,, \quad \varepsilon = \text{Const.}\, b_{kL}^{1/2}\, y\,, \quad C = b_{kL}^{-9}\,, \quad f^*(x) \le \text{Const.}\, y$$
$$\text{and} \quad |\varrho| \le \text{Const.}\, (C^*(\tilde{p}_0^*) + b_{kL}^{1/2} y) \le \text{Const.}\, y\,.$$

Lemma 3.3. yields

$$||S_{\xi_0}(x, \omega_0^*)| - |S_\xi(x, \omega_0^*)|| \le \text{Const.}\, (C^*(\tilde{p}_0^*) + b_k y + b_k^{-2} R^s b_{kL}^{1/2} y + b_k^{-2} R^{-1/2} y)\,.$$

and choosing $R = b_k^{-6}$ we obtain

$$||S_{\xi_0}(x, \omega_0^*)| - |S_\xi(x, \omega_0^*)|| \le \text{Const.}\, (C^*(\tilde{p}_0^*) + b_k y)\,,$$

which is the desired estimate.

Proof of Lemma 3.3. Defining g by $g(t) = f(t) - \varrho e^{i\tilde{n}\cdot t}$ we get

$$||S_{\xi_0}(x, \omega_0^*)| - |S_\xi(x, \omega_0^*)|| \le \left| \int_{\omega_0^*} k(x-t)\,(1 - e^{i(\xi-\xi_0)\cdot(x-t)}) e^{-i\xi_0\cdot t} g(t) dt \right| +$$
$$+ 2|\varrho| \sup_{\eta, \omega} \left| \int_\omega k(t) e^{i\eta\cdot t} dt \right|,$$

where the sup is taken over all $\eta \in \mathbf{R}^s$ and all cubes ω with $0 \in \frac{1}{2}\omega$. The last term is not greater than

$$\text{Const.}\, |\varrho| \left(1 + \sup_{\substack{r>0 \\ \eta\in\mathbf{R}^s}} \left| \int_{|t|\le r} k(t) e^{i\eta\cdot t} dt \right|\right).$$

According to [1], pp. 89—90, $\int_{|t|\le r} k(t) e^{i\eta\cdot t}\, dt$ is uniformly bounded in r and η and hence the last term in the above inequality is majorized by Const. $|\varrho|$.

We introduce some auxiliary functions. Let $\varphi \in C^\infty(\mathbf{R}^s)$ vanish for $|t| \ge 300\sqrt{s}$, be equal to 1 for $|t| \le 200\sqrt{s}$ and satisfy $0 \le \varphi(t) \le 1$ for all $t \in \mathbf{R}^s$. Assume $\delta(\omega_0^*) = 2\pi \cdot 2^{-\nu}$ and define K by $K(t) = k(t)(1 - e^{i(\xi-\xi_0)\cdot t})\varphi(2^\nu t)$, $t \in \mathbf{R}^s$. Let $H(t) = C_0 J_{s+1/2}(|t|)\, |t|^{-s-1/2}$, $t \in \mathbf{R}^s$, where $J_{s+1/2}$ is the Bessel function of order $s + 1/2$ and C_0 is a constant chosen so that $\int_{\mathbf{R}^s} H(t) dt = 1$. The following properties of H and its Fourier transform $\hat{H}$ are well-known. (See e.g. [10], pp. 51—52.)

$$H(t) = O(1)\,, \quad |t| \to 0 \tag{3.2}$$

$$H(t) = O(|t|^{-s-1})\,, \quad |t| \to \infty \tag{3.3}$$

$$\hat{H}(u) = (1 - |u|^2)^{(s+1)/2}\,, \quad |u| \le 1\,, \quad \text{and} \quad \hat{H}(u) = 0\,, \quad |u| > 1 \tag{3.4}$$

Setting $A(x) = \int_{\omega_0^*} K{*}H_{2^\nu R}(x-t)\, e^{-i\xi_0\cdot t} g(t)dt$ and

$$B(x) = \int_{\omega_j^*} (K(x-t) - K{*}H_{2^\nu R}(x-t))\, e^{-i\xi_0\cdot t} g(t)dt$$

we have

$$\int_{\omega_0^*} k(x-t)\,(1 - e^{i(\xi-\xi_0)\cdot(x-t)})\, e^{-i\xi_0\cdot t} g(t)dt = A(x) + B(x)\,.$$

For the estimation of $A(x)$ first observe that for all $u \in \mathbf{R}^s$

$$|\hat{K}(u)| \leq \int_{\mathbf{R}^s} |K(t)|dt \leq \text{Const.}\ |\xi_0 - \xi|\ 2^{-\nu} \int_{\mathbf{R}^s} |t|^{-s+1}\varphi(t)dt \leq \text{Const.}\ A\,.$$

From (3.4) it follows that

$$K{*}H_{2^\nu R}(t) = (2\pi)^{-s} \int_{|u|\leq 2^\nu R} \hat{K}(u) \left(1 - \frac{|u|^2}{(2^\nu R)^2}\right)^{(s+1)/2} e^{iu\cdot t}\, du\,, \quad t \in \mathbf{R}^s\,,$$

and using Fubini's theorem we obtain

$$A(x) = (2\pi)^{-s} \int_{|u|\leq 2^\nu R} \hat{K}(u) \left(1 - \frac{|u|^2}{(2^\nu R)^2}\right)^{(s+1)/2} e^{iu\cdot x} \left\{ \int_{\omega_0^*} e^{-i(u+\xi_0)\cdot t} g(t)dt \right\} du\,.$$

If $R \leq C$ the hypothesis of the lemma implies that

$$\left| \int_{\omega_j^*} e^{-i(u+\xi_0)\cdot t} g(t)dt \right| \leq \text{Const.}\ 2^{-\nu s}\varepsilon\,, \quad |u| \leq 2^\nu R\,,$$

and hence

$$|A(x)| \leq \text{Const.}\ AR^s\varepsilon\,, \quad R \leq C\,. \tag{3.5}$$

To estimate $B(x)$ we introduce the function

$$\Phi(z\,,h) = \int_{|t|\leq h} |K(z) - K(z-t)|dt\,, \quad z \in \mathbf{R}^s\,, \quad h > 0\,.$$

We claim that

$$\Phi(z\,,h) \leq \text{Const.}\ A2^\nu h^{s+1/2}|z|^{-s+1/2}\,, \quad |z| \leq 2^{-\nu}\,100\sqrt{s}\,, \quad h > 0\,. \tag{3.6}$$

For $h \leq |z|/2$ we use the mean value theorem and the estimate

$$|D_iK(z)| \leq \text{Const.}\ A2^\nu|z|^{-s}\,, \quad 0 < |z| \leq 2^{-\nu}\,200\sqrt{s}\,, \quad i = 1\,,2\,,\ldots\,,s\,, \tag{3.7}$$

to prove (3.6). (3.7) is easily obtained by a direct computation of $D_i K$. For $h > |z|/2$ the estimate $|K(z)| \leq \text{Const.}\, A 2^\nu |z|^{-s+1}$ is used to establish (3.6).

We will now prove that

$$|K(z) - K*H_{2^\nu R}(z)| \leq \text{Const.}\, AR^{-1/2}\, 2^{\nu/2} |z|^{-s+1/2} \quad \text{for} \quad |z| \leq 2^{-\nu}\, 100 \sqrt{s}\,. \tag{3.8}$$

We have

$$K(z) - K*H_{2^\nu R}(z) = \int\limits_{|u| \leq (2^\nu R)^{-1}} (K(z) - K(z-u))\, H_{2^\nu R}(u) du + $$
$$+ \int\limits_{|u| > (2^\nu R)^{-1}} (K(z) - K(z-u))\, H_{2^\nu R}(u) du\,.$$

From (3.2) it follows that the first term is less than $\text{Const.}\,(2^\nu R)^s \Phi(z\,,\,(2^\nu R)^{-1})$ and using (3.6) we see that this can be majorized by the right hand side of (3.8). Using (3.3), introducing polar coordinates and performing a partial integration we can estimate the second term with $\text{Const.}\,(2^\nu R)^{-1} \int\limits_{(2^\nu R)^{-1}} \Phi(z\,,\,r)\, r^{-s-2} dr$. (3.6) implies that this is less than $\text{Const.}\, AR^{-1/2} 2^{\nu/2} |z|^{-s+1/2}$ and hence the proof of (3.8) is complete.

By use of (3.8) $B(x)$ can be estimated:

$$|B(x)| \leq \text{Const.}\, AR^{-1/2} 2^{\nu/2} \int\limits_{\omega_0^*} |x-t|^{-s+1/2} |g(t)| dt \leq \text{Const.}\, AR^{-1/2} g^*(x)\,.$$

The definition of g implies that $g^*(x) \leq f^*(x) + |\varrho|$ and we obtain $|B(x)| \leq \text{Const.}\, AR^{-1/2}(f^*(x) + |\varrho|)$. This completes the proof of the lemma.

We remark that at the end of the proof of the basic result there are constructed sequences $\{\xi_j\}$ and $\{\omega_j^*\}$, where $\xi_j \in \mathbf{R}_+^s$ and $\omega_j^* \in B^*$, corresponding to the sequences $\{n_j\}$ and $\{\omega_j^*\}$ in Section 11 in [7]. The proof in several variables is different from the one in [7] in that we do not require that $\xi_j \in Z_+^s(\omega_j^*)$ for each j.

With the modifications mentioned in this section the method in [7] gives the result

$$m\{x \in T_s\,;\, m\chi_F(x) > y\} \leq B_p^p\, y^{-p}\, mF\,, \quad y > 0\,, \quad 1 < p < \infty\,, \tag{3.9}$$

where $B_p \leq \text{Const.}\, p^2/(p-1)$. Choosing p suitably (depending on y) we obtain Lemma 1.2.

4. Odd kernels without smoothness

In this section we will show that if the kernel k is odd, then the operator M associated with k is bounded on $L^p(T_s)$, $1 < p < \infty$, even if k does not satisfy regularity conditions such as (1.3). Let $\mathscr{A}'$ denote the class of all complex-valued kernels k which satisfy the following three conditions

k is positively homogeneous of degree $-s$ (4.1)

k is integrable over the unit sphere and $\int_S k(x)d\sigma(x) = 0$ (4.2)

k is odd. (4.3)

Kernels in the class $\mathscr{A}'$ are special cases of kernels treated by Calderón and Zygmund in [2]. (See Theorem 3, p. 290.)

THEOREM 4.1. *Let $k \in \mathscr{A}'$ and let M be defined as in (1.4). Then the inequalities (B), (C) and (D) in Theorem 1.1. hold for M.*

Proof. It is not difficult to see that Mf is measurable if $f \in L \log L(T_s)$. In the proof of the theorem we will use the results for the operator M^* in [7]. It is easy to see that the results in [7] hold even if we replace M^* by the operator M_1, defined by

$$M_1 f(x) = \sup_{\xi \in R} \left| \int_{-\pi}^{\pi} \frac{e^{-i\xi t} f(t)}{x - t} dt \right|, \quad x \in (-\pi, \pi).$$

We first give the proof of (C). Assume that $f \in L^p(T_s)$ and extend f to $\mathbf{R}^s$ by setting it equal to zero outside T_s. Let

$$S_\xi^\varepsilon(x) = \int_{|x-t|>\varepsilon} k(x-t) e^{-i\xi\cdot t} f(t)dt, \quad x \in T_s,$$

and let $S_\xi(x)$ be the pointwise limit of $S_\xi^\varepsilon(x)$, when ε tends to zero. Using the fact that k is odd we get

$$S_\xi^\varepsilon(x) = \int_{|x-t|>\varepsilon} k(x-t)e^{-i\xi\cdot t} f(t)dt = \int_{|y|>\varepsilon} k(y) e^{-i\xi\cdot(x-y)} f(x-y)dy =$$

$$= e^{-i\xi\cdot x} \int_S \left(\int_\varepsilon^\infty k(ty') e^{i\xi\cdot y' t} f(x - ty') t^{s-1} \, dt \right) d\sigma(y') =$$

$$= e^{-i\xi\cdot x} \tfrac{1}{2} \int_S k(y') \left(\int_{|t|>\varepsilon} t^{-1} \, e^{i\xi\cdot y' t} f(x - ty')dt \right) d\sigma(y').$$

Letting ε tend to zero we obtain

$$S_\xi(x) = e^{-i\xi\cdot x} \tfrac{1}{2} \int_S k(y') \left(\int_{\mathbf{R}} t^{-1} \, e^{i\xi\cdot y' t} f(x - ty')dt \right) d\sigma(y')$$

for almost every $x \in T_s$ and all $\xi \in \mathbf{R}^s$. We therefore have

$$Mf(x) \leq \tfrac{1}{2} \int_S |k(y')| \left(\sup_{\eta \in \mathbf{R}} \left| \int_{\mathbf{R}} t^{-1} \, e^{i\eta t} f(x - ty')dt \right| \right) d\sigma(y').$$

Using Minkowski's inequality for integrals and Theorem 1 in [7] we obtain $\|Mf\|_p \leq \frac{1}{2} \int_S |k(y')| d\sigma(y')\, C_p \|f\|_p$ and (C) is proved.

The proof of (B) is similar and is therefore omitted.

To prove (D) we will show that the inequality (3.9) holds for M with $B_p \leq \text{Const.}\, p^3/(p-1)^2$. Setting

$$M_1 \chi_F(x, y') = \sup_{\eta \in R} \left| \int_{\mathbf{R}} t^{-1} e^{i\eta t} \chi_F(x - ty')\, dt \right|$$

we get $M\chi_F(x) \leq \frac{1}{2} \int_S |k(y')| M_1 \chi_F(x, y')\, d\sigma(y')$. From the basic result in [7], pp. 235—236, it follows that

$m\{x \in T_s\,;\, M_1\chi_F(x, y') > \lambda\} \leq (\text{Const.})^p p^{2p} (p-1)^{-p} \lambda^{-p} mF$ for $\lambda > 0$, $1 < p < \infty$, $y' \in S$. For the moment let f^* denote the nondecreasing rearrangement of a function f defined on T_s and let f^{**} be defined as in [8], p. 257, with $r = 1$, $M = T_s$. Using the inequality (2.2) in [8], p. 258, we get

$$\begin{aligned}
&\sup_{t>0} t^{1/p} (M\chi_F)^*(t) \leq \sup_{t>0} t^{1/p} (M\chi_F)^{**}(t) \leq \\
&\leq \tfrac{1}{2} \int_S |k(y')| \{\sup_{t>0} t^{1/p} (M_1 \chi_F(\cdot\,, y'))^{**}(t)\} d\sigma(y') \leq \\
&\leq \tfrac{1}{2} p(p-1)^{-1} \int_S |k(y')| \{\sup_{t>0} t^{1/p} (M_1 \chi_F(\cdot\,, y'))^{*}(t)\} d\sigma(y') \leq \\
&\leq \tfrac{1}{2} p(p-1)^{-1}\, \text{Const.}\, p^2 (p-1)^{-1} (mF)^{1/p} \int_S |k(y')| d\sigma(y') = \\
&= \text{Const.}\, p^3\, (p-1)^{-2} (mF)^{1/p}\,.
\end{aligned}$$

This inequality yields (3.9) with $B_p \leq \text{Const.}\, p^3 (p-1)^{-2} = O(p)$, $p \to \infty$. Using this estimate of B_p we can complete the proof of (D) as in [11], pp. 569—570.

5. Extensions and applications

In this section we will give some extensions and applications of Theorems 1.1 and 4.1. We first remark that a careful examination of the proofs of Theorems 1.1 and 4.1 shows that the theorems hold also if we replace M by the operator $\check{M}$ defined by

$$\check{M}f(x) = \sup_{\varepsilon>0,\, \xi\in\mathbf{R}^s} \left| \int_{T_s\setminus\{t\,;\,|x-t|<\varepsilon\}} k(x-t)e^{-i\xi\cdot t} f(t)dt \right| .$$

We also point out that the results for the operator M_1 proved in [7] and [11] are special cases of the corresponding results for M in Theorem 1.1. This can be seen by taking $s = 2$ and choosing k as a suitable odd kernel.

So far we have only studied functions defined in T_s, but now we will define the analogue of the operator M for functions defined in $\mathbf{R}^s$. For $f \in L^p(\mathbf{R}^s)$, $1 < p < \infty$, let Nf be defined by

$$Nf(x) = \sup_{\xi\in\mathbf{R}^s} \left| \int_{\mathbf{R}^s} k(x-t)e^{-i\xi\cdot t} f(t)dt \right| , \quad x \in \mathbf{R}^s ,$$

where $k \in \mathcal{A}$ or $\mathcal{A}'$. Then the following theorem holds.

THEOREM 5.1. *If k belongs to $\mathcal{A}$ or $\mathcal{A}'$, then*

$$\|Nf\|_p \leq C_p \|f\|_p , \quad 1 < p < \infty , \tag{5.1}$$

where the norms are taken with respect to $\mathbf{R}^s$.

Proof. Let Q denote the cube $\{x \in \mathbf{R}^s\,;\, |x_i| \leq 1,\ i = 1, \ldots, s\}$. We will use the fact that Theorems 1.1 and 4.1 hold with T_s replaced by Q, if M is defined by $Mf(x) = \sup_{\xi} \left| \int_Q k(x-t)e^{-i\xi\cdot t} f(t)dt \right|$, $x \in Q$. Let $f \in L^p(\mathbf{R}^s)$ and let

$F_n(x) = \sup_{\xi} \left| \int_{nQ} k(x-t)e^{-i\xi\cdot t} f(t)dt \right|$, $x \in nQ$, and set $F_n(x)$ equal to zero for $x \notin nQ$, $n = 1, 2, \ldots$ Performing a change of variable and using the homogeneity of k we get

$$F_n(x) = M(f(n\,\cdot))\,(n^{-1}x) , \quad x \in nQ .$$

Hence Theorem 1.1 or 4.1 with T_s replaced by Q yields

$$\int_{nQ} |F_n(x)|^p \, dx \leq C_p^p \int_{nQ} |f(x)|^p \, dx .$$

If we use the fact that $Nf(x) \leq \varliminf_{n\to\infty} F_n(x)$, (5.1) now follows from the above estimate and Fatou's lemma.

We will now use Theorem 5.1 to prove a convergence result.

THEOREM 5.2. *Let μ be a bounded Borel measure in $\mathbf{R}^s$ and assume that μ has no point mass at the origin. Suppose that k belongs to $\mathcal{A}$ or $\mathcal{A}'$ and that* $\lim_{\substack{\varepsilon\to 0\\ \lambda\to\infty}} \int_{\varepsilon\leq|x|\leq\lambda} \hat{\mu}(x)\, k(x)dx = 1$. *Let K be defined by $K(x) = \hat{\mu}(x)\, k(x)$, $x \neq 0$. Then*

$\lim\limits_{R\to\infty}\int\limits_{\mathbf{R}^s} K_R(x-t)f(t)dt = f(x)$ *for a.e.* x *in* $\mathbf{R}^s$ *if* $f \in L^p(\mathbf{R}^s)$ *for some* p *with* $1 < p < \infty$.

Proof. First observe that $K_R(x) = \hat{\mu}(Rx)k(x)$. Let $f \in L^p(\mathbf{R}^s)$. Lebesgue's theorem on dominated convergence implies that for a.e. $x \in \mathbf{R}^s$ $\lim\limits_{\varepsilon\to 0}\int\limits_{|x-t|>\varepsilon} K_R(x-t)f(t)dt$ exists and equals $\int\limits_{\mathbf{R}^s} e^{-iR\xi\cdot x}\left(\int\limits_{\mathbf{R}^s} k(x-t)e^{iR\xi\cdot t}f(t)\,dt\right)d\mu(\xi)$ for all $R > 0$. We denote this limit by $S_Rf(x)$ and set $S^*f(x) = \sup\limits_{R>0}|S_Rf(x)|$. It follows that $S^*f(x) \leq \|\mu\|Nf(x)$, where $\|\mu\|$ is the total mass of μ. Theorem 5.1 yields $\|S^*f\|_p \leq C_p\|\mu\|\,\|f\|_p$, which implies that the theorem will be proved if we can show that $\lim\limits_{R\to\infty} S_Rf(x) = f(x)$ for all x if $f \in C^\infty(\mathbf{R}^s)$ and has compact support. Let f be of this type and fix $x \in \mathbf{R}^s$. Take λ so large that the support of f is contained in a ball with center x and radius λ. Using the hypothesis we have

$$1 = \int\limits_{|x-t|\leq\lambda} K_R(x-t)dt + \lim_{\varrho\to\infty}\int\limits_{R\lambda\leq|y|\leq\varrho} K(y)dy\,,$$

and hence

$$S_Rf(x) - f(x) = \int\limits_{|x-t|\leq\lambda} K_R(x-t)\,(f(t)-f(x))dt - f(x)\lim_{\varrho\to\infty}\int\limits_{R\lambda\leq|y|\leq\varrho} K(y)dy\,.$$

The second term obviously tends to zero when R tends to infinity and the first term equals

$$\int\limits_{\mathbf{R}^s}\left\{\int\limits_{|x-t|\leq\lambda} e^{iR\xi\cdot t}k(x-t)\,(f(t)-f(x))dt\right\} e^{-iR\xi\cdot x}\,d\mu(\xi)\,. \tag{5.2}$$

For $\xi \neq 0$ it follows from Riemann — Lebesgue's lemma that the inner integral tends to zero when R tends to infinity. For all ξ and R the inner integral is bounded by $\int\limits_{|x-\,|\leq\lambda} |k(x-t)|\,|f(t)-f(x)|dt$, which is finite. Since μ has no point mass at the origin, Lebesgue's theorem on dominated convergence proves that (5.2) tends to zero when R tends to infinity. This completes the proof of the theorem.

We remark that if $k \in \mathscr{A}$, then the equality $\lim\limits_{\substack{\varepsilon\to 0\\ \lambda\to\infty}}\int\limits_{\varepsilon\leq|x|\leq\lambda}\hat{\mu}(x)k(x)dx = 1$ is implied by the condition

$$\int\limits_{\mathbf{R}^s}\hat{k}(x)d\mu(x) = 1\,. \tag{5.3}$$

We will now give an example of a measure μ and kernels k, which satisfy the conditions in Theorem 5.2.

Example 5.3. Let μ be a discrete measure with point mass $2^{-s}\prod_1^s \varepsilon_i$ at the points $(\varepsilon_1, \ldots, \varepsilon_s)$, $\varepsilon_i = \pm 1$, $i = 1, 2, \ldots, s$, and with no mass at any other points. Assume that $k \in \mathscr{A}$ and that $\hat{k}(\varepsilon_1, \ldots, \varepsilon_s) = \prod_1^s \varepsilon_i$ for $\varepsilon_i = \pm 1$. Then, if $1 < p < \infty$ and $f \in L^p(\mathbf{R}^s)$,

$$\lim_{R\to\infty} (-i)^s \int_{\mathbf{R}^s} \prod_1^s \sin R(x_j - t_j) k(x-t) f(t) dt = f(x) \quad \text{a.e. in } \mathbf{R}^s. \tag{5.4}$$

Proof. It is easy to see that $\int_{\mathbf{R}^s} \hat{k}(x)\, d\mu(x) = 1$ and $\hat{\mu}(x) = (-i)^s \prod_1^s \sin x_j$, and hence (5.4) follows from Theorem 5.2.

The existence of kernels k in $\mathscr{A}$ which satisfy the conditions in the above example follows from the fact that every function in $C^\infty(\mathbf{R}^s \setminus \{0\})$, which is homogeneous of degree zero and has mean value zero over the unit sphere, is the Fourier transform $\hat{k}$ of some C^∞ Calderón — Zygmund kernel k. (See [3], p. 312.)

We also want to point out that if $k \in \mathscr{A}$, then the assumption that $f \in L^p(\mathbf{R}^s)$ in Theorem 5.2 can be replaced by the condition that f has compact support and $\int_{\mathbf{R}^s} |f(x)| \log^+ |f(x)| \log^+ \log^+ |f(x)| dx$ is finite. The proof of this is analogous to the proof of (A) in [11]. If $k \in \mathscr{A}'$ it is sufficient that f has compact support and that $f(x)(\log^+ |f(x)|)^2$ is integrable, which follows from (B) in Theorem 4.1.

In the case $s = 1$ we will give an alternative formulation of Theorem 5.2. Let $\mathcal{K} = \{K\,;\, K(x) = \frac{1}{x}\hat{\mu}(x),\ x \in \mathbf{R}\setminus\{0\},$ where μ is a bounded Borel measure on $\mathbf{R}$ without point mass at the origin and $-i\pi \int_{\mathbf{R}} \operatorname{sgn}(x) d\mu(x) = 1\}$. Since $-i\pi \operatorname{sgn}(x)$ is the Fourier transform of x^{-1} the last condition is just the analogue of (5.3). Theorem 5.2 for $s = 1$ with $k(x)$ replaced by x^{-1} implies that if $f \in L^p(\mathbf{R})$, $1 < p < \infty$, and $K \in \mathcal{K}$, then $\lim_{R\to\infty} f * K_R(x) = f(x)$ for a.e. x in $\mathbf{R}$. If $K \in \mathcal{K}$ we define its Fourier transform by $\hat{K}(y) = \lim_{\substack{\varepsilon\to 0 \\ N\to\infty}} \int_{\varepsilon \le |x| \le N} e^{-iyx} K(x) dx$. It follows that $\hat{K}(y) = -i\pi \int_{\mathbf{R}} \operatorname{sgn}(y+t) d\mu(t)$ and a calculation shows that the Fourier transforms of the kernels in $\mathcal{K}$ are precisely the functions L of bounded variation, which satisfy the following three conditions:

$$L(y) = \tfrac{1}{2}(L(y+) + L(y-)) , \quad y \in \mathbf{R} \tag{5.5}$$

$$L(0) = 1 \quad \text{and} \quad L \quad \text{is continuous at the origin} . \tag{5.6}$$

$$\lim_{y\to+\infty} L(y) + \lim_{y\to-\infty} L(y) = 0 \tag{5.7}$$

We obtain the following summability result.

COROLLARY 5.4. *Let* $1 < p \leq 2$ *and* $f \in L^p(\mathbf{R})$. *Assume that* L *is a function of a bounded variation, which satisfies (5.6). Then* $\lim_{n\to\infty} \frac{1}{2\pi} \int_{\mathbf{R}} L\left(\frac{y}{n}\right) \hat{f}(y)\, e^{ixy} dy = f(x)$ *for a.e.* $x \in \mathbf{R}$.

Proof. We need only show that condition (5.7) can be removed. This can be verified if we consider $\int_0^\infty L\left(\frac{y}{n}\right)\hat{f}(y)e^{ixy}dy$ and $\int_{-\infty}^0 L\left(\frac{y}{n}\right)\hat{f}(y)e^{ixy}dy$ separately.

6. The square partial sums of multiple Fourier series

In this section we study the square partial sums of a function $f \in L^1(T_s)$, given by

$$S_n f(x) = \sum_{|k_i|\leq n} c_k\, e^{ik\cdot x} = \pi^{-s} \int_{T_s} \prod_{i=1}^{s} D_n(x_i - t_i) f(t) dt ,$$

where c_k are the Fourier coefficients of f and D_n denotes the Dirichlet kernel. Let $M_s f(x) = \sup_{n\in Z_+} |S_n f(x)|$, $x \in T_s$. Then the following theorem holds.

THEOREM 6.1.

$$\textit{If} \int_{T_s} |f(x)|\,(\log^+|f(x)|)^s \log^+\log^+ |f(x)|dx < \infty \textit{ then } \lim_{n\to\infty} S_n f(x) = f(x) \textit{ a.e. in } T_s . \tag{6.1}$$

$$\|M_s f\|_1 \leq \text{Const.} \int_{T_s} |f(x)|\,(\log^+ |f(x)|)^{s+1} dx + \text{Const.} \tag{6.2}$$

$$\|M_s f\|_p \leq C_p \|f\|_p , \quad 1 < p < \infty . \tag{6.3}$$

Proof. We first treat the case $s = 2$. Let $1 < p < \infty$ and define for $f \in L^p(\mathbf{R}^2)$ and $\xi \in \mathbf{R}$ $S'_\xi f$ by

$$S'_\xi f(x) = \iint_{\mathbf{R}^2} \frac{e^{-i\xi(t_1+t_2)} f(t_1, t_2)}{(x_1 - t_1)(x_2 - t_2)}\, dt_1\, dt_2 , \quad x \in \mathbf{R}^2 .$$

To prove (6.3) it is sufficient to prove that the operator M'_s defined by $M'_s f(x) =$

$= \sup_{\xi \in \mathbf{R}} |S'_\xi f(x)|$, $x \in \mathbf{R}^2$, is a bounded operator on $L^p(\mathbf{R}^2)$. We first claim that if $f \in L^p(\mathbf{R}^2)$ then for a.e. x in $\mathbf{R}^2$

$$\iint_{\mathbf{R}^2} \frac{f(t_1, t_2)}{(x_1 - t_1)(x_2 - t_2)} dt_1 \, dt_2 + \pi^2 f(x) = \int_{\mathbf{R}} \frac{1}{x_1 + x_2 - u_1} \left(\int_{\mathbf{R}} \frac{f(u_1 - u_2, u_2)}{x_2 - u_2} du_2 \right) du_1 + \\ + \int_{\mathbf{R}} \frac{1}{x_1 + x_2 - u_2} \left(\int_{\mathbf{R}} \frac{f(u_1, u_2 - u_1)}{x_1 - u_1} du_1 \right) du_2, \tag{6.4}$$

where the integrals are to be taken in the principal value sense. To establish (6.4) it is enough to prove it for f in a dense subclass of $L^p(\mathbf{R}^2)$. This can be done by computing the Fourier transform of both sides of (6.4) for $f \in L^2(\mathbf{R}^2)$ or we can proceed in the following way. Let $f(x_1, x_2) = f_1(x_1) f_2(x_2)$, where f_1 and $f_2 \in C^\infty(\mathbf{R})$ and have compact support. We obviously have

$$\frac{f(t_1, t_2)}{(x_1 - t_1)(x_2 - t_2)} = \frac{f(t_1, t_2)}{(x_1 + x_2 - (t_1 + t_2))(x_2 - t_2)} + \frac{f(t_1, t_2)}{(x_1 + x_2 - (t_1 + t_2))(x_1 - t_1)},$$

for $t_1 \neq x_1$, $t_2 \neq x_2$, $t_1 + t_2 \neq x_1 + x_2$. (6.4) follows if we integrate this relation over the region $\{(t_1, t_2); |x_1 - t_1| > \varepsilon, \ |x_2 - t_2| > \varepsilon, \ |x_1 + x_2 - (t_1 + t_2)| > \delta\}$, perform a change of variables in the integrals obtained on the right hand side and let first ε and then δ tend to zero. The term $\pi^2 f(x)$ enters because of the fact that $\lim_{\delta \to 0} \iint_{\substack{|u+v| \leq \delta \\ \delta \leq |u-v| \leq 1}} \frac{du \, dv}{uv} = -\pi^2$.

Now fix $f \in L^p(\mathbf{R}^2)$. Replacing $f(t_1, t_2)$ by $e^{-i\xi(t_1+t_2)} f(t_1, t_2)$ in (6.4) and taking the supremum over all $\xi \in \mathbf{R}$ we get

$$M'_s f(x) \leq \sup_\xi \left| \int_{\mathbf{R}} \frac{e^{-i\xi u_1}}{x_1 + x_2 - u_1} \left(\int_{\mathbf{R}} \frac{f(u_1 - u_2, u_2)}{x_2 - u_2} du_2 \right) du_1 \right| + \\ + \sup_\xi \left| \int_{\mathbf{R}} \frac{e^{-i\xi u_2}}{x_1 + x_2 - u_2} \left(\int_{\mathbf{R}} \frac{f(u_1, u_2 - u_1)}{x_1 - u_1} du_1 \right) du_2 \right| + \pi^2 |f(x)|. \tag{6.5}$$

Using the L^p estimates of the Hilbert transform and the operator M_1 defined in Section 4 we obtain $\|M'_s f\|_{p, \mathbf{R}^2} \leq C_p \|f\|_{p, \mathbf{R}^2}$, and the proof of (6.3) is complete.

(6.1) and (6.2) can be proved by use of the above method and the results in [7] and [11] for functions belonging to classes close to L^1.

In the case $s > 2$ we use the following analogue of (6.4):

$$\iint_{\mathbf{R}^s} \cdots \int \frac{f(t_1, \ldots, t_s)}{(x_1 - t_1) \ldots (x_s - t_s)} dt_1 \ldots dt_s + a_s f(x) = \sum_{i=1}^{s} F_i(x), \tag{6.6}$$

where

$$F_i(x) = \int_{\mathbf{R}} \frac{1}{\sum_1^s x_j - u_i} \left(\int\int_{\mathbf{R}^{s-1}} \cdots \int \frac{f(u_1, \ldots, u_{i-1}, u_i - \sum_{j \neq i} u_j, u_{i+1}, \ldots, u_s)}{(x_1 - u_1) \ldots (x_{i-1} - u_{i-1})(x_{i+1} - u_{i+1}) \ldots (x_s - u_s)} \right.$$

$$\left. du_1 \ldots du_{i-1} du_{i+1} \ldots du_s \right) du_i \,, \quad i = 1, 2, \ldots, s \,,$$

and $a_s = \begin{cases} 0, & \text{if } s \text{ is odd} \\ \pi^s, \text{ if } s = 4k + 2, & k = 0, 1, 2, \ldots \\ -\pi^s, \text{ if } s = 4k, & k = 1, 2, \ldots \end{cases}$

(6.6) is most easily proved by computation of Fourier transforms. For example if $f \in L^2(\mathbf{R}^s)$ then

$$\hat{F}_i(y) = (-i\pi)^s \operatorname{sgn}(y_i) \prod_{j \neq i} \operatorname{sgn}(y_j - y_i) \hat{f}(y) \,, \quad \text{a.e. } y \in \mathbf{R}^s \,.$$

As an application of the method which gave the estimate of M_s' we will now prove that the condition $k \in \mathcal{A}$ in Theorem 5.2 can be replaced by the assumption that k is a Calderón — Zygmund kernel for which $k \log^+ |k|$ is integrable over the unit sphere. First define the operator M_Θ' for $0 \leq \Theta \leq 2\pi$ by

$$M_\Theta' f(x) = \sup_{\xi \in \mathbf{R}} \left| \int\int_{\mathbf{R}^2} \frac{e^{-i\xi(\cos\Theta\, t_1 + \sin\Theta\, t_2)}}{(x_1 - t_1)(x_2 - t_2)} f(t_1, t_2) dt_1\, dt_2 \right| \,,$$

$$x \in \mathbf{R}^2 \,, f \in L^p(\mathbf{R}^2) \,, 1 < p < \infty \,.$$

M_Θ' can be estimated by an obvious modification of the method which was used to estimate M_s' and we obtain

$$\|M_\Theta' f\|_{p,\mathbf{R}^2} \leq C_p \|f\|_{p,\mathbf{R}^2} \,, \quad 1 < p < \infty \,, \tag{6.7}$$

where C_p is independent of Θ.

Let k be a kernel of the type described above and define the operator N_ξ, $\xi \in \mathbf{R}^s$, by

$$N_\xi f(x) = \sup_{R > 0} \left| \int_{\mathbf{R}^s} k(x - t) e^{iR\xi \cdot t} f(t)\, dt \right| , \quad x \in \mathbf{R}^s \,, \; f \in L^p(\mathbf{R}^s) \,, \; 1 < p < \infty \,.$$

Using (6.7) and the estimates in [2], p. 304, we can prove that

$$\|N_\xi f\|_{p,\mathbf{R}^s} \leq C_p \|f\|_{p,\mathbf{R}^s} \,, \quad 1 < p < \infty \,, \tag{6.8}$$

where C_p is independent of ξ. The crucial step in the proof of (6.8) is the observation that if k_1 and k_2 are odd Calderón — Zygmund kernels, then

$$\int_{|y|>\varepsilon} k_1(y)\left(\int_{\mathbf{R}^s} k_2(z)e^{i\eta\cdot(x-y-z)}f(x-y-z)dz\right)dy =$$

$$= e^{i\eta\cdot x}\frac{1}{4}\int_S\int_S k_1(y')\, k_2(z')\left(\int_{|t|>\varepsilon}\left(\int_{\mathbf{R}}\frac{e^{-i(\eta\cdot y't+\eta\cdot z'u)}}{tu}f(x-ty'-uz')du\right)dt\right)d\sigma(y')\,d\sigma(z')$$

for almost every $x \in \mathbf{R}^s$ and every $\eta \in \mathbf{R}^s$ and $\varepsilon>0$.

We now use (6.8) to extend Theorem 5.2. With the same notations as in the theorem we obtain

$$S^*f(x) \leq \int_{\mathbf{R}^s} N_\xi f(x)|d\mu(\xi)|\,,$$

and Minkowski's inequality for integrals yields $\|S^*f\|_p \leq C_p\|\mu\|\,\|f\|_p$. The rest of the proof is the same as in Theorem 5.2.

7. The rectangular partial sums of double Fourier series

For $f \in L^1(T_2)$ let the rectangular partial sums be defined by

$$S_{mn}f(x,y) = \sum_{k=-m}^{m}\sum_{l=-n}^{n} c_{kl}\, e^{i(kx+ly)} = \pi^{-2}\int\int_{T_2} D_m(x-t)D_n(y-u)f(t,u)dtdu\,,\ (x,y) \in T_2\,,$$

where c_{kl} are the Fourier coefficients of f.

THEOREM 7.1. *If $f \in L^p(T_2)$ for some $p > 1$, then*

$$S_{mn}f(x\,,y) = o\,(\log\min\,(m\,,n))\,,\quad m\,,n \to \infty\,,\quad \textit{for a.e.}\quad (x\,,y) \in T_2\,.$$

Proof. If $g \in L^1(T_1)$ we denote its partial sums by $S_ng(x) = S_n(g\,;x) = S_n(g(\cdot)\,;x)$ and set $S_1^*g(x) = S_1^*(g\,;x) = S_1^*(g(\cdot)\,;x) = \sup_n |S_ng(x)|\,,\ x \in T_1$. Extend g to the real axis by setting it equal to zero outside T_1. Using well-known estimates of the Dirichlet kernel we can easily prove that

$$|S_ng(x)| \leq \text{Const.}\log n\ g^*(x)\,,\quad x \in T_1\,,\quad n \geq 2\,, \tag{7.1}$$

where g^* denotes the Hardy — Littlewood maximal function of g. Let $1 < p < \infty$ and $f \in L^p(T_2)$. Since

$$S_{mn}f(x\,,y) = \pi^{-1}\int_{T_1} D_m(x-t)\left(\pi^{-1}\int_{T_1} D_n(y-u)f(t\,,u)du\right)dt\,,$$

(7.1) implies that

$$|S_{mn}f(x\,,y)| \leq \text{Const.}\log m\sup_{x\in\omega}\frac{1}{|\omega|}\int_\omega\left|\int_{T_1} D_n(y-u)f(t\,,u)du\right|dt\,,\quad (x\,,y) \in T_2\,,$$

$$m \geq 2\,,$$

where $|\omega|$ denotes the length of an interval ω and we have set f equal to zero outside T_2. From the above inequality it follows that

$$\sup_{m\geq 2,n} (|S_{mn}f(x, y)|/\log m) \leq \text{Const.} \sup_{x\in\omega} \frac{1}{|\omega|} \int_{\omega} S_1^*(f(t, \cdot); y)dt, \quad (x, y) \in T_2.$$

Using the L^p estimates of the Hardy — Littlewood maximal function and the operator S_1^*, we can prove that the L^p norm of the left hand side above is less than $C_p\|f\|_p$. By approximation of f with trigonometric polynomials we can use this fact to prove that $S_{mn}f(x, y) = o(\log m)$ a.e. From the symmetry it follows that $S_{mn}f(x, y) = o(\log n)$ a.e. and hence $S_{mn}f(x, y) = o(\log \min(m, n))$ a.e.

We remark that it is easy to show that the assumption $f \in L^p(T_2)$ for some $p > 1$ in the above theorem can be replaced by $f \in L(\log L)^2(T_2)$.

We will now give a condition on the Fourier coefficients of a function $f \in L^2(T_2)$, which is sufficient for the convergence a.e. of the rectangular partial sums. First define S_2^* by $S_2^* f(x, y) = \sup_{m,n} |S_{mn}f(x, y)|$ for $(x, y) \in T_2$ and $f \in L^1(T_2)$.

THEOREM 7.2. *Assume that the Fourier coefficients* c_{mn} *of a function* $f \in L^2(T_2)$ *satisfy* $\sum_{m,n} |c_{mn}|^2(\log \min(|m| + 2, |n| + 2))^2 < \infty$. *Then* $\lim_{m,n\to\infty} S_{mn}f(x, y) = f(x, y)$ *a.e. in* T_2 *and*

$$\|S_2^* f\|_2 \leq \text{Const.} \{\sum_{m,n} |c_{mn}|^2 (\log \min(|m| + 2, |n| + 2))^2\}^{1/2}.$$

Proof. Let $l_{mn} = (\log \min(|m| + 2, |n| + 2))^{-1}$ and let $\Delta l_{mn} = l_{mn} + l_{m+1,n+1} - l_{m+1,n} - l_{m,n+1}$. We furthermore set $l_k = l_{kk}$ and $\Delta l_k = l_k - l_{k+1}$. Let f satisfy the assumptions in the theorem and let g be the function in $L^2(T_2)$, which has Fourier coefficients $l_{mn}^{-1} c_{mn}$. Define T^* for $h \in L^1(T_2)$ by $T^*h = \sup_{m,n} (|S_{mn}h| l_{mn})$. If we use the fact that $\Delta l_{mn} = \Delta l_m$ for $m = n$ and vanishes for $m \neq n$, a partial summation yields

$$S_{mn}f = \sum_{k=0}^{\min(m,n)-1} S_{kk}g\, \Delta l_k + \sum_{k=0}^{\min(m,n)-1} S_{mk}g\, \Delta l_k + \sum_{k=0}^{\min(m,n)-1} S_{kn}g\, \Delta l_k + O(T^*g).$$

Denote the three first terms on the right by A, B and C respectively. Define σ_k and σ^* by $\sigma_k h = (k + 1)^{-1} \sum_{j=0}^{k} S_{jj}h$ and $\sigma^*h = \sup_k |\sigma_k h|$, $h \in L^1(T_2)$. Another partial summation shows that $|A| \leq \text{Const.}\, \sigma^*g$. We also define P^* and Q^* by

$$P^*h(x, y) = \sup_n \int_{T_1} K_n(y - u) \left\{ \sup_m \left| \int_{T_1} D_m(x - t)\, h(t, u)dt \right| \right\} du$$

and

$$Q^*h(x, y) = \sup_m \int_{T_1} K_m(x - t) \left\{ \sup_n \left| \int_{T_1} D_n(y - u)\, h(t, u)du \right| \right\} dt,$$

where $h \in L^1(T_2)$, $(x, y) \in T_2$ and K_n denotes the Fejér kernel. A partial summation in the expression for B yields

$$|B| \leq \text{Const.} \sup_k \{(k+1)^{-1} \sum_{j=0}^{k} S_{mj} g\} \leq \text{Const.}\ P^* g$$

and analogously it follows that $|C| \leq \text{Const.}\ Q^* g$.

We have shown that $S_2^* f \leq \text{Const.}\ (\sigma^* g + P^* g + Q^* g + T^* g)$. Using the boundedness in $L^2(T_1)$ of the operator S_1^* we can prove that P^* and Q^* are bounded operators on $L^2(T_2)$. Also T^* is bounded by the proof of Theorem 7.1 and the same holds for σ^* by Theorem 6.1. We therefore get

$$\|S_2^* f\|_2 \leq \text{Const.}\ \|g\|_2 = \text{Const.}\ (\sum_{m,n} |c_{mn}|^2\, l_{mn}^{-2})^{1/2}\ .$$

The convergence of $S_{mn} f$ follows from this inequality and the proof is complete.

Theorem 7.2 is slightly stronger than the theorem in [9], p. 95, in which Kaczmarz proves that $\sum_{m,n} |c_{mn}|^2 \log(|m|+2) \log(|n|+2) < \infty$ is a sufficient condition for the convergence a.e. of S_{mn}.

We give one more theorem on the convergence of the rectangular partial sums.

THEOREM 7.3. *Let $\{m_k\}_0^\infty$ be a sequence of integers such that $m_0 = 0$, $m_1 = 1$ and $m_{k+1}/m_k \geq q > 1$, $k = 1, 2, \ldots$ Assume that $p > 1$ and $f \in L^p(T_2)$. Then $\lim_{k,n\to\infty} S_{m_k,n} f(x, y) = f(x, y)$ for a.e. $(x, y) \in T_2$.*

Proof. Let $g \in L^1(T_1)$ and define $\Delta_0 g = S_0 g$ and $\Delta_k g = S_{m_k} g - S_{m_{k-1}} g$, $k = 1, 2, \ldots$ For fixed $y \in T_1$, we define $f'(x, y) = \sum_{k=0}^{\infty} \Delta_{2k+1}(f(\cdot, y); x)$ and $f''(x, y) = \sum_{k=0}^{\infty} \Delta_{2k}(f(\cdot, y); x)$. From Theorem (4.24) in [19], p. 233, it follows that f' and f'' are well-defined a.e. and that

$$f(x, y) = f'(x, y) + f''(x, y) \quad \text{a.e.} \tag{7.2}$$

and

$$\|f'\|_p \leq C_p \|f\|_p\,, \quad \|f''\|_p \leq C_p \|f\|_p\,, \tag{7.3}$$

where the norms are taken with respect to T_2. We define G_n by

$$G_n(x, y) = S_n(f(x, \cdot); y)$$

and we let G_n' and G_n'' be defined by the same formula with f replaced by f' and f''. (7.2) yields

$$S_{mn} f(x, y) = S_m(G_n'(\cdot, y); x) + S_m(G_n''(\cdot, y); x)\ .$$

From the definition of f' it follows that for fixed y as a function of x G_n' has a Fourier series with gaps. Hence the inequality (1.20) in [19], p. 164, implies that

$$\sup_k |S_{m_k}(G'_n(\cdot\,, y)\,; x)| \leq \text{Const.} \sup_m \int_{T_1} K_m(x-t)|G'_n(t\,, y)|dt \leq \text{Const.}\ P^*f'(x\,, y)\,,$$

where P^* is the operator defined in the proof of Theorem 7.2. Defining R^* by $R^*h = \sup_{k,n} |S_{m_k,n}h|$, $h \in L^1(T_2)$, we get $R^*f \leq \text{Const.}\,(P^*f' + P^*f'')$. The boundedness of the operator P^* on $L^p(T_2)$ combined with (7.3) yields $\|R^*f\|_p \leq C_p\|f\|_p$, and the theorem follows from this estimate.

8. Bochner — Riesz summability of multiple Fourier series

In this section let T_s denote the cube $\{x \in \mathbf{R}^s\,;\, |x_i| \leq \pi,\ \ i = 1\,, \ldots, s\}$. Let α be equal to the critical index $\frac{1}{2}(s-1)$ and, for $f \in L^1(T_s)$, let $S^\alpha_R f(x) = \sum_{|n|<R}\left(1 - \frac{|n|^2}{R^2}\right)^\alpha c_n e^{in\cdot x}$, where c_n are the Fourier coefficients of f. We define S^* by $S^*f(x) = \sup_{R>0} |S^\alpha_R f(x)|$. Hence for $s = 1$ S^* equals S^*_1. The operator S^* has been studied by E. M. Stein [12].

The operator M_1, defined in Section 4 by $M_1f(x) = \sup_{\xi\in\mathbf{R}}\left|\int_{-\pi}^{\pi} \frac{e^{-i\xi t}f(t)}{x-t}\,dt\right|$, has been used to estimate S^*_1. (See [7].) We will show that for all odd values of s there is a close connection between the operators S^* and M_1.

Assume that the functions Φ and Φ_1 defined for non-negative real numbers satisfy the following three conditions:

Φ is non-negative, convex, $\Phi(0) = 0$ and $\Phi(u)/u \to \infty$, $u \to \infty$ (cf. [18], p. 25). (8.1)

Either Φ_1 satisfies the same conditions as Φ and the inequality

$$\Phi_1(2u) \leq \text{Const.}\ \ \Phi_1(u)\,, u \geq 0, \ \text{ or } \ \Phi_1(u) \equiv u\,. \tag{8.2}$$

$$\Phi_1(u \log u) \leq \text{Const.}\ \Phi(u), \ \ u \geq 1\,. \tag{8.3}$$

Let $L_\Phi(T_s)$ denote the class of all measurable functions f on T_s for which $\int_{T_s} \Phi(|f(x)|)dx$ is finite. Then the following theorem holds.

THEOREM 8.1. *Suppose that* $\int_{T_1} \Phi_1(M_1f(x))dx \leq A \int_{T_1} \Phi(|f(x)|)dx + B$ *for all* $f \in L_\Phi(T_1)$ *for some constants* A *and* B. *Then, if* s *is odd, there exist constants* A' *and* B' *such that for* $f \in L_\Phi(T_s)$

$$\int_{T_s} \Phi_1(S^*f(x))dx \leq A' \int_{T_s} \Phi(|f(x)|)dx + B'\,.$$

Proof. Assume that f has period 2π in each variable and that $f \in L_\Phi(T_s)$. We have $S_R^\alpha f(x) = (2\pi)^{-s} \int_{T_s} D_R(y) f(x-y)dy$, where $D_R(x) = \sum_{|n|<R} \left(1 - \frac{|n|^2}{R^2}\right)^\alpha e^{in\cdot x}$. Let $g(x)$ equal $f(x)$ for $x \in 2T_s$ and let g vanish outside $2T_s$. Before considering S^*f we will study the functions $\sigma_R g(x) = (2\pi)^{-s} \int_{\mathbf{R}^s} H_R(y)\, g(x-y)dy$ and $\sigma^* g(x) = \sup_{R>0} |\sigma_R g(x)|$, where $H_R(x) = \int_{|y|<R} \left(1 - \frac{|y|^2}{R^2}\right)^\alpha e^{ix\cdot y} dy$. In [2], p. 308, Calderón and Zygmund observed that H_R can be written $H_R(x) = \psi(R|x|)|x|^{-s}$, where ψ is an odd Fourier — Stieltjes transform of a measure μ on the real line. Introducing polar coordinates and using the fact that ψ is odd we get

$$\sigma_R g(x) = (2\pi)^{-s} \int_{\mathbf{R}^s} \psi(R|y|)\ |y|^{-s} g(x-y)dy =$$

$$= (2\pi)^{-s} \int_S \left(\int_0^\infty \psi(Rt)\, t^{-1} g(x - ty')dt \right) d\sigma(y') =$$

$$= (2\pi)^{-s}\, \tfrac{1}{2} \int_S \left(\int_{\mathbf{R}} \psi(Rt)\, t^{-1} g(x - ty')dt \right) d\sigma(y') .$$

Denoting the inner integral by $\tilde{g}_R(x, y')$ we have

$$\tilde{g}_R(x, y') = \int_{\mathbf{R}} \left(\int_{\mathbf{R}} e^{-iRt\xi}\, t^{-1} g(x - ty')dt \right) d\mu(\xi)$$

and

$$\sup_R |\tilde{g}_R(x, y')| \leq \|\mu\| \sup_{\xi \in \mathbf{R}} \left| \int_{\mathbf{R}} e^{i\xi t}\, t^{-1} g(x - ty')dt \right| . \tag{8.4}$$

It follows that $\sigma^* g(x) \leq \text{Const.} \int_S \sup_R |\tilde{g}_R(x, y')| d\sigma(y')$ and (8.2) and Jensen's inequality for convex functions yield

$$\Phi_1(\sigma^* g(x)) \leq \text{Const.} \int_S \Phi_1(\sup_R |\tilde{g}_R(x, y')|)\, d\sigma(y') . \tag{8.5}$$

Using (8.4) and the assumption in the theorem we see that

$$\int_{T_s} \Phi_1(\sup |\tilde{g}_R(x, y')|)\, dx \leq \text{Const.} \int_{T_s} \Phi(|g(x)|)\, dx + \text{Const.} \leq$$

$$\leq \text{Const.} \int_{T_s} \Phi(|f(x)|)\, dx + \text{Const.}, \quad y' \in S .$$

Hence, integrating (8.5) we obtain

$$\int_{T_s} \Phi_1(\sigma^* g(x)) dx \leq \text{Const.} \int_{T_s} \Phi(|f(x)|)\, dx + \text{Const.} \tag{8.6}$$

It remains to compare S^*f with σ^*g. Defining $\Delta_R(x) = D_R(x) - H_R(x)$ we get

$$S^\alpha_R f(x) - \sigma_R g(x) = (2\pi)^{-s} \int_{T_s} \Delta_R(y) f(x-y) dy - (2\pi)^{-s} \int_{\mathbf{R}^s \setminus T_s} H_R(y)\, g(x-y) dy\,, \quad x \in T_s .$$

Using the estimate of Δ_R on p. 103 and the relation (4.4) on p. 105 in [13], we obtain

$$S^*f(x) \leq \sigma^* g(x) + \text{Const.} \int_{T_s} |f(t)| \log^+ |f(t)| dt + \text{Const.}, \quad x \in T_s .$$

Jensen's inequality combined with (8.2) and (8.3) now yields

$$\Phi_1(S^*f(x)) \leq \text{Const.} \left\{ \Phi_1(\sigma^* g(x)) + \int_{T_s} \Phi(|f(t)|) dt + 1 \right\}, \quad x \in T_s .$$

The theorem follows if we integrate this inequality and use (8.6).

Taking $\Phi_1(u) = u$, $\Phi(u) = u(\log^+ u)^2$ and using Theorem 2 in [7] we see that if s is odd then Theorem 8.1 implies the well-known result ([12], pp. 96—97)

$$\int_{T_s} S^*f(x) dx \leq \text{Const.} \int_{T_s} |f(x)| (\log^+ |f(x)|)^2 dx + \text{Const.} \tag{8.7}$$

One consequence of Theorem 8.1 is that if the estimate

$$\int_{T_1} M_1 f(x) dx \leq \text{Const.} \int_{T_1} |f(x)| (\log^+ |f(x)|)^2 dx + \text{Const.}$$

can be improved in the sense that $\Phi(u) = u(\log^+ u)^2$ can be replaced by a function $\Psi(u)$ such that $\lim_{u \to \infty} (\Psi(u) / \Phi(u)) = 0$, then (8.7) can be improved in the same way.

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