A proof of the Bieberbach conjecture

by

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In 1916, L. Bieberbach [2] conjectured that the inequality

$$|\alpha_n| \leq n|\alpha_1|$$

holds for every power series $\Sigma_{n=1}^{\infty} \alpha_n z^n$ with constant coefficient zero which represents a function with distinct values at distinct points of the unit disk. He also conjectured that equality holds with $n>1$ only for a constant multiple of the Koebe function

$$\frac{z}{(1+\omega z)^2}$$

where ω is a constant of absolute value one.

Bieberbach [2] verified the Bieberbach conjecture for the second coefficient. The Bieberbach conjecture for the third coefficient was verified by K. Löwner [9] in 1923. In 1955, P. R. Garabedian and M. Schiffer [7] verified the Bieberbach conjecture for the fourth coefficient. The Bieberbach conjecture for the sixth coefficient was verified in 1968 by R. N. Pederson [13] and, independently, by M. Ozawa [12]. In 1972, Pederson and Schiffer [14] verified the Bieberbach conjecture for the fifth coefficient. No other case of the Bieberbach conjecture has previously been verified.

A proof of the Bieberbach conjecture is now obtained for all remaining coefficients. Two other conjectures are also verified.

In 1936, M. S. Robertson [17] conjectured that the inequality

$$|\beta_1|^2+|\beta_2|^2+\ldots+|\beta_n|^2 \leq n|\beta_1|^2$$

holds for every odd power series $\Sigma_{n=1}^{\infty} \beta_n z^{2n-1}$ which represents a function with distinct values at distinct points of the unit disk. Such a power series is obtained from any

10–858285 *Acta Mathematica* 154. Imprimé le 27 Fevrier 1985

power series $\Sigma\, \alpha_n z^n$ with constant coefficient zero which represents a function with distinct values at distinct points of the unit disk through the identity

$$\left(\sum \beta_n z^{2n-1}\right)^2 = \sum \alpha_n z^{2n}.$$

An elementary argument shows that the Robertson conjecture implies the Bieberbach conjecture [17].

The logarithmic coefficients of a power series $f(z)$ with constant coefficient zero which represents a function with distinct values at distinct points of the unit disk are defined by the expansion

$$f(z) = zf'(0)\exp\left(2\sum_{n=1}^{\infty} \gamma_n z^n\right).$$

If $g(z)=\Sigma\, \beta_n z^{2n-1}$ is an odd power series such that $g(z)^2=f(z^2)$, then

$$\sum \beta_n z^{n-1} = \beta_1 \exp\left(\sum \gamma_n z^n\right).$$

In 1967, N. A. Lebedev and I. M. Milin [10] obtained the inequality

$$\frac{1}{r+1}\sum_{n=1}^{r+1} |\beta_n|^2 \leq |\beta_1|^2 \exp\left(\frac{1}{r+1}\sum_{n=1}^{r} (r+1-n)\left(n|\gamma_n|^2 - \frac{1}{n}\right)\right).$$

Equality holds if, and only if, a complex number ω of absolute value one exists such that $\gamma_n=\omega^n/n$ for $n=1,\ldots,r$. And in 1971, Milin [11] conjectured that the inequality

$$\sum_{n=1}^{r} (r+1-n)\, n|\gamma_n|^2 \leq \sum_{n=1}^{r} (r+1-n)\frac{1}{n}$$

holds for every positive integer r. Because of the Lebedev–Milin inequality, the Milin conjecture implies the Robertson conjecture and the Bieberbach conjecture.

A proof of the first three cases of the Milin conjecture was obtained by A. Ž. Grinšpan [8]. A proof of the remaining cases of the Milin conjecture is now obtained. The proof depends on a continuous application of the Riemann mapping theorem which is due to Löwner [9].

Löwner used the method to prove the Bieberbach conjecture for the third coefficient. In this approach the problem is to propagate information by means of a differential equation. For this purpose information has to be coded in a convenient form and then carried from one end of an interval to the other. An introduction to the methods which are now used will eventually appear in the author's monograph on "Square summable power series", which culminates in the proof of the Bieberbach conjecture.

Thanks are given to the members of the Leningrad Seminar in Geometric Function Theory for their confirmation of the proof of the Bieberbach conjecture during the author's stay at the V. A. Steklov Mathematical Institute in April, May and June of 1984.

For three weeks before the author's arrival, E. G. Emel'ianov prepared the seminar by presenting a paper [5] of the author, written in the fall of 1982 and brought to Leningrad by S. V. Hruščev, which contains an earlier form of the estimation theory. Emel'ianov was the first to confirm the proof of the Bieberbach conjecture, and he facilitated the work of the seminar by discovering a variant of the proof which required no knowledge of the motivating background in functional analysis.

It was Emel'ianov's argument which I. M. Milin and A. Ž. Grinšpan accepted as a proof of the Bieberbach conjecture. The author then worked with the seminar leader, G. V. Kuz'mina, to consolidate the findings of the seminar as represented by the written reports of Emel'ianov and Milin. These conclusions of the seminar are presented as a proof of the Bieberbach conjecture in an Academy of Sciences preprint [6].

Thanks are also due to the members of the Leningrad Seminar in Functional Analysis, in particular to V. P. Havin, S. V. Hruščev, and N. K. Nikol'skii, for examining the theories which naturally lead to the proof of the Bieberbach conjecture and which will appear in "Square summable power series".

It was the opinion of both Leningrad seminars that the proof of the Bieberbach conjecture should have an independent publication for the convenience of those who have no further interest in the theory. It was thought that such a publication, far from detracting from the main theory, would serve as an enticement to read a fuller treatment when it became available.

The author thanks Academician L. D. Faddeev for making available the facilities of the Mathematical Institute under the terms of the exchange agreement between the Academy of Sciences of the USA and the Academy of Sciences of the USSR.

The relevant information from the Löwner theory will first be stated. A power series $g(z)$ is said to be subordinate to a power series $f(z)$ if $g(z)=f(B(z))$ for a power series $B(z)$ with constant coefficient zero which represents a function which is bounded by one in the unit disk. If $f(z)$ and $g(z)$ are power series which represent functions which have distinct values at distinct points of the unit disk, then $g(z)$ is subordinate to

$f(z)$ if, and only if, the region onto which $g(z)$ maps the unit disk is contained in the region onto which $f(z)$ maps the unit disk.

A Löwner family is a family of power series $F(t, z)$, indexed by a positive parameter t, such that each series $F(t, z)$ has constant coefficient zero, has coefficient of z equal to t, and represents a function with distinct values at distinct points of the unit disk, and such that $F(a, z)$ is subordinate to $F(b, z)$ when $a<b$.

Assume that $f(z)$ and $g(z)$ are power series with constant coefficient zero and coefficient of z positive which represent functions with distinct values at distinct points of the unit disk. If $g(z)$ is subordinate to $f(z)$, then $f(z)$ and $g(z)$ are members of a Löwner family of power series $F(t, z)$.

Assume given a Löwner family of power series $F(t, z)$. Then the coefficients of $F(t, z)$ are absolutely continuous functions of t which satisfy the Löwner differential equation

$$t\frac{\partial}{\partial t}F(t, z) = \varphi(t, z)\, z\frac{\partial}{\partial z}F(t, z)$$

where $\varphi(t, z)$ is a power series with constant coefficient one which represents a function with positive real part in the unit disk for every index t, and the coefficients of $\varphi(t, z)$ are measurable functions of t.

Assume given a family of power series $\varphi(t, z)$ with constant coefficient one, t positive, which represent functions with positive real part in the unit disk. If the coefficients of $\varphi(t, z)$ are measurable functions of t, then a unique Löwner family of power series $F(t, z)$ exists which satisfies the Löwner differential equation with the given coefficient function $\varphi(t, z)$.

The present definition of a Löwner family differs from that of Löwner [9] who uses the logarithm of t as a parameter. Thus Löwner regards the underlying semigroup of substitution transformations as additive rather than multiplicative. And he makes use of the family for a smaller range of the parameter. The present variant of the Löwner theory is developed in previous work [3].

Some Hilbert spaces of power series, which arise in the theory of the Grunsky transformation [4], are used in estimating logarithmic coefficients. Assume that σ_n is a given function of positive integers n, with nonnegative numbers as values, such that $\sigma_{n+1} \leq \sigma_n$ for every n. Define $\mathcal{G}_\sigma$ to be the Hilbert space of equivalence classes of power series $f(z)=\Sigma\, a_n z^n$ with constant coefficient zero such that

$$\|f(z)\|^2_{\mathcal{G}_\sigma} = \sum n\sigma_n |a_n|^2$$

is finite. Equivalence of power series $f(z)$ and $g(z)$ with constant coefficient zero means that the coefficient of z^n in $f(z)$ is equal to the coefficient of z^n in $g(z)$ when σ_n is positive.

A family of spaces $\mathcal{G}_{\sigma(t)}$, $t\geq 1$, is said to be admissible if $\sigma_n(t)$ is a nonincreasing and absolutely continuous function of t and if the differential equation

$$\sigma_n(t)+\frac{t\sigma_n'(t)}{n}=\sigma_{n+1}(t)-\frac{t\sigma_{n+1}'(t)}{n+1}$$

holds for every postive integer n.

These conditions allow an estimate of logarithmic coefficients of bounded functions.

THEOREM 1. *Assume that an admissible family of spaces $\mathcal{G}_{\sigma(t)}$ is given such that $\sigma_1(t)$ is not identically zero but $\sigma_n(t)$ is eventually identically zero. Then the inequality*

$$\left\|\log\frac{B(z)}{zB'(0)}+f(B(z))\right\|^2_{\mathcal{G}_{\sigma(a)}}\leq\|f(z)\|^2_{\mathcal{G}_{\sigma(b)}}+4\sum_{n=1}^{\infty}\frac{\sigma_n(a)-\sigma_n(b)}{n}$$

holds for every element $f(z)$ of $\mathcal{G}_{\sigma(b)}$ and for every power series $B(z)$ with constant coefficient zero which represents a function which is bounded by one and has distinct values at distinct points of the unit disk, $1\leq a=b|B'(0)|$. Equality holds with $B'(0)$ positive if, and only if, a complex number ω of absolute value one exists such that

$$\frac{B(z)}{(1+\omega B(z))^2}=\frac{B'(0)\,z}{(1+\omega z)^2}$$

and such that the coefficient of z^n in $f(z)$ is equal to the coefficient of z^n in

$$-2\log(1+\omega z)$$

when $\sigma_n(t)$ is not identically zero.

Proof of Theorem 1. Since $B(z)$ can be replaced by $B(\lambda z)$ for a complex number λ of absolute value one, it can be assumed that the coefficient of z in $B(z)$ is positive. Then a Löwner family of power series $F(t, z)$ exists such that the identity $F(a, z)=F(b, B(z))$ is satisfied for given numbers a and b such that $1\leq a=bB'(0)$. The Löwner family can for example be chosen so that $F(b, z)$ is a constant multiple of z.

When $0<a\leq b<\infty$, a unique power series $B(b, a, z)$ with constant coefficient zero exists, which represents a function which is bounded by one and has distinct values at distinct points of the unit disk, such that

$$F(a, z) = F(b, B(b, a, z)).$$

The coefficients of $B(b, a, z)$ are absolutely continuous functions of a which satisfy the Löwner equation

$$a\frac{\partial}{\partial a}B(b, a, z) = \varphi(a, z)\, z\frac{\partial}{\partial z}B(b, a, z).$$

Note that the identity

$$\frac{\sigma_n(a)-\sigma_n(b)}{n}+\frac{\sigma_{n+1}(a)-\sigma_{n+1}(b)}{n+1} = \int_a^b \frac{\sigma_n(t)-\sigma_{n+1}(t)}{t}\,dt$$

is a consequence of the differential equations for the functions $\sigma_n(t)$. Since $\sigma_n(t)$ is eventually identically zero, it follows that

$$\int_a^b \frac{\sigma_1(t)-t\sigma_1'(t)}{t}\,dt = 2\sum_{n=1}^{\infty}\frac{\sigma_n(a)-\sigma_n(b)}{n}.$$

Hold b fixed. The desired inequality is now verified by showing that the expression

$$\left\|\log\frac{B(b, a, z)}{za/b}+f(B(b, a, z))\right\|^2_{\mathcal{G}_{\sigma(a)}}-2\int_a^b \frac{\sigma_1(t)-t\sigma_1'(t)}{t}\,dt$$

is a nondecreasing function of a, $a \leq b$. This is done by showing that the expression is an absolutely continuous function of a whose derivative is nonnegative almost everywhere.

For a computation of the expression, write

$$h(a, z) = \log\frac{B(b, a, z)}{za/b}+f(B(b, a, z))$$

for $a \leq b$, and observe that the differential equation

$$a\frac{\partial}{\partial a}h(a, z) = \varphi(a, z)\, z\frac{\partial}{\partial z}h(a, z)+\varphi(a, z)-1$$

is satisfied. Write

$$h(a, z) = \sum_{n=1}^{\infty} h_n(a)\, z^n.$$

The function of a which is to be shown nondecreasing is

$$\sum_{n=1}^{\infty} h_n(a)\bar{h}_n(a)\, n\sigma_n(a) - 2\int_a^b \frac{\sigma_1(t)-t\sigma_1'(t)}{t}\,dt.$$

Indeed the indefinite integral and each term in the sum is absolutely continuous. The desired conclusion is verified by showing that the result of formally applying the operator $a(\partial/\partial a)$ is nonnegative.

The expression to be shown nonnegative is

$$\sum_{n=1}^{\infty} ah_n'(a)\bar{h}_n(a)\, n\sigma_n(a) + \sum_{n=1}^{\infty} h_n(a)\, a\bar{h}_n(a)\, n\sigma_n(a)$$
$$+\sum_{n=1}^{\infty} h_n(a)\bar{h}_n(a)\, na\sigma_n'(a) + 2\sigma_1(a) - 2a\sigma_1'(a).$$

Use is now made of the Herglotz representation of a function which is analytic and has positive real part in the unit disk. For each real number a, a unique nonnegative measure $\mu(a,\cdot)$ exists on the Borel subsets of the unit circle such that the identity

$$\varphi(a,z) = \int \frac{1+\omega z}{1-\omega z}\, d\mu(a,\omega)$$

is satisfied in the sense of formal power series.

It is convenient to introduce the notation

$$s_n(a,\omega) = nh_n(a) + \omega(n-1)h_{n-1}(a) + \ldots + \omega^{n-1}h_1(a)$$

with the interpretation $s_0(a,\omega)=0$. An elementary calculation shows that

$$a\frac{\partial}{\partial a}h_n(a) = \int [2\omega^n + s_n(a,\omega) + \omega s_{n-1}(a,\omega)]\, d\mu(a,\omega)$$

and that

$$h_n(a) = \frac{s_n(a,\omega) - \omega s_{n-1}(a,\omega)}{n}.$$

The expression to be shown nonnegative is the integral with respect to $\mu(a,\cdot)$ of the sum

$$\sum_{n=1}^{\infty} [2\omega^n + s_n(a,\omega) + \omega s_{n-1}(a,\omega)]\, [s_n(a,\omega) - \omega s_{n-1}(a,\omega)]^{-}\sigma_n(a)$$

$$+\sum_{n=1}^{\infty}[s_n(a,\omega)-\omega s_{n-1}(a,\omega)]\,[2\omega^n+s_n(a,\omega)+\omega s_{n-1}(a,\omega)]^-\sigma_n(a)$$

$$+\sum_{n=1}^{\infty}|s_n(a,\omega)-\omega s_{n-1}(a,\omega)|^2\frac{a\sigma_n'(a)}{n}+2\sigma_1(a)-2a\sigma_1'(a)$$

$$=\sum_{n=1}^{\infty}2[\omega^n\bar{s}_n(a,\omega)-\omega^{n-1}\bar{s}_{n-1}(a,\omega)]\,\sigma_n(a)$$

$$+\sum_{n=1}^{\infty}2[\omega^{-n}s_n(a,\omega)-\omega^{1-n}s_{n-1}(a,\omega)]\,\sigma_n(a)$$

$$+\sum_{n=1}^{\infty}2[s_n(a,\omega)\,\bar{s}_n(a,\omega)-s_{n-1}(a,\omega)\,\bar{s}_{n-1}(a,\omega)]\,\sigma_n(a)$$

$$+\sum_{n=1}^{\infty}|\omega^{-n}s_n(a,\omega)-\omega^{1-n}s_{n-1}(a,\omega)|^2\frac{a\sigma_n'(a)}{n}+2\sigma_1(a)-2a\sigma_1'(a)$$

$$=\sum_{n=1}^{\infty}2\omega^n\bar{s}_n(a,\omega)\,[\sigma_n(a)-\sigma_{n+1}(a)]$$

$$+\sum_{n=1}^{\infty}2\omega^{-n}s_n(a,\omega)\,[\sigma_n(a)-\sigma_{n+1}(a)]$$

$$+\sum_{n=1}^{\infty}2s_n(a,\omega)\,\bar{s}_n(a,\omega)\,[\sigma_n(a)-\sigma_{n+1}(a)]$$

$$+\sum_{n=1}^{\infty}|\omega^{-n}s_n(a,\omega)-\omega^{1-n}s_{n-1}(a,\omega)|^2\frac{a\sigma_n'(a)}{n}+2\sigma_1(a)-2a\sigma_1'(a)$$

$$=\sum_{n=1}^{\infty}2|1+\omega^{-n}s_n(a,\omega)|^2\,[\sigma_n(a)-\sigma_{n+1}(a)]$$

$$+\sum_{n=1}^{\infty}|\omega^{-n}s_n(a,\omega)-\omega^{1-n}s_{n-1}(a,\omega)|^2\frac{a\sigma_n'(a)}{n}-2a\sigma_1'(a)$$

$$=\sum_{n=1}^{\infty}-2|1+\omega^{-n}s_n(a,\omega)|^2\left[\frac{a\sigma_n'(a)}{n}+\frac{a\sigma_{n+1}'(a)}{n+1}\right]$$

$$+\sum_{n=1}^{\infty}|\omega^{-n}s_n(a,\omega)-\omega^{1-n}s_{n-1}(a,\omega)|^2\frac{a\sigma_n'(a)}{n}-2a\sigma_1'(a)$$

$$=\sum_{n=1}^{\infty}-2|1+\omega^{-n}s_n(a,\omega)|^2\frac{a\sigma_n'(a)}{n}$$

$$+\sum_{n=1}^{\infty}-2|1+\omega^{1-n}s_{n-1}(a,\omega)|^2\frac{a\sigma_n'(a)}{n}$$

$$+\sum_{n=1}^{\infty}|\omega^{-n}s_n(a,\omega)-\omega^{1-n}s_{n-1}(a,\omega)|^2\frac{a\sigma_n'(a)}{n}$$

$$=\sum_{n=1}^{\infty}-|2+\omega^{-n}s_n(a,\omega)+\omega^{1-n}s_{n-1}(a,\omega)|^2\frac{a\sigma_n'(a)}{n}$$

$$\geqslant 0.$$

Note that all summations which appear are actually finite because of the hypothesis that $\sigma_n(t)$ is eventually identically zero. There is no difficulty in justifying the needed interchanges of summation with differentiation or integration. Note that, when $\sigma_n(t)$ is not identically zero, $\sigma_n'(t)$ is nonzero, except possibly at isolated points, because $\sigma_n(t)$ is a polynomial in $1/t$, as is shown in the proof of Theorem 3.

Equality holds if, and only if,

$$\int |2+\omega^{-n}s_n(a,\omega)+\omega^{1-n}s_{n-1}(a,\omega)|^2 d\mu(a,\omega)=0$$

whenever $\sigma_n'(a)$ is nonzero. Since $s_0(a,\omega)=0$ and $s_1(a,\omega)=h_1(a)$, the identity

$$\int |2+\omega^{-1}h_1(a)|^2 d\mu(a,\omega)=0$$

holds when $\sigma_1'(a)$ is not zero. It follows that $\mu(a,\cdot)$ is the measure with mass one concentrated at some point $\omega(a)$ and that $h_1(a)=-2\omega(a)$. The differential equation satisfied by $h(a,z)$ now implies that $\omega(a)=\omega$ is independent of a. These conclusions do apply since $\sigma_1(t)$ is not identically zero by hypothesis. Since

$$\varphi(a,z)=\frac{1+\omega z}{1-\omega z},$$

the desired form of $B(b,a,z)$ follows. The identity

$$2+\omega^{-n}s_n(a,\omega)+\omega^{1-n}s_{n-1}(a,\omega)=0$$

holds when $\sigma_n(t)$ is not identically zero. An inductive argument shows that

$$nh_n(a)=2(-1)^n\omega^n.$$

Thus, the coefficient of z^n in $h(z)$ is equal to the coefficient of z^n in

$$-2\log(1+\omega z).$$

This completes the proof of the theorem.

An estimate of logarithmic coefficients of unbounded functions is obtained on passing to a limit.

THEOREM 2. *Assume that an admissible family of spaces* $\mathcal{G}_{\sigma(t)}$ *is given such that* $\sigma_1(t)$ *is not identically zero but* $\sigma_n(t)$ *is eventually identically zero. Then the inequality*

$$\left\| \log \frac{F(z)}{zF'(0)} \right\|^2_{\mathcal{G}_{\sigma(a)}} \leq 4 \sum_{n=1}^{\infty} \frac{\sigma_n(a)}{n}$$

holds for every power series $F(z)$ *with constant coefficient zero which represents a function which has distinct values at distinct points of the unit disk,* $1 \leq a$. *Equality holds if, and only if, a complex number* ω *of absolute value one exists such that*

$$F(z) = \frac{F'(0)\, z}{(1+\omega z)^2}.$$

Proof of Theorem 2. Since $F(z)$ can be replaced by $F(\lambda z)$ for a complex number λ of absolute value one, it can be assumed that the coefficient of z in $F(z)$ is positive. Then a Löwner family of power series $F(t, z)$ exists which contains $F(z)$. Define the series $B(b, a, z)$ as in the proof of Theorem 1. Apply the estimate of Theorem 1 with

$$f(z) = \log \frac{F(b, z)}{bz}$$

and $B(z) = B(b, a, z)$. The inequality reads

$$\left\| \log \frac{F(a, z)}{az} \right\|^2_{\mathcal{G}_{\sigma(a)}} \leq \left\| \log \frac{F(b, z)}{bz} \right\|^2_{\mathcal{G}_{\sigma(b)}} + 4 \sum_{n=1}^{\infty} \frac{\sigma_n(a) - \sigma_n(b)}{n}.$$

If $F(z)$ represents a function which is bounded in the unit disk, then the choice of $F(b, z)$ can be made equal to a constant multiple of z and with $F(z) = F(a, z)$. The desired estimate of logarithmic coefficients follows immediately. If $F(z)$ represents an unbounded function in the disk, then

$$F(z) = \lim_{t \uparrow 1} F(tz)$$

is a limit of power series which represent bounded functions. The desired estimate of logarithmic coefficients now follows generally.

To determine the cases of equality, return to the Löwner family of functions $F(t, z)$. The identity

$$\lim_{t\to\infty} \left\| \log \frac{F(t,z)}{tz} \right\|_{\mathscr{G}_{\sigma(t)}} = 0$$

now follows because

$$\lim_{t\to\infty} \sigma_n(t) = 0$$

for every positive integer n. (See the proof of Theorem 3 for the form of the functions $\sigma_n(t)$ as polynomials in $1/t$ with constant coefficient zero.)

Another derivation of the inequality

$$\left\| \log \frac{F(a,z)}{az} \right\|^2_{\mathscr{G}_{\sigma(a)}} \leqslant 4 \sum_{n=1}^{\infty} \frac{\sigma_n(a)}{n}$$

is obtained from the inequality at the start of the proof by the arbitrariness of b. If equality holds, it holds in that previous inequality whenever $a \leqslant b$. By Theorem 1, a complex number ω of absolute value one exists such that

$$\frac{B(b,a,z)}{(1+\omega B(b,a,z))^2} = \frac{za/b}{(1+\omega z)^2},$$

a condition which implies that the coefficient $\varphi(t, z)$ in the Löwner equation is equal to

$$\frac{1+\omega z}{1-\omega z}$$

for $a \leqslant t \leqslant b$. The number ω is clearly independent of b.

By Theorem 1, it also follows that the coefficient of z^n in

$$\log \frac{F(b,z)}{bz}$$

is equal to the coefficient of z^n in

$$-2 \log(1+\omega z)$$

when $\sigma_n(t)$ is not identically zero. Because of the identity

$$\log \frac{F(a,z)}{az} = \log \frac{F(B(b,a,z))}{bB(b,a,z)} + \log \frac{B(b,a,z)}{az/b}$$

where $F(a,z)=F(z)$, the coefficient of z^n in

$$\log \frac{F(z)}{zF'(0)}$$

is equal to the coefficient of z^n in

$$-2\log(1+\omega z)$$

when $\sigma_n(t)$ is not identically zero. It follows that the coefficient of z^{n+1} in $F(z)$ is equal to the coefficient of z^{n+1} in

$$\frac{zF'(0)}{(1+\omega z)^2}$$

when $\sigma_n(t)$ is not identically zero. Since $\sigma_1(t)$ is not identically zero by hypothesis, $F(z)$ gives a case of equality in the Bieberbach conjecture for the second coefficient. As Bieberbach [2] shows, it follows that

$$F(z)=\frac{zF'(0)}{(1+\omega z)^2}.$$

This completes the proof of the theorem.

An inequality which is due to Richard Askey and George Gasper [1] is used to construct admissible families of spaces $\mathcal{G}_{\sigma(t)}$ such that $\sigma_n(t)$ is eventually identically zero. The notation

$$F(a,b,c;d,e;z)=1+\frac{a\cdot b\cdot c}{1\cdot d\cdot e}z+\frac{a(a+1)\,b(b+1)\,c(c+1)}{1\cdot 2\cdot d(d+1)\,e(e+1)}z^2+\dots$$

is used for the generalized hypergeometric series.

THEOREM 3. *Assume that $\mathcal{G}_\sigma$ is a given space such that the inequality*

$$\sigma_n-\sigma_{n+1}\geqslant\sigma_{n+1}-\sigma_{n+2}$$

holds for every positive integer n and such that σ_n is eventually zero. Then an admissible family of spaces $\mathcal{G}_{\sigma(t)}$ exists such that $\sigma_n(t)$ is eventually identically zero and such that

$$\sigma_n(1)=\sigma_n$$

for every positive integer n.

Proof of Theorem 3. Note that the hypotheses on the numbers σ_n are equivalent to the existence of nonnegative numbers p_r, all but a finite number of which are zero, for positive integers r such that the identity

$$\sigma_n = \sum_{r=1}^{\infty} p_r \max(r+1-n, 0)$$

holds for every positive integer n. Since the identity can be written

$$\sigma_n = \sum_{r=n}^{\infty} p_r(r+1-n),$$

it implies that

$$\sigma_n - \sigma_{n+1} = \sum_{r=n}^{\infty} p_r.$$

The required numbers p_r are unique and are given by

$$p_r = (\sigma_r - \sigma_{r+1}) - (\sigma_{r+1} - \sigma_{r+2}).$$

The solution of the system of differential equations is obtained in the form

$$\sigma_n(t)/n = \Delta_n(t) + \frac{(2n+2)}{(-1)}\Delta_{n+1}(t) + \frac{(2n+3)(2n+4)}{(-1)(-2)}\Delta_{n+2}(t) + \frac{(2n+4)(2n+5)(2n+6)}{(-1)(-2)(-3)}\Delta_{n+3}(t) + \ldots$$

where $\Delta_n(t)$ satisfies the elementary differential equation

$$t\Delta_n'(t) = -n\Delta_n(t)$$

with solution

$$\Delta_n(t) = \Delta_n(1)\,t^{-n}.$$

There is no difficulty about convergence because $\Delta_n(t)$ is eventually zero.

It is possible to solve for the functions $n\Delta_n(t)$ in terms of the functions $\sigma_n(t)$ because a square matrix with zeros below the diagonal and ones on the diagonal has an inverse of the same form. Thus, each function $n\Delta_n(t)$ is expressible as a linear combination of the functions $\sigma_k(t)$ with $k \geqslant n$, and the coefficient of $\sigma_n(t)$ is one. The coefficient of $\sigma_k(t)$ when $k>n$ is determined inductively in the unique way such that the differential equation for $\Delta_n(t)$ is satisfied. The resulting identity is

$$n\Delta_n(t) = \sigma_n(t) + \frac{(2n)}{1}\sigma_{n+1}(t) + \frac{(2n)(2n+1)}{1\cdot 2}\sigma_{n+2}(t) + \frac{(2n)(2n+1)(2n+2)}{1\cdot 2\cdot 3}\sigma_{n+3}(t) + \dots .$$

It is applied when $t=1$ to determine $\Delta_n(t)$.

By the remarks at the start of the proof, it is sufficient to give a proof of the theorem in the case that some positive integer r exists such that the identity

$$\sigma_n = \max(r+1-n, 0)$$

holds for every positive integer n. Then $\sigma_n(t)$ and $\Delta_n(t)$ are identically zero when $n>r$. When $n=1, \dots, r$, the identity

$$n\Delta_n(t) = (r+1-n)F(n-r, 2n; n-r-1; 1)\, t^{-n} = \frac{\Gamma(r+n+2)}{\Gamma(r+1-n)\Gamma(2n+2)} t^{-n}$$

is satisfied. It follows that the identity

$$\sigma_n(t)/n = \frac{\Gamma(r+n+2)}{\Gamma(r+1-n)\Gamma(2n+2)} \int_t^\infty \sum_{k=0}^{\infty} \lambda_k s^{-k-n-1}\, ds$$

is satisfied, where

$$\frac{\Gamma(r+n+2)}{\Gamma(r+1-n)\Gamma(2n+2)}\lambda_k = \frac{(2n+k+1)\dots(2n+2k)}{(-1)\dots(-k)} \frac{\Gamma(r+n+k+2)}{\Gamma(r+1-n-k)\Gamma(2n+2k+2)}.$$

Then $\lambda_0 = 1$ and, when $k>0$,

$$\frac{\lambda_k}{\lambda_{k-1}} = \frac{(n-r+k-1)(r+n+k+1)(n+k-\frac{1}{2})}{k(2n+k)(n+k+\frac{1}{2})}.$$

This obtains the identity

$$\sigma_n(t)/n = \frac{\Gamma(r+n+2)}{\Gamma(r+1-n)\Gamma(2n+2)} \int_t^\infty F(n-r, r+n+2, n+\tfrac{1}{2}; 2n+1, n+\tfrac{3}{2}; s^{-1})\, s^{-n-1}\, ds$$

when $n=1, \dots, r$. The theorem now follows because Askey and Gasper (in the proof of their third theorem) show that the integrand is nonnegative.

The Milin conjecture is a consequence of Theorems 2 and 3. Equality holds in the inequality of the conjecture only for a constant multiple of the Koebe function. Not only is the inequality conjectured by Bieberbach generally true, but equality holds only for a constant multiple of the Koebe function, as he conjectured.

The proof of the Milin conjecture here given differs from the argument verified by the Leningrad Seminar in Geometric Function Theory [6] in that it avoids approximation by the mapping functions of special slit regions and determines when equality holds in the inequality conjectured by Milin. The cases of equality were determined by Emel'ianov using the methods of the Leningrad Seminar [6], but this result was not included in the preprint because it is more easily obtained by the present method, which was known to the author in June 1984 but which could not have been included in the preprint without excessive delay.

A similar improvement of the preprint argument was discovered independently by C. H. FitzGerald and Ch. Pommerenke (informal communication). This variant of the proof was widely distributed in July and August of 1984 and was instrumental in obtaining general acceptance of it. Their argument removes some of the approximation required by the preprint argument by using a linear form of the Löwner equation, rather than the nonlinear form of the preprint. But their argument applies directly only to slit regions and relies on an approximation theorem, due to Löwner [9], for the general validity of the estimates. This complicates the determination of when equality holds.

The present technique for avoiding approximation uses a general form of the Löwner equation which is a speciality of Pommerenke [15] and which appears in his monograph on univalent functions [16]. The author takes this opportunity to thank him for having written this stimulating introduction to the theory.

The author thanks Walter Gautschi for supplying him with information about the work of Askey and Gasper which concluded the proof of the Bieberbach conjecture at the end of February 1984. The proof of the Bieberbach conjecture had otherwise been completed at the end of January. The Purdue computer was used by Gautschi in February to verify the remaining inequalities up to the thirtieth coefficient. This information served as an encouragement for further work as the Bieberbach conjecture had previously been generally doubted for odd coefficients, starting with the nineteenth.

The confirmation of the proof of the Bieberbach conjecture by the Leningrad Seminar in Geometric Function Theory took place in the middle of May 1984.

The events connected with the verification of the Bieberbach conjecture demonstrate the value of interdisciplinary and international cooperation!

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Received September 25, 1984
Received in revised form October 26, 1984