On the vanishing of and spanning sets for Poincaré series for cusp forms

by

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§ 0. Introduction and summary of results

0.1. Let Γ be a finitely generated non-elementary Kleinian group with region of discontinuity $\Omega=\Omega(\Gamma)$ and limit set $\Lambda=\Lambda(\Gamma)$. Let $\lambda(z)\,|dz|$ be the Poincaré metric on Ω (normalized to have constant negative curvature -1). Fix $q\in\mathbf{Z}$, $q\geqslant 2$. A *cusp form for* Γ *of weight* $(-2q)$ is a holomorphic function φ on Ω that satisfies

$$\varphi(\gamma z)\,\gamma'(z)^q=\varphi(z),\quad \text{all } z\in\Omega, \text{ all } \gamma\in\Gamma, \tag{0.1.1}$$

and

$$\iint_{\Omega/\Gamma}\lambda(z)^{2-q}|\varphi(z)\,dz\wedge d\bar z|<\infty. \tag{0.1.2}$$

Condition (0.1.2) is equivalent to

$$\sup\{\lambda(z)^{-q}|\varphi(z)|;\ z\in\Omega\}<\infty. \tag{0.1.3}$$

Denote by $\mathbf{A}_q(\Omega,\Gamma)$ the space of cusp forms for Γ of weight $(-2q)$.

If $\infty\in\Omega$, then for $\varphi\in\mathbf{A}_q(\Omega,\Gamma)$,

$$\varphi(z)=O(|z|^{-2q}),\quad z\to\infty.$$

We will be studying spaces of rational functions. A rational function f will be considered to be *holomorphic at* ∞ if

$$f(z)=O(|z|^{-2q}),\quad z\to\infty.$$

We will consider it to have a *simple pole at* ∞ if

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$$\lim_{z \to \infty} z^{2q-1} f(z)$$

is finite and non zero.

It will be convenient to work with the following set that contains the limit set. Let $\Lambda_q = \Lambda_q(\Gamma)$ denote the union of the limit set Λ with the set of fixed points $z_0 \in \Omega$ of elliptic elements whose stabilizers in Γ are subgroups of order ν *not* a factor of $q-1$; that is, with ν satisfying

$$q-1 \not\equiv 0 \pmod{\nu}. \tag{0.1.4}$$

Note that Λ_2 contains all elliptic fixed points in Ω; while Λ_3, for example, does not contain the elliptic fixed points in Ω whose stabilizers are of order 2.

We let $\mathscr{R}_q(\Lambda_q)$ denote the space of rational functions f that satisfy the following conditions:

(i) f is holomorphic on $\mathbf{C} \cup \{\infty\} \setminus \Lambda_q$,
(ii) all the poles of f at finite points in Λ_q are simple,
(iii) $f(z) = O(|z|^{-2q})$, $z \to \infty$ if $\infty \notin \Lambda_q$, and
(iv) $f(z) = O(|z|^{-(2q-1)})$, $z \to \infty$ if $\infty \in \Lambda_q$.

In view of the convention introduced earlier, we can reformulate the definition of $\mathscr{R}_q(\Lambda_q)$. A rational function $f \in \mathscr{R}_q(\Lambda_q)$ if and only if all the poles are simple and are located in Λ_q.

To simplify notation, we introduce an operator on functions as follows. If A is a Möbius transformation and $2n, 2m \in \mathbf{Z}$ with $n+m \in \mathbf{Z}$, then for every function f on a domain D, we define

$$(A_{n,m}^* f)(z) = f(Az) A'(z)^n \, \overline{A'(z)^m}, \quad z \in A^{-1}(D). \tag{0.1.5}$$

Abbreviate $A_{n,0}^*$ by A_n^*.

We note that $\mathscr{R}_q(\Lambda_q)$ is invariant under the set of linear operators $\{A_q^*; A \in \Gamma\}$.

We can hence define for $f \in \mathscr{R}_q(\Lambda_q)$,

$$f(\infty) = (-1)^q \lim_{z \to \infty} z^{2q} f(z). \tag{0.1.6}$$

This is a well defined concept; the limit takes values in the extended plane.

If S is a subset of Λ_q, we let

$$\mathscr{R}_q(S) = \{f \in \mathscr{R}_q(\Lambda_q); f \text{ is holomorphic on } \Lambda_q \setminus S\}.$$

If $f\in \mathcal{R}_q(\Lambda_q)$, the then *Poincaré series*

$$\sum_{\gamma\in\Gamma} f(\gamma z)\,\gamma'(z)^q, \quad z\in\Omega, \tag{0.1.7}$$

converges absolutely and uniformly on compact subsets of Ω, and yields a cusp form $\Theta_q f\in \mathbf{A}_q(\Omega,\Gamma)$ (see Proposition 1.5). Bers [4] has shown that

$$\Theta_q\colon \mathcal{R}_q(\Lambda)\to \mathbf{A}_q(\Omega,\Gamma)$$

is surjective. The starting point of this investigation was the following quantitative improvement of Bers' result.

THEOREM 1. *Let $a_1,\ldots,a_{2q-1}$ be $(2q-1)$ distinct points in Λ_q. Let $\gamma_0=I$, $\gamma_1,\ldots,\gamma_N$ be generators for Γ. Let*

$$S=\{\gamma_j(a_k);\ k=1,\ldots,2q-1, j=0,\ldots,N\}. \tag{0.1.8}$$

Then

$$\Theta_q(\mathcal{R}_q(S))=\mathbf{A}_q(\Omega,\Gamma).$$

Note that

$$\dim \mathcal{R}_q(S)\leqslant (2q-1)\,N. \tag{0.1.9}$$

We will see that in many instances it is possible, as a consequence of Theorem 1, to choose from a presentation for Γ, a set S consisting of precisely $(2q-1)+\dim \mathbf{A}_q(\Omega,\Gamma)$ points, the points in S depending on only *finitely* many choices (see § 7 and § 8) so that

$$\Theta_q\colon \mathcal{R}_q(S)\to \mathbf{A}_q(\Omega,\Gamma)$$

is an isomorphism. The points of S will depend holomorphically on moduli and will give a global trivialization of the vector bundle of cusp forms of weight $(-2q)$ over the deformation space of Γ (see § 11).

In [17] bases for $\mathbf{A}_2(\Omega,\Gamma)$ were obtained by different methods for Γ geometrically finite function groups. Hejhal [11] has obtained spanning sets for Poincaré series for Fuchsian groups, also by different methods. Bases for Schottky groups were found by Bers [5], for arbitrary q, and Hejhal [10], for low values of q. Wolpert [27] obtained bases for $q=2$ and Γ Fuchsian, by studying deformations of conformal structures. The methods of this paper are cohomological (see § 0.2). The idea for Theorem 1 is already present in Bers [4], where cohomological ideas were used but not pushed to the limit, and Hejhal [11, p. 357, example 2], where an explicit basis is constructed for $q=2$ and Γ a generic Fuchsian group of type $(2,0)$. The reader is referred to [17] for more bibliographical remarks about this and some other problems considered in this paper.

Remark. Using the notation introduced in (0.1.5), we can rewrite (0.1.7) as

$$\Theta_q f = \sum_{\gamma \in \Gamma} \gamma_q^* f, \quad f \in \mathcal{R}_q(\Lambda_q).$$

0.2. We turn now to the more interesting *vanishing problem* raised by Poincaré [24, p. 249] (see also Petersson [23] and Hejhal [11]): Let Δ be a Γ-invariant union of components of the Kleinian group Γ. Find necessary and sufficient conditions for $\Theta_q f | \Delta$ to vanish (identically) for a given $f \in \mathcal{R}_q(\Lambda_q)$.

To begin the discussion of the vanishing problem, we introduce the subspace

$$\mathbf{A}_q(\Delta, \Gamma) = \{\varphi \in \mathbf{A}_q(\Omega, \Gamma);\ \varphi = 0 \text{ on } \Omega \setminus \Delta\},$$

and proceed to describe the Eichler cohomology groups.

Let D be an arbitrary Γ-invariant subset of $\mathbf{C} \cup \{\infty\}$. We let Γ *act on the right on* functions F on D by the formula

$$F \cdot \gamma = \gamma_{1-q}^* F, \quad \gamma \in \Gamma.\text{(}^1\text{)} \tag{0.2.1}$$

The vector space Π_{2q-2} of polynomials in one complex variable of degree at most $2q-2$ is invariant under this action; that is,

$$p \cdot \gamma \in \Pi_{2q-2} \quad \text{whenever } p \in \Pi_{2q-2} \text{ and } \gamma \in \Gamma. \tag{0.2.2}$$

Of course, formula (0.2.2) is valid on $\mathbf{C} \cup \{\infty\}$. A mapping $\chi: \Gamma \to \Pi_{2q-2}$ is a *cocycle* provided

$$\chi(\gamma_1 \circ \gamma_2) = \chi(\gamma_1) \cdot \gamma_2 + \chi(\gamma_2), \quad \gamma_1, \gamma_2 \in \Gamma.$$

Such a cocycle is a *coboundary* if

$$\chi(\gamma) = p \cdot \gamma - p, \quad \gamma \in \Gamma,$$

for some fixed $p \in \Pi_{2q-2}$. The *cohomology space* $H^1(\Gamma, \Pi_{2q-2})$ is defined as the

(¹) The action of Γ on Π_{2q-2} and on Eichler integrals (defined below) will be denoted by a dot $(\cdot)$. This should not be confused with composition of mappings denoted by the usual symbol $(\circ)$.

vector space of cocycles modulo the vector space of coboundaries. We remark that if Γ is generated by N elements, then

$$\dim H^1(\Gamma, \Pi_{2q-2}) \leq (2q-1)(N-1), \tag{0.2.3}$$

and equality holds in (0.2.3) whenever Γ is a free group on N generators. Inequality (0.2.3) should be compared with (0.1.9).

Let $A \in \Gamma$ be a parabolic element. A cocycle χ is *parabolic with respect to* A provided there is a $v \in \Pi_{2q-2}$ such that

$$\chi(A) = v \cdot A - v. \tag{0.2.4}$$

A cocycle is called *parabolic* if it is parabolic with respect to all parabolic elements of Γ; while it is called Δ-*parabolic* if it is parabolic with respect to every parabolic element in Γ determined by a puncture on Δ/Γ (here Δ is, as before, an invariant union of components of Γ). The image in $H^1(\Gamma, \Pi_{2q-2})$ of the set of parabolic (respectively, Δ-parabolic) cocycles is denoted by $PH^1(\Gamma, \Pi_{2q-2})$ (respectively, $PH^1_\Delta(\Gamma, \Pi_{2q-2})$).

A function F on D (a Γ-invariant subset of $\mathbf{C} \cup \{\infty\}$) will be called an *Eichler integral* (*for* Γ, *with support on* D, *of order* $1-q$) provided that for each $\gamma \in \Gamma$, there is a polynomial $\chi(\gamma) \in \Pi_{2q-2}$ so that

$$F \cdot \gamma - F = \chi(\gamma) | D. \tag{0.2.5}$$

In this case χ is a cocycle for Γ and we will call it, the *period of the Eichler integral* F, $\operatorname{pd} F$. Every cocycle is the period of some smooth Eichler integral supported on Ω (see [14, pp. 180–186]).(²) We will be mostly interested in Eichler integrals supported on Λ_q.

We note that every $p \in \Pi_{2q-2}$ is an Eichler integral (supported on $\mathbf{C} \cup \{\infty\}$) and that the value of an Eichler integral at ∞ is given by

$$F(\infty) = (-1)^{1-q} \lim_{z \to \infty} z^{2-2q} F(z), \tag{0.2.6}$$

provided that this limit exists.

For $\varphi \in \mathbf{A}_q(\Omega, \Gamma)$, $\mu = \lambda^{2-2q} \bar{\varphi}$ is called *canonical generalized Beltrami differential* for Γ.(³) A continuous function F on $\mathbf{C}$ is called a *potential* for μ provided

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(²) In [14], the term Eichler integral was reserved for holomorphic and meromorphic functions F that satisfy (0.2.5). It seems appropriate to consider more general functions under the same title.

(³) In [14], generalized Beltrami differentials were called generalized Beltrami coefficients. It is traditional for $q=2$ to call a Beltrami differential, a Beltrami coefficient, if and only if its norm is less than one.

$$F(z) = O(|z|^{2q-2}), \quad z \to \infty, \tag{0.2.7}$$

and

$$\frac{\partial F}{\partial \bar{z}} = \mu \tag{0.2.8}$$

in the sense of generalized derivatives. Such a potential F is said to *vanish at* ∞ if (0.2.7) is strengthened to

$$F(z) = o(|z|^{2q-2}), \quad z \to \infty.$$

Let $a_1, \dots, a_{2q-1}$ be $(2q-1)$ distinct points in Λ_q (as in Theorem 1). Then for $z \in \mathbf{C}$,

$$F(z) = \frac{(z-a_1)\dots(z-a_{2q-1})}{2\pi i} \iint_{\Omega} \frac{\mu(\zeta)\, d\zeta \wedge d\bar{\zeta}}{(\zeta-z)(\zeta-a_1)\dots(\zeta-a_{2q-1})} \tag{0.2.9}$$

defines the unique potential $F=F_\varphi$ for μ that vanishes at a_k, $k=1, \dots, 2q-1$. We introduce the convention that if $a_k=\infty$ for some k, then the terms $(z-a_k)$ and $(\zeta-a_k)$ are dropped from the formula (0.2.9).

We define $\beta^*(\varphi)$ to be the cohomology class of $\mathrm{pd}\, F_\varphi$. We obtain this way the *Bers map*

$$\beta^*: \mathbf{A}_q(\Delta, \Gamma) \to PH^1(\Gamma, \Pi_{2q-2}).\text{(4)} \tag{0.2.10}$$

The map β^* is conjugate linear, and injective. The injectivity of β^* is the crucial fact that will yield (among other things) Theorem 1. This deep result is due to Ahlfors [1] for $q=2$ and to Bers [3] for $q>2$. A reference for most of the material of this section is the monograph [14].

Having fixed $(2q-1)$ distinct points $a_1, \dots, a_{2q-1}$ in Λ_q, we define

$$\mathcal{F}_{1-q}(\Delta, \Gamma) = \{\text{restrictions to } \Lambda_q \text{ of potentials } F_\psi \text{ that vanish at } a_k,\ k=1, \dots, 2q-1, \text{ with } \psi \in \mathbf{A}_q(\Delta, \Gamma)\}.$$

We construct next a basis for $\mathcal{R}_q(\Lambda_q)$ *adapted* to the $(2q-1)$ distinct points $a_1, \dots, a_{2q-1}$. For $z \in \Lambda_q$, $z \neq a_j$, $z \neq \infty$, $\zeta \in \Omega$, set

$$f(z, \zeta) = \frac{-1}{2\pi} \frac{1}{\zeta - z} \prod_{j=1}^{2q-1} \frac{z-a_j}{\zeta-a_j}, \tag{0.2.11}$$

(with the usual convention if $a_j=\infty$ for some j) and if $\infty \in \Lambda_q$, $\infty \neq a_j$,

(4) It is shown in [12], [13] that β^* is the splitting map for a map $\beta: H^1(\Gamma, \Pi_{2q-2}) \to \mathbf{A}_q(\Delta, \Gamma)$.

$$f(\infty,\zeta)=(-1)^q\prod_{j=1}^{2q-1}\frac{1}{\zeta-a_j}. \tag{0.2.12}$$

If $f\in\mathcal{R}_q(\Lambda_q)$, then we can find $m\geq 1$ distinct points

$$b_1,\dots,b_m\in\Lambda_q\setminus\{a_1,\dots,a_{2q-1}\}$$

and complex numbers $\beta_1,\dots,\beta_m$ so that

$$f(\zeta)=\sum_{j=1}^{m}\beta_j f(b_j,\zeta),\quad \zeta\in\mathbf{C}. \tag{0.2.13}$$

The points $b_1,\dots,b_m$ and the constants $\beta_1,\dots,\beta_m$ are uniquely determined by f. We define a surjective linear map

$$\mathcal{K}\colon\mathcal{R}_q(\Lambda_q)\to\mathcal{F}_{1-q}(\Omega,\Gamma)^*$$

from $\mathcal{R}_q(\Lambda_q)$ to the dual space $\mathcal{F}_{1-q}(\Omega,\Gamma)^*$ of $\mathcal{F}_{1-q}(\Omega,\Gamma)$ by the formula

$$\mathcal{K}(f)(F)=\sum_{j=1}^{m}\beta_j F(b_j),\quad F\in\mathcal{F}_{1-q}(\Omega,\Gamma), \tag{0.2.14}$$

where $f\in\mathcal{R}_q(\Lambda_q)$ is given by (0.2.13).

Remark. In defining $\mathcal{K}$ we must insist that the same $(2q-1)$ distinct points be used for normalization of the functions in $\mathcal{F}_{1-q}(\Omega,\Gamma)$ (that is, all the potentials must vanish at these points) *and* for the basis for $\mathcal{R}_q(\Lambda_q)$ (that is, all functions appearing in the basis must have poles at these points).

THEOREM 2. *Given $f\in\mathcal{R}_q(\Lambda_q)$, then*

$$\Theta_q f|\Delta=0 \quad\Leftrightarrow\quad \mathcal{K}(f)|\mathcal{F}_{1-q}(\Delta,\Gamma)=0.$$

Since $\mathcal{K}$ is such a simple algebraic operator, the solution to the vanishing problem is hence reduced to describing a basis for the space $\mathcal{F}_{1-q}(\Delta,\Gamma)$. The above theorem is a little more than an exercise in definition chasing. However, *when combined with Theorems* 3 *and* 4, it will turn out to be a powerful tool in attacking the vanishing problem (see, for example, § 0.7).

0.3. In many cases, the spaces $\mathcal{F}_{1-q}(\Omega,\Gamma)$ can be determined by essentially algebraic data.

THEOREM 3. *If the Bers map*

$$\beta^*\colon \mathrm{A}_q(\Omega, \Gamma) \to PH^1(\Gamma, \Pi_{2q-2}) \tag{0.3.1}$$

is surjective, then $\mathscr{F}_{1-q}(\Omega, \Gamma)$ *can be determined algebraically from the parabolic cocycles for* Γ.

We must explain the content of the theorem. Let $a_1, \ldots, a_{2q-1}$ be fixed points of loxodromic elements of Γ. Given a basis for the parabolic cocycles for Γ, we can from this basis construct the values, at the loxodromic fixed points (*and* certain other points), of functions that form a basis for $\mathscr{F}_{1-q}(\Omega, \Gamma)$. (See § 4.1.)

In many cases (for Schottky groups, § 7, and quasi-Fuchsian groups, § 8, for example), the description of $PH^1(\Gamma, \Pi_{2q-2})$ involves only linear algebra—the evaluation of kernels of linear operators, mostly. Thus, for groups with rather simple algebraic presentations, and for rational functions with poles only at (loxodromic) fixed points, Poincaré's vanishing problem has a purely algebraic solution.

0.4. The Bers map β^* of (0.3.1) is surjective whenever Γ is a Fuchsian, quasi-Fuchsian, or Schottky group [14, p. 215]. Nakada [22] has shown that for $q=2$, the map β^* is surjective for every geometrically finite function group.(5) Thus in principle, given such a group Γ, one should be able to construct a finite algorithm to decide whether or not $\Theta_q f=0$ for a given $f \in \mathscr{R}_q(\Lambda_q)$. We state the most explicit construction of such an algorithm in

THEOREM 4. *Let* Γ *be a Schottky group or a finitely generated Fuchsian or quasi-Fuchsian group of the first kind given by a standard presentation on a canonical set of generators. Let* $f \in \mathscr{R}_q(\Lambda_q)$ *have poles only at fixed points. Then there exists a finite algebraic algorithm that determines whether or not* $\Theta_q f=0$.

Theorem 4 contains a solution to Poincaré's vanishing problem. The first algebraic decision procedure for the solution of the vanishing problem was obtained by Hejhal [10], [11]. His methods apply to a more limited class of groups (covering groups of compact Riemann surfaces of genus $g>1$ and Schottky groups) and low values of q. Our approach has the advantage of greater algebraic simplicity (see § 0.7). Transcendental methods involving information about Weierstrass point can lead to vanishing theorems (see Petersson [23]). There are also formal reasons why a Poincaré series vanishes. See Ljan [18] and Metzger [20], as well as § 6.

(5) The results of [21] should yield the same result for all q.

0.5. Let Γ be a finitely generated Fuchsian group of the first kind acting on the unit disk Δ. Then $\Lambda=\partial\Delta$, the unit circle, and $\Omega=\{z\in\mathbf{C}\cup\{\infty\};|z|\neq1\}$. It is of interest to determine when a Poincaré series vanishes only on Δ. This involves determining $\mathcal{F}_{1-q}(\Delta,\Gamma)$ from $\mathcal{F}_{1-q}(\Omega,\Gamma)$.

THEOREM 5. *Let Γ be a finitely generated Fuchsian group of the first kind acting on the unit disk Δ. Then there exists a positive integer $m=m(\Gamma,q)$ such that for $F\in\mathcal{F}_{1-q}(\Omega,\Gamma)$, we have*

$$F\in \mathcal{F}_{1-q}(\Delta,\Gamma) \quad\Leftrightarrow\quad \int_0^{2\pi} e^{i(1-k-2q)\theta}F(e^{i\theta})\,d\theta = 0, \quad \textit{for } k=0,1,\ldots,m.$$

0.6. Combining Theorems 1 and 2 can lead to interesting consequences. An example is the theorem below.

Definition. A point $a\in\Lambda_q$ will be called a *q-uniqueness* point (or a *uniqueness* point) if either

(a) a is a fixed point of a loxodromic element of Γ, or

(b) $a\in\Omega$, or

(c) the maximal finite cyclic subgroup stabilizing a has order ν satisfying (0.1.4).

The set of q-uniqueness points for Γ will be denoted by Λ_q^0. Note that Λ_q^0 is a dense subset of Λ_q.

Remarks. (1) Non fixed points are never uniqueness points. We shall call them *non-uniqueness* points.

(2) Parabolic and elliptic fixed points may or may not be uniqueness points.

THEOREM 6. *Let Γ be a finitely generated non-elementary Kleinian group. Assume that*

$$\beta^*\colon \mathbf{A}_q(\Omega,\Gamma)\to H^1(\Gamma,\Pi_{2q-2})$$

is surjective. Let $\Lambda_0\subset\Lambda_q$ be a non-empty Γ-invariant set consisting only of q-uniqueness points. Then

$$\Theta_q\colon \mathcal{R}_q(\Lambda_0)\to \mathbf{A}_q(\Omega,\Gamma)$$

is surjective and its kernel is spanned by

$$\{f-A_q^*f;\ f\in \mathcal{R}_q(\Lambda_0),\ A\in\Gamma\}.$$

0.7. In this paragraph we illustrate the main ideas in the proofs of Theorems 3 and 4, and outline the algebraic procedure for determining whether or not a Poincaré series of a rational function vanishes. Let Γ be a non-elementary finitely generated Kleinian group with the following two properties.

(i) the map β^* of (0.3.1) is surjective, and
(ii) one can construct algebraically a basis for $PH^1(\Gamma, \Pi_{2q-2})$.

Fix $(2q-1)$ distinct points $a_1, \ldots, a_{2q-1}$ in Λ. Assume there are loxodromic elements L_j in Γ with $L_j(a_j)=a_j$, $j=1, \ldots, 2q-1$. Let $f \in \mathcal{R}_q(\Lambda)$ be given by (0.2.13) and assume that there are loxodromic elements $T_j \in \Gamma$ with $T_j(b_j)=b_j$, $j=1, \ldots, m$. How do we determine whether or not φ, the Poincaré series of f, vanishes on Ω? Note that (see (2.4.1))

$$\varphi(\zeta) = \sum_{j=1}^{m} \beta_j \varphi(b_j, \zeta), \quad \zeta \in \Omega,$$

where $\varphi(b_j, \cdot)$ is the Poincaré series of $f(b_j, \cdot)$. Let $\{\psi_k\}_{k=1}^{d}$ be any convenient basis for $A_q(\Omega, \Gamma)$. *This basis is needed in the proof of the theorems; not in the algorithm for deciding whether* $\varphi=0$ *or* $\varphi \neq 0$. Let F_k be a potential for $\lambda^{2-2q}\bar{\psi}_k$ that vanishes at a_j, $j=1, \ldots, 2q-1$, and use the Petersson scalar product (2.4.2) to conclude that (see (2.4.3))

$$F_k(b_j) = \langle \varphi(b_j, \cdot), \psi_k \rangle_\Gamma.$$

Thus

$$\varphi = 0 \iff \sum_{j=1}^{m} \beta_j F_k(b_j) = 0, \quad k = 1, \ldots, d.$$

Let $P_k = \operatorname{pd} F_k$, $k=1, \ldots, d$. Let $\{Q_k\}_{k=1}^{d}$ be *any explicit* basis for $PH^1(\Gamma, \Pi_{2q-2})$; that is, the cohomology classes of these cocycles project to a basis for $PH^1(\Gamma, \Pi_{2q-2})$. It involves no loss of generality to assume that $\beta^*(\psi_k)=Q_k$, $k=1, \ldots, d$. To complete the algorithm we must compute $F_k(b_j)$ algebraically.

To evaluate P_k, observe that for $k=1, \ldots, d$, there is an $R_k \in \Pi_{2q-2}$, such that

$$Q_k(T)(z) = P_k(T)(z) + R_k(Tz)\, T'(z)^{1-q} - R_k(z),$$

for all $T \in \Gamma$, all $z \in \mathbf{C}$. Hence

$$Q_k(L_j)(a_j) = 0 + R_k(a_j)\, [L_j'(a_j)^{1-q} - 1],$$

for $j=1, \ldots, 2q-1$. Knowledge of $R_k(a_j)$ for $1 \leq j \leq 2q-1$, determines R_k. We can now compute $P_k(T)(z)$, hence in particular $P_k(T_j)(b_j)$. This computation can be quite lengthy if T_j is a long word in the generators of Γ. In any event, we are almost home:

$$F_k(T_j z)\,T_j'(z)^{1-q}-F_k(z)=P_k(T_j)(z),$$

$$F_k(b_j)\,[T_j'(b_j)^{1-q}-1]=P_k(T_j)(b_j),$$

$$F_k(b_j)=\frac{P_k(T_j)(b_j)}{T_j'(b_j)^{1-q}-1},\quad 1\leqslant j,\ k\leqslant d.$$

Remark. If Γ is Fuchsian and $\Gamma\subset PSL(2,\mathbf{R})$, then for symmetric rational functions (those $f\in\mathcal{R}_q(\Lambda_q)$ with $\overline{f(\bar z)}=f(z)$, all $z\in\mathbf{C}$), the vanishing of $\Theta_q f$ on Ω is equivalent to its vanishing on the upper half plane (because $\overline{\Theta_q f(\bar z)}=\Theta_q f(z)$, all $z\in\Omega$). Hence for such functions Theorem 5 is *not* needed.

Bibliographical remarks. This work is based on the development of the Eichler cohomology machinery in the fundamental papers of Ahlfors [1], [2], and Bers [3], as well as in the author's papers [12], [13]. The main results of these five papers are the principal subjects of the author's mostly expository monograph [14]. For the convenience of the reader, references will most often cite [14] rather than the original sources. Slightly weaker versions of most of the above results were announced in [15].

For Fuchsian groups an essential difference between our methods and those of [11] is that we work with Poincaré series while Hejhal uses relative Poincaré series. Readers interested in pursuing the precise relationships between these methods and those of Hejhal should consult [11] equations (65), (77), (111) and page 373 lines 19–25. Additional remarks on the comparison of the two approaches will appear in Hejhal's forthcoming elaboration of [11]. It should be pointed out that both approaches are powerful tools for effectively

(i) deciding when a Poincaré series of a rational function vanishes (identically), and

(ii) constructing explicit bases for the spaces of cusp form consisting (respectively) of Poincaré series and relative Poincaré series.

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§ 1. Some preliminaries (computations involving rational functions)

1.1. If Γ is a finitely generated non-elementary Kleinian group with region of discontinuity $\Omega=\Omega(\Gamma)$ and limit set $\Lambda=\Lambda(\Gamma)$, then for any Möbius transformation A, $A\Gamma A^{-1}$ is a finitely generated non-elementary Kleinian group with region of discontinuity $A(\Omega)$ and limit set $A(\Lambda)$. Furthermore, $A(\Lambda_q)=A(\Lambda_q(\Gamma))=\Lambda_q(A\Gamma A^{-1})$. The operators

$$A_q^*\colon \mathbf{A}_q(A(\Omega), A\Gamma A^{-1}) \to \mathbf{A}_q(\Omega, \Gamma)$$

and

$$A_q^*\colon \mathcal{R}_q(A(\Lambda_q)) \to \mathcal{R}_q(\Lambda_q)$$

are (surjective) $\mathbf{C}$-linear isomorphisms.

If we denote the Poincaré series operator Θ_q for the group G by $\Theta_{q,G}$, then the following is a commutative diagram:

$$\begin{array}{ccc} \mathcal{R}_q(A(\Lambda_q)) & \xrightarrow{A_q^*} & \mathcal{R}_q(\Lambda_q) \\ \downarrow{\scriptstyle \Theta_{q,A\Gamma A^{-1}}} & & \downarrow{\scriptstyle \Theta_{q,\Gamma}} \\ \mathbf{A}_q(A(\Omega), A\Gamma A^{-1}) & \xrightarrow{A_q^*} & \mathbf{A}_q(\Omega, \Gamma) \end{array}$$

The above considerations show that it suffices to prove Theorem 1 under the hypothesis that $\infty \notin \Lambda_q$. Furthermore, the definition of $f(\infty)$ for $f \in \mathcal{R}_q(\Lambda_q)$ given by (0.1.6) makes the equation

$$(A_q^* f)(z) = f(Az)A'(z)^q$$

valid even at $z=\infty$.

1.2. Let D be a Γ-invariant set (for example, Ω, Λ or Λ_q). A function F defined on $A(D)$ is an Eichler integral for $A\Gamma A^{-1}$ that vanishes at $a \in D$, if and only if $A_{1-q}^* F = F \cdot A$ is an Eichler integral for Γ supported on D that vanishes at a. Furthermore, if Δ is a Γ-invariant union of components of Γ, then

$$A_{1-q}^*\colon \mathcal{F}_{1-q}(A(\Delta), A\Gamma A^{-1}) \to \mathcal{F}_{1-q}(\Delta, \Gamma)$$

is an isomorphism. Here the Eichler integrals in $\mathcal{F}_{1-q}(A(\Delta), A\Gamma A^{-1})$ are normalized to vanish at Aa_j, while those in $\mathcal{F}_{1-q}(\Delta, \Gamma)$ vanish at a_j, $j=1, \dots, 2q-1$. Let us introduce the pairing

$$\mathcal{R}_q(\Lambda_q) \times \mathcal{F}_{1-q}(\Omega, \Gamma) \to \mathbf{C} \tag{1.2.1}$$

defined by

$$(f, F)_\Gamma \mapsto \mathcal{K}(f)(F), \tag{1.2.2}$$

where $\mathcal{K}$ is defined by (0.2.14).

PROPOSITION. *For* $f \in \mathcal{R}_q(A(\Lambda_q))$ *and* $F \in \mathcal{F}_{1-q}(A(\Omega), A\Gamma A^{-1})$, *we have*

$$(f, F)_{A\Gamma A^{-1}} = (A_q^* f,\ A_{1-q}^* F)_\Gamma. \tag{1.2.3}$$

The above proposition will be proven in § 1.4, after we derive some useful relations between rational functions in § 1.3.

1.3. Let us rewrite the functions f introduced in § 0.2 in the following form:

$$f_\alpha(a, \zeta) = \frac{-1}{2\pi} \frac{1}{\zeta - a} \prod_{j=1}^{2q-1} \frac{a - a_j}{\zeta - a_j}, \tag{1.3.1}$$

where $\alpha = (a_1, \ldots, a_{2q-1}) \in \mathbf{C}^{2q-1}$ has distinct entries and $a \neq a_j$, $j = 1, \ldots, 2q-1$. Let A be a Möbius transformation and write

$$a = Ab, \quad a_j = Ab_j, \quad \beta = (b_1, \ldots, b_{2q-1}). \tag{1.3.2}$$

Using the fundamental identity

$$(Az - A\zeta)^2 = (z - \zeta)^2 A'(z) A'(\zeta), \tag{1.3.3}$$

we conclude that

$$\begin{aligned} f_\alpha(a, A\zeta) A'(\zeta)^q &= \frac{-1}{2\pi} \frac{1}{A\zeta - a} \prod_{j=1}^{2q-1} \frac{a - a_j}{A\zeta - a_j} A'(\zeta)^q \\ &= \frac{-1}{2\pi} \frac{1}{A\zeta - Ab} \prod_j \frac{Ab - Ab_j}{A\zeta - Ab_j} A'(\zeta)^q \\ &= \frac{-1}{2\pi} \frac{A'(b)^{q-1}}{\zeta - b} \prod_j \frac{b - b_j}{\zeta - b_j}; \end{aligned} \tag{1.3.4}$$

or equivalently

$$f_{A\beta}(Ab, A\zeta) A'(b)^{1-q} A'(\zeta)^q = f_\beta(b, \zeta). \tag{1.3.5}$$

We introduce the convention (which agrees with earlier usage) that $f = f_\alpha$. Thus the last equation can be rewritten as

$$f(a, A\zeta) A'(\zeta)^q = f_\beta(b, \zeta) A'(b)^{q-1}. \tag{1.3.6}$$

Comparing residues we see that

$$f(a, A\zeta)A'(\zeta)^q = A'(b)^{q-1}\left[f(b,\zeta) - \sum_{k=1}^{2q-1}\left(\prod_{j\neq k}\frac{b-b_j}{b_k-b_j}\right)f(b_k,\zeta)\right]. \qquad (1.3.7)$$

Remark. If $b=a_l$ for some l, then the term $f(b,\zeta)$ is to be dropped from formula (1.3.7). Similarly, if $b_k=a_l$ for some pair of indices k, l, then the $f(b_k,\zeta)$ term is to be dropped.

To verify (1.3.7), we define the function

$$h(\zeta) = f_\beta(b,\zeta) - f_\alpha(b,\zeta) + \sum_{k=1}^{2q-1}\left(\prod_{j\neq k}\frac{b-b_j}{b_k-b_j}\right)f_\alpha(b_k,\zeta),$$

following the convention in the above remark (only for the index α). We observe that h is a rational function with simple poles, and note further that it has zero residue at $\zeta=b$ and $\zeta=b_k$ provided $b\neq a_l$ and $b_k\neq a_l$, $1\leqslant k$, $l\leqslant 2q-1$. Thus h has at most simple poles at $\zeta=a_j$, $j=1,\dots,2q-1$ and vanishes to at least order $2q$ at infinity. Thus h is identically zero.

1.4. We are now ready to prove Proposition 1.2. Fix $F\in\mathscr{F}_{1-q}(A(\Omega), A\Gamma A^{-1})$. By linearity, it suffices to verify (1.2.3) for a basis for $\mathscr{R}_q(A(\Lambda_q))$. We assume that Eichler integrals in $\mathscr{F}_{1-q}(A(\Omega), A\Gamma A^{-1})$ vanish at $a_j\in A(\Lambda_q)$, $j=1,\dots,2q-1$. Thus for f given by (1.3.1) with $a\in A(\Lambda_q)\setminus\{a_1,\dots,a_{2q-1}\}$

$$(f, F)_{A\Gamma A^{-1}} = F(a),$$

and using (1.3.5), we see that

$$\begin{aligned}(A_q^* f,\ A_{1-q}^* F)_\Gamma &= A'(b)^{q-1}(A_{1-q}^* F)(b)\\ &= A'(b)^{q-1}F(Ab)A'(b)^{1-q} = F(Ab) = F(a).\end{aligned}$$

Remarks. (1) As a consequence of Proposition 1.2, it suffices to prove Theorem 2 under the assumption that $\infty\notin\Lambda_q$.

(2) The pairing (1.2.1) can be extended to a bigger space of Eichler integrals. Let D be a Γ-invariant subset of $\mathbf{C}\cup\{\infty\}$. Let $a_1,\dots,a_{2q-1}$ be $(2q-1)$ distinct points in D. We define

$$\mathbf{F}_{1-q}(D,\Gamma) = \{F \text{ is an Eichler integral on } D;\ F(a_j)=0,\ j=1,\dots,2q-1\}.$$

Clearly, $\mathbf{F}_{1-q}(\Lambda_q,\Gamma)\supset\mathscr{F}_{1-q}(\Omega,\Gamma)$ and formula (1.2.2) defines a pairing

$$\mathcal{R}_q(\Lambda_q)\times\mathbf{F}_{1-q}(\Lambda_q,\Gamma)\to\mathbf{C}.$$

Proposition 1.2 remains valid under the assumption that $F\in\mathbf{F}_{1-q}(A(\Lambda_q),A\Gamma A^{-1})$.

1.5.

PROPOSITION. *The Poincaré series operator Θ_q maps $\mathcal{R}_q(\Lambda_q)$ onto $\mathbf{A}_q(\Omega,\Gamma)$.*

Proof. Bers [4] has shown that Θ_q maps $\mathcal{R}_q(\Lambda)$ onto $\mathbf{A}_q(\Omega,\Gamma)$. Hence it remains to show that $\Theta_q f$ converges uniformly on compact subsets of Ω for $f\in\mathcal{R}_q(\Lambda_q)$ with a pole in Ω, that $\Theta_q f$ is regular everywhere, and satisfies the cusp condition (see [14, p. 117] for a definition and proof) at each puncture. Since $\mathcal{R}_q(\Lambda_q)$ is a subspace of the Banach space of integrable functions on Ω with respect to the measure $\lambda(z)^{2-q}|dz\wedge d\bar{z}|$, $\Theta_q f$ converges uniformly on compact subsets of Ω (see [14, pp. 121–123]) and satisfies

$$\iint_{\Omega/\Gamma}\lambda(z)^{2-q}|(\Theta_q f)(z)\,dz\wedge d\bar{z}|\leq\iint_{\Omega}\lambda(z)^{2-q}|f(z)\,dz\wedge d\bar{z}|.$$

To show that $\Theta_q f$ is regular on Ω, we may assume that f is given by (1.3.1) with $a=0\in\Lambda_q\setminus\Lambda$ and $a_j\in\Lambda$ for $j=1,\dots,2q-1$. Furthermore, we normalize Γ so that the stabilizer of 0 is generated by

$$E(z)=Kz,\quad \text{all } z\in\mathbf{C}\cup\{\infty\},$$

where

$$K=e^{2\pi i/\nu},$$

and ν satisfies (0.1.4).

Let G be a subgroup of Γ; the Poincaré series operator $\Theta_q=\Theta_{q,\Gamma}$ for the group Γ can always be factored as

$$\Theta_{q,\Gamma}=\Theta_{q,\Gamma/G}\circ\Theta_{q,G},$$

where $\Theta_{q,G}$ is the Poincaré series operator for the group G, and $\Theta_{q,\Gamma/G}$ is the *relative* Poincaré series operator defined on G-invariant functions (functions f such that $A_q^*f=f$ for all $A\in G$) by the formula

$$\Theta_{q,\Gamma/G}f=\sum_{\gamma\in\Gamma/G}\gamma_q^*f,$$

where the sum runs over left coset representatives for Γ modulo G.

If we let G be the cyclic subgroup generated by E, it suffices, in order to finish the proof of the proposition, to show that $\Theta_{q,G}f$ is regular at zero. We start with

$$f(\zeta)=\frac{1}{2\pi}\frac{1}{\zeta}\prod_{j=1}^{2q-1}\frac{a_j}{\zeta-a_j},$$

and observe that

$$f(E\zeta)E'(\zeta)^q=\frac{1}{2\pi}\frac{1}{\zeta}K^{q-1}\prod_{j=1}^{2q-1}\frac{a_j}{K\zeta-a_j}.$$

This function is holomorphic on $\Omega\setminus\{0\}$ (because $K^{-1}a_j=E^{-1}(a_j)\in\Lambda$, for $j=1,\ldots,2q-1$). Thus we must show that the residue at 0 of

$$(\Theta_{q,G}f)(\zeta)=\sum_{j=0}^{\nu-1}f(E^j\zeta)(E^j)'(\zeta)^q,\quad \zeta\in\Omega,$$

is equal to 0. We note that the residue of $(E^j)_q^*f$ at 0 is

$$\frac{1}{2\pi}(-1)^{2q-1}(K^{q-1})^j=\frac{-1}{2\pi}(K^{q-1})^j.$$

Thus the residue of $\Theta_{q,G}f$ at 0 is

$$\frac{-1}{2\pi}\sum_{j=0}^{\nu-1}(K^{q-1})^j=\frac{-1}{2\pi}\frac{1-K^{(q-1)\nu}}{1-K^{q-1}}=0$$

by (0.1.4).

Remark. If f is given by (1.3.1) with $a_j\in\Lambda$, $j=1,\ldots,2q-1$, and $a\notin\Lambda_q$, then $\Theta_q f$ is holomorphic on $\Omega\setminus\Gamma a$, satisfies the cusp condition at each puncture on Ω/Γ, and has a simple pole at a with residue $-\nu$ (ν=order of stabilizer of a in Γ). The q-differential obtained by projecting $\Theta_q f$ to Ω/Γ is regular everywhere except the image of a (where it has a pole of order q), the images of the other elliptic fixed points (if the stabilizer of such a point has order μ, then the pole has order $\leq[q-q/\mu]$), and the punctures (where it has a pole of order $\leq q-1$).(6)

(6) The symbol $[x]$, defined for $x\in\mathbf{R}$, stands for the greatest integer $\leq x$. It should be noted that $q-1=\lim_{\mu\to\infty}[q-q/\mu]$. Hence, we define $[q-q/\infty]=q-1$.

§ 2. Spanning sets for Poincaré series (proof of Theorem 1)

2.1. We begin the proof of Theorem 1. By the results of § 1.1, we may assume, without loss of generality, that $S \subseteq \mathbf{C}$.

We start with the injectivity of the Bers map β^* of (0.3.1) and conclude that

$$\dim \mathscr{F}_{1-q}(\Omega, \Gamma) = \dim \mathbf{A}_q(\Omega, \Gamma) = d.$$

We assume that $d > 0$, since otherwise the conclusion of Theorem 1 is satisfied. Proposition 1.5 showed that $\Theta_q(\mathscr{R}_q(S)) \subset \mathbf{A}_q(\Omega, \Gamma)$, and thus we must show that $\Theta_q(\mathscr{R}_q(S))$ contains d linearly independent cusp forms.

It was shown in [14, Lemma 2.5 of Chapter V] that for $\varphi \in \mathbf{A}_q(\Omega, \Gamma)$, $\beta^*(\varphi) = 0$ if and only if the potential F_φ for $\mu = \lambda^{2-2q}\bar{\varphi}$ constructed by formula (0.2.9) vanishes identically on Λ. We need

LEMMA. *For* $\varphi \in \mathbf{A}_q(\Omega, \Gamma)$,

$$\varphi = 0 \quad \Leftrightarrow \quad \beta^*\varphi = 0 \quad \Leftrightarrow \quad F_\varphi | \Lambda_q = 0.$$

Proof. For the convenience of the reader we repeat the arguments of [14] that yield the above result. Assume that $\beta^*\varphi = 0$. Thus there exists a $p \in \Pi_{2q-2}$ so that

$$\chi_\varphi(\gamma) = F_\varphi \cdot \gamma - F_\varphi = p \cdot \gamma - p, \quad \text{all } \gamma \in \Gamma$$

(abbreviate χ_φ by χ and F_φ by F). Thus

$$(F-p) \cdot \gamma = (F-p), \quad \text{all } \gamma \in \Gamma. \tag{2.1.1}$$

For $z_0 \in \Lambda_q \setminus \Lambda$, let E be the generator of the stabilizer of z_0. Then $E'(z_0)^{1-q} \neq 1$ by (0.1.4). Thus from (2.1.1), we see that

$$(F-p)(z_0) = 0.$$

The same conclusion holds (by the same argument) at the loxodromic fixed points; and thus by the continuity of $F-p$, and the density of the loxodromic fixed points in Λ, for all $z_0 \in \Lambda$. Thus $F-p$ vanishes on Λ_q. But $F(a_j) = 0$ for $j = 1, \ldots, 2q-1$, and thus $p = 0$ and $F | \Lambda_q = 0$.

The converse is, of course, obvious. The first equivalence is a restatement of the injectivity of β^*.

2.2. We continue with the proof of (and notation introduced by) Theorem 1.

LEMMA. *For* $\varphi \in \mathbf{A}_q(\Omega, \Gamma)$,

$$\varphi = 0 \quad \Leftrightarrow \quad \beta^*\varphi = 0 \quad \Leftrightarrow \quad F_\varphi | S = 0.$$

Proof. It suffices to show that $\chi=\chi_\varphi=0$ whenever $F|S=0$ $(F=F_\varphi)$. We compute for the generator γ_j, $j=1,\dots,N$ of Γ,

$$\chi(\gamma_j)(a_k)=F(\gamma_j a_k)\gamma_j'(a_k)^{1-q}-F(a_k)=0$$

for $k=1,\dots,2q-1$. Hence the polynomial $\chi(\gamma_j)\in\Pi_{2q-2}$ vanishes at $2q-1$ distinct points, and must be the zero polynomial. We conclude that χ vanishes on the generators. Hence $\chi=0$, and the injectivity of β^* shows that $\varphi=0$.

COROLLARY. *The restriction of functions in $\mathcal{F}_{1-q}(\Omega,\Gamma)$ to S is an isomorphism.*

2.3. We now select d distinct points $b_1,\dots,b_d$ in $S\setminus\{a_1,\dots,a_{2q-1}\}$ and d functions $F_1,\dots,F_d\in\mathcal{F}_{1-q}(\Omega,\Gamma)$ so that

$$F_j(b_k)=\delta_{jk},\quad 1\le j,\ k\le d. \tag{2.3.1}$$

(Here δ_{jk} is the Kronecker delta function.) For the convenience of the reader, we reproduce a standard argument.

Choose any point $b_1\in S$ such that not every $F\in\mathcal{F}_{1-q}(\Omega,\Gamma)$ vanishes at b_1. Choose $F_1\in\mathcal{F}_{1-q}(\Omega,\Gamma)$ so that $F_1(b_1)\neq 0$. Without loss of generality $F_1(b_1)=1$. By induction we show that having chosen $F_1,\dots,F_n\in\mathcal{F}_{1-q}(\Omega,\Gamma)$ and $b_1,\dots,b_n\in S$, $1\le n<d$ so that the $n\times n$ matrix

$$(F_j(b_k)),\quad 1\le j,\ k\le n,$$

is the identity matrix, then we can select b_{n+1}, $\tilde F_1,\dots,\tilde F_{n+1}$ so that

$$(\tilde F_j(b_k)),\quad 1\le j,\ k\le n+1,$$

is the identity matrix and the linear span of $\tilde F_1,\dots,\tilde F_{n+1}$ is the same as the linear span of $F_1,\dots,F_n$, $\tilde F_{n+1}$. Let $\mathbf{B}\subset\mathcal{F}_{1-q}(\Omega,\Gamma)$ be the linear span of $F_1,\dots,F_n$. Then $\dim\mathbf{B}=n$. Let

$$e\colon\mathcal{F}_{1-q}(\Omega,\Gamma)\to\mathbf{C}^n$$

be defined by $e(F)=(F(b_1),\dots,F(b_n))$. By the induction hypothesis e is surjective. Hence

$$\dim\operatorname{Ker}e=d-n>0,$$

and there is a point $b_{n+1}\in S$ and a function $\tilde F_{n+1}\in\operatorname{Ker}e$ such that $\tilde F_{n+1}(b_{n+1})=1$. Finally we define

$$\tilde{F}_j = F_j - F_j(b_{n+1})\tilde{F}_{n+1}, \quad j = 1, \dots, n.$$

Definition. Given a finite set S with $\{a_1, \dots, a_{2q-1}\} \subsetneqq S \subset \Lambda_q$, we let $b_1, \dots, b_n$ be the distinct points in $S \setminus \{a_1, \dots, a_{2q-1}\}$. We define the *evaluation map*

$$e_S \colon \mathcal{F}_{1-q}(\Omega, \Gamma) \to \mathbf{C}^n$$

by

$$e_S(F) = (F(b_1), \dots, F(b_n)), \quad F \in \mathcal{F}_{1-q}(\Omega, \Gamma).$$

2.4. The next step is to consider the rational functions $f(z, \zeta)$ defined by (0.2.11) and (0.2.12) for $z \neq a_j$, $j = 1, \dots, 2q-1$, $z \in \Lambda_q$ and $\zeta \in \mathbf{C}$, and to study the Poincaré series

$$\varphi(z, \zeta) = \sum_{\gamma \in \Gamma} f(z, \gamma\zeta)\, \gamma'(\zeta)^q, \quad \zeta \in \Omega. \tag{2.4.1}$$

We claim that the cusp forms

$$\varphi(b_1, \cdot), \dots, \varphi(b_d, \cdot)$$

form a basis for $\mathbf{A}_q(\Omega, \Gamma)$.

To verify this claim, we introduce the final ingredient in the proof. The *Petersson scalar product* on $\mathbf{A}_q(\Omega, \Gamma)$ is given by

$$\langle \varphi, \psi \rangle_\Gamma = i \iint_{\Omega/\Gamma} \lambda(z)^{2-2q} \varphi(z) \overline{\psi(z)}\, dz \wedge d\bar{z}. \tag{2.4.2}$$

A calculation shows that the potential $F = F_\psi$ for $\mu = \lambda^{2-2q}\bar{\psi}$, $\psi \in \mathbf{A}_q(\Omega, \Gamma)$, satisfies

$$\begin{aligned} F_\psi(b_k) &= \frac{(b_k - a_1) \dots (b_k - a_{2q-1})}{2\pi i} \iint_\Omega \frac{\lambda(\zeta)^{2-2q} \overline{\psi(\zeta)}\, d\zeta \wedge d\bar{\zeta}}{(\zeta - b_k)(\zeta - a_1) \dots (\zeta - a_{2q-1})} \\ &= i \iint_\Omega f(b_k, \zeta)\, \lambda(\zeta)^{2-2q} \overline{\psi(\zeta)}\, d\zeta \wedge d\bar{\zeta} \\ &= i \iint_{\Omega/\Gamma} \varphi(b_k, \zeta)\, \lambda(\zeta)^{2-2q} \overline{\psi(\gamma)}\, d\zeta \wedge d\bar{\zeta} \\ &= \langle \varphi(b_k, \cdot), \psi \rangle_\Gamma, \quad k = 1, \dots, d. \end{aligned} \tag{2.4.3}$$

Thus every point b_k, $k = 1, \dots, d$, defines a conjugate linear functional on $\mathbf{A}_q(\Omega, \Gamma)$ by the formula

$$\mathbf{A}_q(\Omega,\Gamma)\ni\psi\mapsto F_\psi(b_k)\in\mathbf{C}.$$

Choose the basis $F_1,\dots,F_d$ of $\mathscr{F}_{1-q}(\Omega,\Gamma)$ that satisfies (2.3.1), and write $F_j=F_{\psi_j}$, $\psi_j\in\mathbf{A}_q(\Omega,\Gamma)$, $j=1,\dots,d$. We see that $\psi_1,\dots,\psi_d$ is a basis for $\mathbf{A}_q(\Omega,\Gamma)$ and

$$\langle\varphi(b_k,\cdot),\psi_j\rangle_\Gamma=F_j(b_k)=\delta_{jk},\quad 1\leq j,\ k\leq d,$$

which shows that $\varphi(b_k,\cdot)$, $k=1,\dots,d$, is also a basis for $\mathbf{A}_q(\Omega,\Gamma)$. This completes the proof of Theorem 1.

2.5. We will need a converse to Theorem 1.

THEOREM. *Let Γ be a finitely generated non-elementary Kleinian group. Let $a_1,\dots,a_{2q-1}$ be $(2q-1)$ distinct points in Λ_q. Let $b_1,\dots,b_d$ be $d>0$ distinct points in $\Lambda_q\setminus\{a_1,\dots,a_{2q-1}\}$. Then $\varphi(b_1,\cdot),\dots,\varphi(b_d,\cdot)$ forms a basis for $\mathbf{A}_q(\Omega,\Gamma)$ if and only if*

$$F\mapsto(F(b_1),\dots,F(b_d))\in\mathbf{C}^d \tag{2.5.1}$$

is an isomorphism of $\mathscr{F}_{1-q}(\Omega,\Gamma)$ onto $\mathbf{C}^d$.

Proof. The proof of Theorem 1 showed sufficiency, and we need only establish necessity. Thus assume that $\varphi(b_1,\cdot),\dots,\varphi(b_d,\cdot)$ forms a basis for $\mathbf{A}_q(\Omega,\Gamma)$. Let $\varphi_j=\varphi(b_j,\cdot)$ and $F_j=F_{\varphi_j}$, $j=1,\dots,d$. Then $F_1,\dots,F_d$ forms a basis for $\mathscr{F}_{1-q}(\Omega,\Gamma)$, Further, the matrix

$$(\langle\varphi_j,\varphi_k\rangle_\Gamma),\quad 1\leq j,\ k\leq d,$$

is non-singular. But as we saw earlier

$$\langle\varphi_j,\varphi_k\rangle_\Gamma=F_k(b_j),\quad 1\leq j,\ k\leq d.$$

Hence the vectors $(F_k(b_1),\dots,F_k(b_d))\in\mathbf{C}^d$, $k=1,\dots,d$ form a basis for $\mathbf{C}^d$.

Remarks. (1) The Petersson scalar product also exhibits an invariance similar to (1.2.3). For any Möbius transformation A, and $\varphi,\psi\in\mathbf{A}_q(A(\Omega),A\Gamma A^{-1})$,

$$\langle\varphi,\psi\rangle_{A\Gamma A^{-1}}=\langle A_q^*\varphi,\ A_q^*\psi\rangle_\Gamma.$$

(2) The map (2.5.1) is e_S for

$$S=\{b_1,\dots,b_d,a_1,\dots,a_{2q-1}\}.$$

§ 3. Cusp forms belonging to a character (a generalization of Theorem 1)

3.1. Let ϱ be a character on Γ; that is, ϱ is a homomorphism of Γ into the unit circle $\{z\in\mathbf{C}; |z|=1\}$. To define the space $\mathbf{A}_q(\Omega,\Gamma,\varrho)$ of *cusp forms of weight* $(-2q)$ *belonging to the character* ϱ, we substitute for (0.1.1), the condition

$$\varphi(\gamma z)\,\gamma'(z)^q\varrho(\gamma)=\varphi(z),\quad \text{all } z\in\Omega,\ \text{all } \gamma\in\Gamma. \tag{3.1.1}$$

The Petersson scalar product (2.4.2) gives $\mathbf{A}_q(\Omega,\Gamma,\varrho)$ a Hilbert space structure, since once again (0.1.2) and (0.1.3) are equivalent. The group Γ now acts on Π_{2q-2} by

$$(p\cdot\gamma)(z)=p(\gamma z)\,\gamma'(z)^{1-q}\varrho(\gamma),\quad p\in\Pi_{2q-2},\ \gamma\in\Gamma,\ z\in\mathbf{C}. \tag{3.1.2}$$

The above formula generalizes (0.2.1). The corresponding cohomology group will be denoted by $H^1(\Gamma,\Pi_{2q-2},\varrho)$.

For $\varphi\in\mathbf{A}_q(\Omega,\Gamma,\varrho)$, formula (0.2.9) again defines a potential for $\mu=\lambda^{2-2q}\bar{\varphi}$, and if (0.2.5) is modified to

$$\chi(\gamma)(z)=F(\gamma z)\,\gamma'(z)^{1-q}\,\overline{\varrho(\gamma)}-F(z),\quad \gamma\in\Gamma,\ z\in\mathbf{C}, \tag{3.1.3}$$

then we obtain a map

$$\beta^*\colon \mathbf{A}_q(\Omega,\Gamma,\varrho)\to H^1(\Gamma,\Pi_{2q-2},\bar{\varrho}).$$

The proof that β^* is injective follows the one given, for example, in [14, pp. 186–191] for the trivial character.

3.2. The Poincaré series operator must be modified to (compare with (0.1.7))

$$(\Theta_q f)(z)=\sum_{\gamma\in\Gamma} f(\gamma z)\,\gamma'(z)^q\varrho(\gamma),\quad z\in\Omega. \tag{3.2.1}$$

With these modifications Theorem 1 is valid for an arbitrary character ϱ, provided $S\subset\Lambda$.

THEOREM. *Let* $a_1,\dots,a_{2q-1}$ *be* $(2q-1)$ *distinct points in* Λ. *Let* $\gamma_0=I$, $\gamma_1,\dots,\gamma_N$ *be generators for* Γ. *Define* S *by* (0.1.8). *Then* $\Theta_q(\mathcal{R}_q(S))=\mathbf{A}_q(\Omega,\Gamma,\varrho)$.

Remarks. (1) It is interesting to note that as long as $S\subset\Lambda$, then the set of rational functions whose Poincaré series span the space of cusp forms does *not* involve the character ϱ.

(2) Were we to consider sets S that contain elliptic fixed points, then we would have to take into account the character ϱ in defining the sets $\Lambda_{q,\varrho}$ that would replace Λ_q.

(3) We will henceforth ignore characters. *Most* of the subsequent analysis carries over for an arbitrary character ϱ. We can also work with forms of weight $(-2q)$, $q\geqslant 2$, with $2q\in\mathbf{Z}$, provided we can choose square roots of the derivatives of the elements of Γ so that

$$\gamma_1'(\gamma_2 z)^{1/2}\gamma_2'(z)^{1/2}=(\gamma_1\circ\gamma_2)'(z)^{1/2},\quad \text{all } \gamma_1,\gamma_2\in\Gamma, \text{ all } z\in\hat{\mathbf{C}}.$$

§ 4. On the vanishing of Poincaré series (proof of Theorem 2)

4.1. Let Γ be a finitely generated non-elementary Kleinian group. In § 0.2 and § 1.4, we considered various classes of Eichler integrals. Two Eichler integrals F_1, F_2 (defined on a common set D) are called *equivalent* or *cohomologous* if their difference F_1-F_2 is (the restriction to D of) a polynomial in Π_{2q-2}. Thus, if F_1 is equivalent to F_2 then $\chi_1=\mathrm{pd}\,F_1$ differs from $\chi_2=\mathrm{pd}\,F_2$ by a coboundary. (Here pd is the period map introduced in § 0.2.) Note further that for $F_1,F_2\in\mathbf{F}_{1-q}(D,\Gamma)$, as defined in § 1.4, F_1 is equivalent to F_2 if and only if $F_1=F_2$; and an arbitrary Eichler integral on D is equivalent to a unique element of $\mathbf{F}_{1-q}(D,\Gamma)$. The following result will motivate the remainder of this section and will be useful in § 5.

We introduce at this point more notation:

$$Z^1(\Gamma,\Pi_{2q-2}) = \text{the vector space of } \Pi_{2q-2}\text{-cocycles for } \Gamma,$$
$$PZ^1_\Delta(\Gamma,\Pi_{2q-2}) = \text{the vector space of } \Delta\text{-parabolic cocycles},$$
$$PZ^1(\Gamma,\Pi_{2q-2}) = \text{the vector space of parabolic cocycles}.$$

As usual, Δ is a Γ-invariant union of components of Ω.

PROPOSITION. *Let Λ_0 be the set of q-uniqueness points for Γ (as defined in* § 0.6) *that are not parabolic fixed points. Then the period map*

$$\mathrm{pd}\colon \mathbf{F}_{1-q}(\Lambda_0,\Gamma)\xrightarrow{\cong} H^1(\Gamma,\Pi_{2q-2})$$

is an isomorphism.

Proof. The injectivity of the period map is proven as in § 2.1. To establish surjectivity, let $\chi\in Z^1(\Gamma,\Pi_{2q-2})$. We shall construct an Eichler integral F on Λ_0 with $\mathrm{pd}\,F=\chi$. Once F is obtained, we choose $p\in\Pi_{2q-2}$ so that

$$F(a_j)=p(a_j), \quad j=1, \ldots, 2q-1.$$

Then $F-p \in \mathrm{F}_{1-q}(\Lambda_0, \Gamma)$, and since $F-p$ is equivalent to F, $\mathrm{pd}(F-p)$ and χ represent the same cohomology class in $H^1(\Gamma, \Pi_{2q-2})$. It remains to construct F from the cocycle χ.

Let $b \in \Lambda_0$. The stabilizer Γ_0 of b in Γ is the commutative group generated by Möbius transformations A, B, where B is trivial or loxodromic, and A is trivial or elliptic. (See § 12.) If B is trivial, then the order ν of A satisfies (0.1.4). Let $p_1=\chi(A)$ and $p_2=\chi(B)$. Since A and B commute, $\chi(A \circ B)=\chi(B \circ A)$ and we conclude that

$$p_1(Bz)B'(z)^{1-q}+p_2(z)=p_2(Az)A'(z)^{1-q}+p_1(z), \quad \text{all } z \in \mathbf{C}. \tag{4.1.1}$$

In particular, for the common fixed point b of A and B,

$$p_1(b)\,[B'(b)^{1-q}-1]=p_2(b)\,[A'(b)^{1-q}-1]. \tag{4.1.2}$$

Note that formulae (4.1.1) and (4.1.2) are valid for arbitrary $A, B \in \Gamma$ with common fixed point $b \in \Lambda_0$. We will draw several conclusions.

Assume that B is non-trivial. Since $|B'(b)| \neq 1$, we can define F at b by the formula

$$F(b)=p_2(b)\,[B'(b)^{1-q}-1]^{-1}.$$

Let $\gamma \in \Gamma$ be an arbitrary loxodromic element fixing $b \in \Lambda_0$. We claim that

$$F(b)=\chi(\gamma)(b)\,[\gamma'(b)^{1-q}-1]^{-1}. \tag{4.1.3}$$

Formula (4.1.3) follows from (4.1.2) since γ commutes with B and $|\gamma'(b)| \neq 1$. Next, if γ is elliptic and fixes b, then (4.1.3) holds whenever $\gamma'(b)^{1-q} \neq 1$.

We claim that for arbitrary $\gamma \in \Gamma$ with fixed point $b \in \Lambda_0$, we have

$$F(b)\,[\gamma'(b)^{1-q}-1]=\chi(\gamma)(b). \tag{4.1.4}$$

We have verified the above formula for all γ with $\gamma'(b)^{1-q} \neq 1$. To show that formula (4.1.4) continues to hold even when $\gamma'(b)^{1-q}=1$, we show that in this case $\chi(\gamma)(b)=0$. This fact follows from (4.1.2) with B loxodromic and $A=\gamma$.

If B is trivial, then $A'(b)^{1-q} \neq 1$ by (0.1.4); it follows that if we define

$$F(b)=p_1(b)\,[A'(b)^{1-q}-1]^{-1},$$

then (4.1.4) holds for every $\gamma \in \Gamma$ that fixes b, since such a γ must be a power of A. This observation follows as in the case of non-trivial B by examining (4.1.2).

We assert next that

$$F(\gamma z)\gamma'(z)^{1-q}-F(z)=\chi(\gamma)(z), \quad \text{all } \gamma\in\Gamma, \text{ all } z\in\Lambda_0. \tag{4.1.5}$$

Since $z\in\Lambda_0$, there is a $g\in\Gamma$ with $gz=z$ and $g'(z)^{1-q}\neq 1$ (as was shown by the arguments above). It follows that

$$(\gamma\circ g\circ\gamma^{-1})(\gamma z)=\gamma z,$$

and

$$(\gamma\circ g\circ\gamma^{-1})'(\gamma z)=g'(z).$$

The cocycle condition yields

$$\chi(\gamma\circ g\circ\gamma^{-1})=[\chi(\gamma)\cdot g+\chi(g)-\chi(\gamma)]\cdot\gamma^{-1}. \tag{4.1.6}$$

Hence we conclude from (4.1.4) that

$$F(z)[g'(z)^{1-q}-1]=\chi(g)(z), \tag{4.1.7}$$

and

$$F(\gamma z)[g'(z)^{1-q}-1]=\chi(\gamma\circ g\circ\gamma^{-1})(\gamma z). \tag{4.1.8}$$

Formula (4.1.5) follows from (4.1.8), (4.1.6), and (4.1.7) by a routine calculation, using once again the fact that $gz=z$. This completes the proof of Proposition 4.1.

Remarks. (1) In the above arguments we have assumed that $\Lambda_0\subset\mathbf{C}$. This can always be achieved by conjugation. In the remainder of this paper, we shall whenever convenient assume without further comment that we have replaced the group by a conjugate so that all calculations are at finite points.

(2) Assume that for $j=1,\ldots,2q-1$, $a_j\in\Lambda_0$ is a fixed point of a loxodromic element $\gamma_j\in\Gamma$. Let $\chi=\operatorname{pd}F$ with $F\in\mathrm{F}_{1-q}(\Lambda_0,\Gamma)$. Then

$$\chi(\gamma_j)(a_j)=F(a_j)\gamma_j'(a_j)^{1-q}-F(a_j)=0.$$

Thus we see that under our assumption

$$H^1(\Gamma,\Pi_{2q-2})\cong\{\chi\in Z^1(\Gamma,\Pi_{2q-2});\chi(\gamma_j)(a_j)=0, \text{ for all } j=1,\ldots,2q-1\}.$$

4.2. Before proceeding to the proof of Theorem 2, we verify that functions (0.2.11), (0.2.12) with $z\in\Lambda_q\setminus\{a_1,\ldots,a_{2q-1}\}$ form a basis of $\mathcal{R}_q(\Lambda_q)$. It is obvious that these functions are linearly independent. Let $f\in\mathcal{R}_q(\Lambda_q)$ and let $b_1,\ldots,b_m$ be the poles of f not in $\{a_1, \ldots, a_{2q-1}\}$. (Recall the convention about poles at ∞ introduced in the introduction.) The dimension of the subspace of $\mathcal{R}_q(\Lambda_q)$ consisting of functions holomorphic except at the points in $\{b_1, \ldots, b_m, a_1, \ldots, a_{2q-1}\}$ is precisely m. The space spanned by the linear combinations in (0.2.13) also has dimension m.

4.3. We are now ready to complete the proof of Theorem 2. Let $f \in \mathcal{R}_q(\Lambda_q)$. Then $\Theta_q f \in \mathbf{A}_q(\Omega, \Gamma) \supset \mathbf{A}_q(\Delta, \Gamma)$. Using the duality of the Petersson scalar product, it suffices to show that

$$\langle \Theta_q f, \psi \rangle_\Gamma = 0 \quad \text{all } \psi \in \mathbf{A}_q(\Delta, \Gamma)$$

if and only if

$$\mathcal{K}(f)(F) = 0 \quad \text{all } F \in \mathcal{F}_{1-q}(\Delta, \Gamma).$$

Now every $F \in \mathcal{F}_{1-q}(\Delta, \Gamma)$ is an F_ψ for some $\psi \in \mathbf{A}_q(\Delta, \Gamma)$. For f given by (0.2.13), we have by (0.2.14) and (2.4.3),

$$\begin{aligned}\mathcal{K}(f)(F_\psi) &= \sum_{j=1}^{m} \beta_j F_\psi(b_j) \\ &= \sum_{j=1}^{m} \beta_j \langle \varphi(b_j, \cdot), \psi \rangle_\Gamma. \\ &= \langle \Theta_q f, \psi \rangle_\Gamma.\end{aligned}$$

Remark. The importance of Theorem 2 is that it allows effective computation to determine whether or not a Poincaré series of a rational function vanishes identically on Ω. This topic will be pursued further in §§ 5, 7, 8, and 10.

4.4. The operators $\mathcal{K}$ and Θ_q allow us to introduce a pairing $\{\cdot, \cdot\}$ on $\mathcal{R}_q(\Lambda_q)$ as follows: Let $f_j \in \mathcal{R}_q(\Lambda_q)$ and let $\psi_j = \Theta_q f_j$, $j = 1, 2$. Define

$$\{f_1, f_2\} = \mathcal{K}(f_1)(F_{\psi_2}) = \langle \Theta_q f_1, \psi_2 \rangle_\Gamma = \langle \psi_1, \psi_2 \rangle_\Gamma.$$

It follows that the pairing is Hermetian in the sense that

$$\overline{\{f_1, f_2\}} = \{f_2, f_1\},$$

and

$$\{f, f\} = 0 \iff \Theta_q f = 0.$$

The Hilbert space obtained by factoring $\mathcal{R}_q(\Lambda_q)$ by the subspace of vectors f of length zero (those $f \in \mathcal{R}_q(\Lambda_q)$ with $\{f, f\}=0$) is canonically isomorphic to $\mathbf{A}_q(\Omega, \Gamma)$.

4.5. We will find it more useful to consider another pairing between subspaces of $\mathcal{R}_q(\Lambda_q)$ and those of $\mathcal{F}_{1-q}(\Omega, \Gamma)$. Let us fix $(2q-1)$ distinct points $a_1, \ldots, a_{2q-1} \in \Lambda_q$. Let $b_1, \ldots, b_m$ be $m>0$ distinct points in $\Lambda_q \setminus \{a_1, \ldots, a_{2q-1}\}$, and let

$$S = \{a_1, \ldots, a_{2q-1}, b_1, \ldots, b_m\}.$$

Define

$$\mathbf{A} = \{\Theta_q f; f \in \mathcal{R}_q(S)\},$$
$$\mathcal{F} = \{F|S; F \in \mathcal{F}_{1-q}(\Omega, \Gamma)\}.$$

Let $f \in \mathcal{R}_q(S)$ be given by (0.2.13). Introduce the pairing

$$\mathbf{A} \times \mathcal{F} \to \mathbf{C}, \tag{4.5.1}$$

by

$$(\varphi, F) = \sum_{j=1}^{m} \beta_j F(b_j), \quad \varphi \in \mathbf{A},\ F \in \mathcal{F}, \tag{4.5.2}$$

where $\varphi = \Theta_q f$.

THEOREM. *The pairing* (4.5.1) *given by* (4.5.2) *is well defined and non-singular. In particular,*

$$\dim \mathbf{A} = \dim \mathcal{F}.$$

Proof. Linearity in both variables of the pairing is clear. It is a consequence of Theorem 2 that (4.5.1) depends only on $\varphi \in \mathbf{A}$ and not on the particular $f \in \mathcal{R}_q(S)$ with $\Theta_q f = \varphi$.

Let $r = \dim \mathcal{F}$. If $r=0$, then we must show that $\mathbf{A}=\{0\}$. If not, there is a j with $1 \leq j \leq m$ and $\varphi(b_j, \cdot) \neq 0$. Thus for $F = F_{\varphi(b_j, \cdot)} \in \mathcal{F}_{1-q}(\Omega, \Lambda)$, we have

$$F(b_j) = \langle \varphi(b_j, \cdot), \varphi(b_j, \cdot) \rangle_\Gamma \neq 0.$$

Thus F is a non-zero element of $\mathcal{F}$. This contradiction shows that $\mathbf{A}=\{0\}$.

Assume now that $r>0$.

Assume that $F \in \mathcal{F}_{1-q}(\Omega, \Gamma)$ and that $(\varphi, F)=0$, all $\varphi \in \mathbf{A}$. Let $k=1, \ldots, m$, and set $\varphi = \varphi(b_k, \cdot)$. Then

$$0 = (\varphi, F) = F(b_k);$$

showing that $F=0$ in $\mathcal{F}$.

Conversely, if for fixed $\varphi \in \mathbf{A}$, $(\varphi, F)=0$ all $F \in \mathcal{F}$, then we must show $\varphi=0$. Now φ can be written as $\Sigma_{k=1}^{m} \beta_k \varphi(b_k, \cdot)$. As in the proof of Theorem 1, we find $r=\dim \mathcal{F}$ points $b_1, \ldots, b_r$ (after relabelling) so that a basis $F_1, \ldots, F_r$ for $\mathcal{F}$ satisfies

$$F_j(b_k) = \delta_{jk}, \quad 1 \leqslant j,\ k \leqslant r \leqslant m.$$

The functions $\varphi(b_1, \cdot), \ldots, \varphi(b_r, \cdot)$ are linearly independent. We have shown that the pairing (4.5.2) is well defined. Thus it suffices to establish the independence of these r functions. We compute

$$(\varphi(b_j, \cdot), F_k) = \delta_{jk}, \quad 1 \leqslant j,\ k \leqslant r.$$

Thus the independence of $\varphi(b_1, \cdot), \ldots, \varphi(b_r, \cdot)$ follows. If $r=m$, there is nothing more to prove.

Assume $0<r<k\leqslant m$. We claim that $\varphi(b_k, \cdot)$ is a linear combination of $\varphi(b_1, \cdot), \ldots, \varphi(b_r, \cdot)$. If not $\varphi(b_k, \cdot)$, $\varphi(b_1, \cdot), \ldots, \varphi(b_r, \cdot)$ can be made part of basis of $\mathbf{A}_q(\Omega, \Gamma)$. By Theorem 2.5, this means that there is an $F \in \mathcal{F}_{1-q}(\Omega, \Gamma)$ such that $F(b_k)=1$ and $F(b_j)=0$, $j=1, \ldots, r$. Now F is a linear combination

$$F = \sum_{k=1}^{r} \alpha_k F_k;$$

and $F(b_j)=0$ implies $\alpha_j=0$, for $j=1, \ldots, r$. Hence $F=0$ on S, and we have obtained a contradiction to $F(b_k)=1$. Thus φ can be written as

$$\varphi = \sum_{k=1}^{r} \beta_k \varphi(b_k, \cdot),$$

and $0=(\varphi, F_j)=\beta_j$, for $j=1, \ldots, r$; showing that $\varphi=0$.

4.6. A *sufficient* condition for the vanishing of the Poincaré series of certain rational functions can be expressed in purely algebraic terms.

THEOREM. *Let Γ be a finitely generated non-elementary Kleinian group. Let $f \in \mathcal{R}_q(\Lambda_0)$, with Λ_0 as defined in Proposition* 4.1. *Define $\mathcal{H}f \in \mathbf{F}_{1-q}(\Lambda_0, \Gamma)^*$ by* (0.2.14). *If $\mathcal{H}f=0$, then $\Theta_q f=0$.*

Proof. The space $\mathbf{F}_{1-q}(\Lambda_0, \Gamma)$ contains the restrictions to Λ_0 of functions in

$\mathscr{F}_{1-q}(\Omega,\Gamma)$, and thus $\mathscr{K}f$ vanishes on $\mathscr{F}_{1-q}(\Omega,\Gamma)$ whenever it vanishes on $\mathbf{F}_{1-q}(\Lambda_0,\Gamma)$.

Remark. Proposition 4.1 showed that $\mathbf{F}_{1-q}(\Lambda_0,\Gamma)$ is determined algebraically from $H^1(\Gamma,\Pi_{2q-2})$. Hence the sufficient condition for the vanishing of a Poincaré series given by the above theorem involves only algebraic data.

4.7. A *generalized Beltrami differential* μ is an equivalence class of measurable functions on Ω satisfying

$$\gamma^*_{1-q,1}\mu=\mu, \quad \text{all } \gamma\in\Gamma,$$

and

$$\lambda^{q-2}|\mu|\in L^\infty(\Omega),$$

where $\lambda(z)\,|dz|$ is the Poincaré metric on Ω. A *potential* for μ is a continuous function F on $\mathbf{C}$ such that (0.2.7) and (0.2.8) hold. For $\varphi\in\mathbf{A}_q(\Omega,\Gamma)$, $\lambda^{2-2q}\bar{\varphi}$ is called a *canonical* generalized Beltrami differential. As we saw earlier the restriction of a potential to Λ_q is an Eichler integral. The space of Eichler integrals obtained by restricting potentials of (arbitrary) generalized Beltrami differentials is not larger than the space of restrictions of potentials of canonical generalized Beltrami differentials ([14, p. 170]), and if we fix $(2q-1)$ distinct points $a_1,\dots,a_{2q-1}$ in Λ_q, then each such Eichler integral is equivalent to a function in $\mathscr{F}_{1-q}(\Omega,\Gamma)$.

A generalized Beltrami differential μ induces a linear functional l on $\mathbf{A}_q(\Omega,\Gamma)$ by

$$\varphi\mapsto i\iint_{\Omega/\Gamma}\varphi(z)\mu(z)\,dz\wedge d\bar{z}.$$

Let μ_1 and μ_2 be two generalized Beltrami differentials, with potentials F_1 and F_2 (respectively) and induced linear functionals l_1 and l_2 (respectively). Then $l_1=l_2$ if and only if F_1 is equivalent to F_2 as Eichler integrals on Λ_q (see [14, pp. 170–171]). The above observations will be useful in computing the values of potentials at parabolic fixed points corresponding to punctures. (See § 5.3.)

§ 5. The structure of the cohomology groups $PH^1(\Gamma,\Pi_{2q-2})$ (proof of Theorem 3)

5.1. Let Δ be a Γ-invariant union of components of the non-elementary finitely generated Kleinian group Γ. Let F be a holomorphic Eichler integral defined on Δ. We call F a *bounded* (holomorphic) Eichler integral provided

$$\varphi = \frac{d^{2q-1}F}{dz^{2q-1}} \in \mathbf{A}_q(\Delta, \Gamma). \tag{5.1.1}$$

If F is a bounded Eichler integral, then the projection of φ to Δ/Γ has a pole of order $\leq q-1$ at each puncture of Δ/Γ. If instead of (5.1.1), we require that φ have a pole of order $\leq q$ at each puncture of Δ/Γ, then F is called a *quasi-bounded* (holomorphic) Eichler integral.

The space of equivalence classes (modulo Π_{2q-2}) of bounded (respectively, quasi-bounded) holomophic Eichler integrals supported on Δ is denoted by $\mathbf{E}^b_{1-q}(\Delta, \Gamma)$ (respectively, $\mathbf{E}^c_{1-q}(\Delta, \Gamma)$). If we choose $(2q-1)$ distinct points $a_1, \ldots, a_{2q-1}$ in Δ (such points can *not* usually be used to normalize potentials in $\mathscr{F}_{1-q}(\Delta, \Gamma)$), then the space $\mathbf{E}^b_{1-q}(\Delta, \Gamma)$ (respectively, $\mathbf{E}^c_{1-q}(\Delta, \Gamma)$) can be identified with the space of bounded (respectively, quasi-bounded) holomorphic Eichler integrals for the group Γ with support on Δ that vanish at a_j, $j=1, \ldots, 2q-1$. The main structure theorem for the Eichler cohomology groups [14, Chapter V] can be formulated as follows. We define for $F_1 \in \mathbf{E}^c_{1-q}(\Delta, \Gamma)$, $F_2 \in \mathscr{F}_{1-q}(\Delta, \Gamma)$,

$$\mathrm{pd}\,(F_1, F_2) = \mathrm{pd}\,F_1 + \mathrm{pd}\,F_2$$

(where pd is the period map on Eichler integrals). Then we have C-linear isomorphisms

$$\mathrm{pd}\colon \mathbf{E}^b_{1-q}(\Delta, \Gamma) \oplus \mathscr{F}_{1-q}(\Delta, \Gamma) \xrightarrow{\cong} PH^1_\Delta(\Gamma, \Pi_{2q-2}),$$

$$\mathrm{pd}\colon \mathbf{E}^c_{1-q}(\Delta, \Gamma) \oplus \mathscr{F}_{1-q}(\Delta, \Gamma) \xrightarrow{\cong} H^1(\Gamma, \Pi_{2q-2}).$$

5.2. The spaces $\mathbf{A}_q(\Delta, \Gamma)$ and $\mathscr{F}_{1-q}(\Delta, \Gamma)$ are conjugate linear isomorphic. In many cases $\mathbf{E}^b_{1-q}(\Omega, \Gamma) = \{0\}$. This condition is equivalent to the surjectivity of the Bers map

$$\beta^*\colon \mathbf{A}_q(\Omega, \Gamma) \to PH^1_\Omega(\Gamma, \Pi_{2q-2}). \tag{5.2.1}$$

Since $PH^1(\Gamma, \Pi_{2q-2}) \subset PH^1_\Omega(\Gamma, \Pi_{2q-2})$, the surjectivity of β^* in (5.2.1) implies the surjectivity of β^* in (0.3.1). *These elementary remarks are extremely important for applications* (see for example §§ 8 and 10). There are groups Γ with $\dim \mathbf{E}^b_{1-q}(\Omega, \Gamma) > 0$, and β^* of (0.3.1) surjective. If for such a group Γ, we know which elements are parabolic (this hypothesis is satisfied by geometrically finite function groups by Maskit's [19] decomposition theorems—see also the presentation in § 9 of [16]), then we can construct algebraically the parabolic cocycles for Γ. From the parabolic cocycles, the values at points in Λ_q^0 of functions in $\mathscr{F}_{1-q}(\Omega,\Gamma)$ can be constructed. We state our next result in the following form.

THEOREM. (a) *Let* Γ *be a finitely generated non-elementary Kleinian group. Assume that the map* β^* *of* (0.3.1) *is surjective. Let* $a_1,\dots,a_{2q-1}$ *by* $(2q-1)$ *distinct points in* Λ_q^0. *The space* $\mathscr{F}_{1-q}(\Omega,\Gamma)$ *is then completely determined from the parabolic cocyles for* Γ.

(b) *Assume that for* $j=1,\dots,2q-1$, $a_j\in\Lambda_q^0$ *is a fixed point of a loxodromic element* $\gamma_j\in\Gamma$ *or an elliptic element* γ_j *of order* v *that satisfies* (0.1.4). *Then the cohomology space* $PH^1(\Gamma,\Pi_{2q-2})$ *is canonically isomorphic to*

$$Z=\{\chi\in PZ^1(\Gamma,\Pi_{2q-2});\ \chi(\gamma_j)(a_j)=0\ \textit{for}\ j=1,\dots,2q-1\}.$$

(c) *The values of functions in* $\mathscr{F}_{1-q}(\Omega,\Gamma)$ *at points in* Λ_q^0 *and at parabolic fixed points corresponding to punctures on* Ω/Γ *can be constructed algebraically from the space of cocycles* Z.

Proof. If $\chi\in PZ^1(\Gamma,\Pi_{2q-2})$, then by Proposition 4.1 (see also Theorem 6.10), there is a unique $F\in \mathrm{F}_{1-q}(\Lambda_q^0,\Gamma)$ so that $\mathrm{pd}\,F$ is cohomologous to χ. The construction of F from χ is completely algebraic. Since χ is a parabolic cocycle, F is the restriction to Λ_q^0 of a unique function in $\mathscr{F}_{1-q}(\Omega,\Gamma)$ because of (i) the surjectivity of β^*, and (ii) the continuity of functions in $\mathscr{F}_{1-q}(\Omega,\Gamma)$. This establishes part (a) of the theorem. Claim (b) follows from Remark 2 of § 4.1. We have already verified most of (c). It remains to show that if $F\in\mathscr{F}_{1-q}(\Omega,\Gamma)$ and $A\in\Gamma$ is parabolic with fixed point $a\in\Lambda$ corresponding to a puncture on Ω/Γ, then $F(a)$ can be computed from $p=\chi(A)$, where $\chi=\mathrm{pd}\,F$. We start with

$$F(Az)\,A'(z)^{1-q}-F(z)=p(z),\quad z\in\Lambda_q. \tag{5.2.2}$$

As a matter of fact, since F is a potential, equation (5.2.2) holds for all $z\in\mathbf{C}\cup\{\infty\}$. Assume that $a\in\mathbf{C}$. Since $A'(a)=1$, (5.2.2) does not immediately determine $F(a)$ from $p(a)$. If F were analytic at a, then we would conclude that

$$F'(Az)\,A'(z)^{2-q}+(1-q)\,F(Az)\,A'(z)^{-q}A''(z)-F'(z)=p'(z)$$

for z near a. It would hence follow that

$$F(a)=\frac{p'(a)}{(1-q)\,A''(a)}. \tag{5.2.3}$$

Note that $A''(a)\neq 0$. Thus we must verify (5.2.3). This will be done in the next section.

5.3. Let F be a potential for the group Γ that vanishes at $(2q-1)$ points in Λ_q^0. We let $C(z)=z+1$ be a parabolic element of Γ generating the stabilizer of the fixed point ∞ corresponding to a puncture on Ω/Γ. We may, without loss of generality, assume that for some $c>0$, the domain $U_c=\{z\in\mathbf{C}; \operatorname{Im} z>c\}$ is precisely invariant under the cyclic group $\langle C\rangle$ in Γ. The potential F satisfies (0.2.7), and, without loss of generality, we may assume, by the remarks in § 4.7, that F is holomorphic in U_c (because every linear functional on $\mathbf{A}_q(\Omega,\Gamma)$ can be induced by a Beltrami differential with support in the image under Γ of a small ball in Ω). Write

$$F(z+1)-F(z)=p(z), \quad z\in U_c, \tag{5.3.1}$$

where $\deg p\leqslant 2q-3$ because $\mathrm{pd}F$ is a parabolic cocycle (see § 6.7 or below). Let $\varphi=d^{2q-1}F/dz^{2q-1}$. Then

$$\varphi(z+1)=\varphi(z), \quad z\in U_c.$$

Hence φ has a Fourier series expansion

$$\varphi(z)=\sum_{n=-\infty}^{\infty}\alpha_n e^{2\pi inz}, \quad z\in U_c.$$

It follows that F can be written as

$$F(z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\beta_n e^{2\pi inz}+v(z), \quad z\in U_c, \tag{5.3.2}$$

where v is a polynomial of degree $\leqslant 2q-1$. But the growth condition (0.2.7) on F implies that $\beta_n=0$ for $n<0$ and that v is of degree of most $2q-2$. Thus

$$F(z)=\sum_{n=1}^{\infty}\beta_n e^{2\pi inz}+v(z), \quad z\in U_c,$$

with $v\in\Pi_{2q-2}$. The value

$$F(\infty)=(-1)^{1-q}a_{2q-2}=v(\infty), \tag{5.3.3}$$

is computed in accordance with (0.2.6), where

$$v(z)=\sum_{j=0}^{2q-2}a_j z^j, \quad z\in\mathbf{C}.$$

From (5.3.2) and (5.3.1), it follows that

$$v(z+1)-v(z)=p(z).$$

Writing

$$p(z)=\sum_{j=0}^{2q-3} b_j z^j, \tag{5.3.4}$$

we see that

$$b_{2q-3}=2(q-1)\,a_{2q-2}=(-1)^{1-q}(2q-2)\,F(\infty). \tag{5.3.5}$$

We claim that the translation (5.3.5) to the finite point $a \in \mathbf{C}$ will yield (5.2.3). With a (and A) given, we choose a Möbius transformation B with $B^{-1} \circ C \circ B = A$. It follows that $B(a)=\infty$, and hence that

$$B(z)=\frac{bz+c}{z-a}.$$

The function $F \cdot B$ is a potential for $B^{-1}\Gamma B$ with $(\mathrm{pd}\,F\cdot B)(A)=p\cdot B$, and we must show that (5.2.3) holds; that is,

$$(F\cdot B)(a)=\frac{(p\cdot B)'(a)}{(1-q)\,A''(a)}. \tag{5.3.6}$$

We compute

$$A(z)=\frac{(ab+a+c)\,z-a^2}{z+(ab-a+c)},$$

$$A''(a)=\frac{-2}{ab+c},$$

(note that $ab+c \neq 0$)

$$(F\cdot B)(z)=F(Bz)\,B'(z)^{1-q}=(-1)^{1-q}(ab+c)^{1-q}(z-a)^{2q-2}$$
$$\times\left\{\sum_{n=1}^{\infty}\beta_n\, e^{2\pi i n B(z)}+\sum_{j=0}^{2q-2} a_j(bz+c)^j(z-a)^{-j}\right\},\quad z\in B^{-1}(U_c),$$

$$(F\circ B)(a)=(-1)^{1-q}(ab+c)^{q-1}a_{2q-2},$$

$$(p\cdot B)(z)=(-1)^{1-q}(ab+c)^{1-q}\sum_{j=0}^{2q-3} b_j(bz+c)^j(z-a)^{2q-2-j},$$

and

$$(p \cdot B)'(a) = (-1)^{1-q}(ab+c)^{q-2}b_{2q-3}.$$

Comparison with (5.3.5) yields (5.3.6).

Remarks. (1) If the parabolic transformation corresponding to a puncture is given by

$$C(z) = z+\alpha,$$

and $p=\mathrm{pd}F$ is given by (5.3.4), then

$$F(\infty) = (-\alpha)^{1-q}b_{2q-3}.$$

The above equality generalizes (5.3.5).

(2) Theorem 5.2 is a strengthened version of Theorem 3.

(3) Parabolic fixed points corresponding to punctures on Ω/Γ can also be used as points at which we normalize the functions in $\mathscr{F}_{1-q}(\Omega, \Gamma)$. For example, in part (a) of Theorem 5.2, we may take one of the a_j to be a fixed point of a parabolic element γ_j corresponding to a puncture on Ω/Γ. The condition $\chi(\gamma_j)(a_j)=0$ in the definition of the space Z is now replaced by $\chi(\gamma_j)'(a_j)=0$. In actual computations (see, for example, § 8.5), this observation will yield significant simplifications.

§ 6. The Γ-invariant linear functionals on $\mathscr{R}_q(\mathbf{\Lambda}_q)$ and $\mathbf{A}_q(\Omega)$ (proof of Theorem 6)

6.1. We begin with the continuous analogue of a discrete problem that we will investigate.

If Ω is the region of discontinuity of a finitely generated non-elementary Kleinian group Γ, we define the Banach space $\mathbf{A}_q(\Omega)$ to consist of holomorphic functions φ on Ω that satisfy

$$\varphi(z) = O(|z|^{-2q}), \quad z \to \infty, \text{ if } \infty \in \Omega, \tag{6.1.1}$$

and

$$\|\varphi\| = \iint_{\Omega} \lambda(z)^{2-q}|\varphi(z)\, dz \wedge d\bar{z}| < \infty, \tag{6.1.2}$$

where $\lambda(z)|dz|$ is, as before, the Poincaré metric on Ω. The space $\mathbf{B}_q(\Omega)$ is the Banach space of holomorphic functions φ on Ω that satisfy (6.1.1) and

$$\|\varphi\|_\infty = \sup\{\lambda(z)^{-q}|\varphi(z)|;\, z \in \Omega\} < \infty. \tag{6.1.3}$$

The *Petersson scalar product*

$$\langle \varphi, \psi \rangle = i \iint_{\Omega} \lambda(z)^{2-2q} \varphi(z) \overline{\psi(z)}\, dz \wedge d\bar{z}$$

is well defined for $\varphi \in \mathbf{A}_q(\Omega)$, $\psi \in \mathbf{B}_q(\Omega)$, and establishes, by a theorem of Bers (see, for example, [14, Chapter III]), a conjugate linear topological isomorphism between the topological dual space $\mathbf{A}_q(\Omega)^*$ of $\mathbf{A}_q(\Omega)$ and $\mathbf{B}_q(\Omega)$.

6.2. We observe that $\mathbf{A}_q(\Omega, \Gamma) \subset \mathbf{B}_q(\Omega)$. For $A \in \Gamma$, the operator A_q^* is a linear invertible isometry of $\mathbf{A}_q(\Omega)$ and $\mathbf{B}_q(\Omega)$. The operator A_q^* is trivial on $\mathbf{A}_q(\Omega, \Gamma)$. Let $l \in \mathbf{A}_q(\Omega)^*$. We say that l is Γ-*invariant* if

$$l(A_q^* \varphi) = l(\varphi), \quad \text{all } A \in \Gamma, \text{ all } \varphi \in \mathbf{A}_q(\Omega). \tag{6.2.1}$$

PROPOSITION. *Let $l \in \mathbf{A}_q(\Omega)^*$ be given by $l(\varphi) = \langle \varphi, \psi \rangle$, for all $\varphi \in \mathbf{A}_q(\Omega)$, and fixed $\psi \in \mathbf{B}_q(\Omega)$. Then l is Γ-invariant if and only if $\psi \in \mathbf{A}_q(\Omega, \Gamma)$.*

Proof. A simple change of variable calculation shows that

$$\langle \varphi, \psi \rangle = \langle A_q^* \varphi, A_q^* \psi \rangle,$$

for all $\varphi \in \mathbf{A}_q(\Omega)$, $\psi \in \mathbf{B}_q(\Omega)$, $A \in \Gamma$. Hence for $\psi \in \mathbf{A}_q(\Omega, \Gamma)$, l is Γ-invariant. Conversely assume that l is Γ-invariant. Thus for $\varphi \in \mathbf{A}_q(\Omega)$, $A \in \Gamma$,

$$\begin{aligned} l(A_q^* \varphi) &= \langle A_q^* \varphi, \psi \rangle = \langle (A^{-1})_q^* \circ A_q^* \varphi, (A^{-1})_q^* \psi \rangle \\ &= \langle \varphi, (A^{-1})_q^* \psi \rangle, \end{aligned}$$

and

$$l(A_q^* \varphi) = l(\varphi) = \langle \varphi, \psi \rangle .$$

By the Bers' isomorphism theorem referred to earlier

$$(A^{-1})_q^* \psi = \psi, \quad \text{all } A \in \Gamma.$$

Hence $\psi \in \mathbf{A}_q(\Omega, \Gamma)$.

6.3. The Poincaré series operator Θ_q is defined on $\mathbf{A}_q(\Omega)$ and maps it onto $\mathbf{A}_q(\Omega, \Gamma)$. The norm of this operator is at most one. (We use the L^1-norm on $\mathbf{A}_q(\Omega, \Gamma)$ given by (0.1.2).)

It is easy to check that

$$\Theta_q \circ A_q^* = \Theta_q, \quad \text{all } A \in \Gamma, \tag{6.3.1}$$

and

$$\langle \varphi, \psi \rangle = \langle \Theta_q \varphi, \psi \rangle_\Gamma, \quad \text{all } \varphi \in \mathbf{A}_q(\Omega), \quad \text{all } \psi \in \mathbf{A}_q(\Omega, \Gamma). \tag{6.3.2}$$

THEOREM (Metzger [20]). *The kernel of*

$$\Theta_q: \mathbf{A}_q(\Omega) \to \mathbf{A}_q(\Omega, \Gamma)$$

is the closure Σ *in* $\mathbf{A}_q(\Omega)$ *of the linear span of*

$$\{\varphi - A_q^* \varphi;\ \varphi \in \mathbf{A}_q(\Omega),\ A \in \Gamma\}.$$

Proof. Formula (6.3.1) shows that

$$\varphi - A_q^* \varphi \in \operatorname{Ker} \Theta_q,$$

for all $\varphi \in \mathbf{A}_q(\Omega)$ and all $A \in \Gamma$. By linearity of Θ_q, the span of such functions lies in the kernel of Θ_q, and by the continuity of Θ_q so does the closure of the span; that is $\Sigma \subset \operatorname{Ker} \Theta_q$.

Conversely, by Hahn–Banach, it suffices to show that every $l \in \mathbf{A}_q(\Omega)^*$ that vanishes on Σ also vanishes on $\operatorname{Ker} \Theta_q$; that is, if $\psi \in \mathbf{B}_q(\Omega)$ and

$$\langle \varphi - A_q^* \varphi, \psi \rangle = 0, \quad \text{all } \varphi \in \mathbf{A}_q(\Omega), \text{ all } A \in \Gamma, \tag{6.3.3}$$

then

$$\langle \varphi, \psi \rangle = 0, \quad \text{all } \varphi \in \operatorname{Ker} \Theta_q. \tag{6.3.4}$$

Equation (6.3.3) shows, by Proportion 6.2, that $\psi \in \mathbf{A}_q(\Omega, \Gamma)$. Hence by (6.3.2), we have (6.3.4).

Remark. Metzger's theorem [20] is a formal consequence of properties of the spaces $\mathbf{A}_q(\Omega)$ and $\mathbf{B}_q(\Omega)$ and the operator Θ_q. Another formal description of $\operatorname{Ker} \Theta_q$ was given by Ljan [18].

6.4. The space $\mathcal{R}_q(\Lambda)$ is a subspace of $\mathbf{A}_q(\Omega)$; see, for example, Bers [4]. For $q=2$, $\mathcal{R}_2(\Lambda)$ is dense in $\mathbf{A}_2(\Omega)$, Ahlfors [1]; see also [14, Chapter IV]. We will for the moment consider $\mathcal{R}_q(\Lambda)$ and $\mathcal{R}_q(\Lambda_q)$ as vector spaces—without a topology. When we speak of continuous functionals on $\mathcal{R}_q(\Lambda)$, we will view $\mathcal{R}_q(\Lambda)$ as a subspace of the Banach space $\mathbf{A}_q(\Omega)$.

Let Λ_0 be a non-empty Γ-invariant subset of Λ_q. By $\mathcal{R}_q(\Lambda_0)^*$ we will denote the space of linear functionals (not necessarily continuous) on $\mathcal{R}_q(\Lambda_0)$. A linear functional $l \in \mathcal{R}_q(\Lambda_0)^*$ will be called Γ-*invariant* if it satisfies (6.2.1) for all $\varphi \in \mathcal{R}_q(\Lambda_0)$ and all $A \in \Gamma$.

PROPOSITION. *Fix* $(2q-1)$-*distinct points in* Λ_0: $a_1, a_2, \dots, a_{2q-1}$. *The vector space* $\mathcal{R}_q(\Lambda_0)^*$ *is canonically isomorphic to the vector space of functions on* Λ_0 *that vanish at* a_j, $j=1, \dots, 2q-1$.

Proof. By conjugation, we may assume $\Lambda_0 \subset \mathbf{C}$. For $b \in \Lambda_0 \setminus \{a_1, \dots, a_{2q-1}\}$ and $l \in \mathcal{R}_q(\Lambda_0)^*$, set

$$F(b) = l(f(b, \cdot)), \tag{6.4.1}$$

for f defined by (0.2.11). Conversely, given F, a function on Λ_0 that vanishes at a_j, $j=1, \dots, 2q-1$, we use (6.4.1) to define the values of $l \in \mathcal{R}_q(\Lambda_0)^*$ on the basis for $\mathcal{R}_q(\Lambda_0)$, specified by (0.2.11).

6.5.

THEOREM. *Let F be a function on* Λ_0 *that vanishes at* a_j, $j=1, \dots, 2q-1$ *and let* l *be the corresponding linear functional on* $\mathcal{R}_q(\Lambda_0)$ *given by* (6.4.1). *The* l *is* Γ-*invariant if and only if* $A^*_{1-q}F - F$ *is the restriction to* Λ_0 *of a polynomial* $p(A) \in \Pi_{2q-2}$, *for all* $A \in \Gamma$ (*that is, if and only if* $F \in \mathrm{F}_{1-q}(\Lambda_0, \Gamma)$).

Proof. We compute using formula (1.3.7), and the notation introduced in §§ 1.3 and 0.2. Fix $A \in \Gamma$. Choose $P \in \Pi_{2q-2}$ such that (recall the definition (1.3.2) and the convention about the index β for the funciton f_β)

$$P(b_k) = -F(b_k), \quad k = 1, \dots, 2q-1.$$

Then for $a \in \Lambda_0 \setminus \{a_1, \dots, a_{2q-1}\}$,

$$l(f(a, \cdot)) = F(a),$$

$$l(A_q^*(f(a, \cdot))) = A'(b)^{q-1}\left\{F(b) - \sum_k \left(\prod_{j \neq k} \frac{b-b_j}{b_k-b_j}\right) F(b_k)\right\}$$

$$= A'(b)^{q-1}\{F(b) + P(b)\}.$$

Assume that $l \in \mathcal{R}_q(\Lambda_0)^*$ is Γ-invariant. Then

$$F(Ab)\, A'(b)^{1-q} - F(b) = P(b), \quad \text{all } b \in \Lambda_0.$$

Since P depends only on l, A, and α and *not* the point b, we see that $F \in \mathbf{F}_{1-q}(\Lambda_0, \Gamma)$.

Conversely, if $F \in \mathbf{F}_{1-q}(\Lambda_0, \Gamma)$, then for $k=1,\dots,2q-1$, $A \in \Gamma$, $p(A)=(\mathrm{pd}\, F)(A)$, we have

$$F(Ab_k)A'(b_k)^{1-q} - F(b_k) = p(A)(b_k).$$

But $F(Ab_k)=F(a_k)=0$; hence $p(A)=P$ and l is Γ-invariant.

Remarks. (1) The proof also shows how to determine the value of $p(A)$.

(2) Given a Γ-invariant linear functional l on $\mathcal{R}_q(\Lambda_0)$, then we have constructed an Eichler integral F on Λ_0 that vanishes at $a_1,\dots,a_{2q-1}$, and from it the cocycle $\chi=\mathrm{pd}\, F$. If the points $\hat{a}_1,\dots,\hat{a}_{2q-1}$ are used to construct an Eichler integral $\hat{F}$ from l, then we obtain a new cocycle $\hat{\chi}=\mathrm{pd}\,\hat{F}$. Now

$$\hat{F}(a) = l(f_{\hat{a}}(a,\cdot)),$$

and

$$\frac{-1}{\zeta-a}\prod_{j=1}^{2q-1}\frac{a-\hat{a}_j}{\zeta-\hat{a}_j} = \frac{-1}{\zeta-a}\prod_{j=1}^{2q-1}\frac{a-a_j}{\zeta-a_j} + \sum_{k=1}^{2q-1}\left(\prod_{j\neq k}\frac{a-\hat{a}_j}{\hat{a}_k-\hat{a}_j}\right)\left(\frac{1}{\zeta-\hat{a}_k}\right)\left(\prod_{j=1}^{2q-1}\frac{\hat{a}_k-a_j}{\zeta-a_j}\right)$$

or

$$f_{\hat{a}}(a,\cdot) = f_a(a,\cdot) - \sum_{k=1}^{2q-1} P_k(a) f_a(\hat{a}_k,\cdot),$$

where P_k is the unique polynomial in Π_{2q-2} with $P_k(a_j)=\delta_{jk}$, j, $k=1,\dots,2q-2$. Hence

$$\hat{F}(a) = F(a) - \sum_{j=1}^{2q-1} P_k(a) F(\hat{a}_k).$$

Let $p \in \Pi_{2q-2}$ be defined by

$$p(a) = \sum_{j=1}^{2q-1} P_k(a) F(\hat{a}_k), \quad a \in \Lambda_0.$$

Thus $p \in \Pi_{2q-2}$ satisfies, $p(a_k)=F(\hat{a}_k)$, $k=1,\dots,2q-1$. We conclude that

$$\hat{F} = F - p;$$

which shows that the cohomology class determined by l is independent of the basis for $\mathcal{R}_q(\Lambda_0)$ used. Also note that even though we assumed that the $\hat{a}_k$'s differed from the a_j's, the formula we derived is valid in general.

6.6. Let us define $\tilde{\mathscr{R}}_q(\Lambda_0)$ to be $\mathscr{R}_q(\Lambda_0)$ factored by the linear span of

$$\{f - A_q^* f;\, f \in \mathscr{R}_q(\Lambda_0),\ A \in F\}.$$

The Γ-invariant linear functionals on $\mathscr{R}_q(\Lambda_0)$ are in one to one canonical correspondence with the linear functionals on $\tilde{\mathscr{R}}_q(\Lambda_0)$, $\tilde{\mathscr{R}}_q(\Lambda_0)^*$. Further, the Poincaré series operator Θ_q is well defined on $\tilde{\mathscr{R}}_q(\Lambda_0)$:

$$\Theta_q\colon \tilde{\mathscr{R}}_q(\Lambda_0) \to \mathbf{A}_q(\Omega, \Gamma). \tag{6.6.1}$$

By Theorem 1, Θ_q of (6.6.1) is surjective, and hence

$$\dim \tilde{\mathscr{R}}_q(\Lambda_0) \geq \dim \mathbf{A}_q(\Omega, \Gamma).$$

6.7.

LEMMA. *Let χ be a Π_{2q-2}-cocycle. Let $A \in \Gamma$ be parabolic with fixed point a. Then χ is parabolic with respect to A if and only if $\chi(A)(a) = 0$.*

Proof. Assume that there is a $v \in \Pi_{2q-2}$ such that (0.2.4) holds. Then

$$\chi(A)(a) = v(Aa)A'(a)^{1-q} - v(a),$$

and since $Aa = a$ and $A'(a) = 1$, we conclude that $\chi(A)(a) = 0$. Conversely we must show that if $\chi(A) = p \in \Pi_{2q-2}$ and $p(a) = 0$, then there is a $v \in \Pi_{2q-2}$ so that (0.2.4) holds. Consider the map

$$\Pi_{2q-2} \ni v \mapsto v \cdot A - v \in \Pi_{2q-2}.$$

This map is linear and its image consists of functions vanishing at a, by the first part of our argument. Let v be in the kernel of this operator. If v vanishes at $x \neq a$, then v also vanishes at $A^n x$, all $n \in \mathbf{Z}$, and such a v must be identically zero. Thus for $v \neq 0$ in the kernel of the above operator, we must have

$$v(z) = (z-a)^k, \quad 0 \leq k \leq 2q-2.$$

Now the condition

$$v(Az)A'(z)^{1-q} = v(z), \quad z \in \mathbf{C},$$

along with the formula for A

$$\frac{1}{Az-a} = \frac{1}{z-a} + K,$$

shows that $k=2q-2$. It is also easy to see that this v is in the kernel of the operator.

The kernel of the operator thus has dimension one, and the image has dimension $2q-2$, and must hence consist of *all* polynomials in Π_{2q-2} that vanish at a.

We have assumed that a is finite. If $a=\infty$, then by conjugation we can reduce this case to the one considered above.

Remark. See [14, Chapter V] for a direct proof of the lemma in case $a=\infty$.

6.8. We need one more fact concerning parabolic cocycles.

LEMMA. *Let Γ_0 be a rank 2 parabolic discrete group of Möbius transformations. Let A and B generate Γ_0. Let χ be a cocycle for Γ_0. Then χ is parabolic with respect to A if and only if it is parabolic with respect to B.*

Proof. It is most convenient to conjugate Γ_0 so that the common fixed point of A and B is at infinity. Without loss of generality we assume that $A(z)=z+1$, $B(z)=z+\tau$, $\operatorname{Im}\tau>0$. We now let $p_1=\chi(A)$, $p_2=\chi(B)$. Assume that χ is parabolic with respect to A. This means that $\deg p_1\leq 2q-3$. The commutativity of Γ_0 implies that $\chi(A\circ B)=\chi(B\circ A)$ or that

$$p_1(z+\tau)-p_1(z)=p_2(z+1)-p_2(z), \quad z\in\mathbf{C}.$$

Now $p_1(z+\tau)-p_1(z)$ is a polynomial in z of degree $\leq 2q-4$; hence so is $p_2(z+1)-p_2(z)$. We conclude that p_2 is at most of degree $2q-3$; showing that χ is parabolic with respect to B.

6.9. Let Λ_0 be a non-empty Γ-invariant subset of Λ_q. Let $b\in\Lambda_q\setminus\Lambda_0$ and set

$$\Lambda_1=\Lambda_0\cup\{\gamma b;\ \gamma\in\Gamma\}=\Lambda_0\cup\Gamma b.$$

LEMMA. *Let $l\in\tilde{\mathcal{R}}_q(\Lambda_0)^*$. Then l extends to an element of $\tilde{\mathcal{R}}_q(\Lambda_1)^*$ whenever b is not a parabolic fixed point of Γ. If b is a parabolic fixed point, the l extends to an element of $\tilde{\mathcal{R}}_q(\Lambda_1)^*$ if and only if the period of the Eichler integral F on Λ_0 corresponding to l is parabolic with respect to some parabolic $A\in\Gamma$ that fixes b.*

Proof. Fix $2q-1$ distinct points $a_1,\dots,a_{2q-1}$ in Λ_0. Let F be the unique Eichler integral supported on Λ_0 that vanishes at a_k, $k=1,\dots,2q-1$ corresponding to the Γ-invariant linear functional l on $\mathcal{R}_q(\Lambda_0)$. Let $\chi=\operatorname{pd}F$. The linear functional l extends to a Γ-invariant linear functional on $\mathcal{R}_q(\Lambda_1)$ if and only if χ is determined by an extension of F to an Eichler integral supported on Λ_1.

Assume that the stabilizer Γ_0 of b in Γ is cyclic of order n ($1 \leq n < \infty$). Let A generate Γ_0. There are two possibilities depending on whether or not n divides $1-q$.

If n does not divide $1-q$, then b is a point of q-uniqueness and F extends uniquely to Γb. If n divides $1-q$, then $A'(b)^{1-q}=1$ and hence $\chi(A)(b)=0$. The last assertion follows from the fact that $\chi(A^n)=0$ and

$$\chi(A^n) = \chi(A) \cdot [A^{n-1}+A^{n-2}+\ldots+1].\text{(}^7\text{)}$$

Hence

$$n\chi(A)(b) = 0.$$

Thus we may arbitrarily define $F(b)$ and observe that (4.1.4) holds for all $\gamma \in \Gamma_0$. The extension of F to Γb is accomplished by

$$F(\gamma b)\gamma'(b)^{1-q} - F(b) = \chi(\gamma)(b), \quad \gamma \in \Gamma. \tag{6.9.1}$$

If Γ_0 contains a loxodromic element, then b is again a q-uniqueness point and the extension of F to Γb is provided by Proposition 4.1, as above.

By the Lemma in § 12, it remains to consider the case where Γ_0 contains a parabolic element. If F extends to Γb then for every parabolic $\gamma \in \Gamma_0$, we have

$$F(\gamma b)\gamma'(b)^{1-q} - F(b) = \chi(\gamma)(b).$$

Since $\gamma b = b$ and $\gamma'(b)=1$, $\chi(\gamma)(b)=0$. By Lemma 6.7, χ is parabolic with respect to every parabolic element fixing b. Conversely, if χ is parabolic with respect to some parabolic element fixing b, it is parabolic with respect to every parabolic element fixing b by Lemma 6.8. The stabilizer Γ_0 of b in Γ is a group generated by a rank 1 or rank 2 parabolic group Γ_1 and an elliptic element A (of order $n=2$, 3, 4, or 6). We use (4.1.4) with $\gamma = A$ to define $F(b)$ if n does not divide $1-q$, and we let $F(b)$ be arbitrary if n divides $1-q$. We need to show that (6.9.1) holds for all $\gamma \in \Gamma_0$. Now an arbitrary $\gamma \in \Gamma_0$ can be written as

$$\gamma = A^k \circ B, \quad k \in \mathbf{Z},\ B \text{ parabolic} \in \Gamma_0.$$

Hence

$$\begin{aligned} \chi(\gamma)(b) &= \chi(A^k)(Bb)B'(b)^{1-q} + \chi(B)(b) \\ &= \chi(A^k)(b). \end{aligned}$$

(7) Not only does Γ act on Π_{2q-2}, but Γ generates an algebra of operators on Π_{2q-2}. Addition is operator sum.

The previous arguments now complete the analysis.

Remarks. (1) The extension of F to Λ_1 is unique if and only if b is a q-uniqueness point. If b is a non-uniqueness point, then F has a one-parameter family of extensions.

(2) If Λ_0 is empty and if we are given a cocycle, χ, then the above procedure shows whether or not χ is the period of an Eichler integral F supported on $\Lambda_1=\Gamma b$, and how to construct F whenever it exists.

6.10. Let Λ_1 be a non-empty Γ-invariant subset of Λ. We will establish the following

THEOREM. (a) *There is a canonical linear map*

$$\varepsilon\colon \tilde{\mathscr{R}}_q(\Lambda_1)^* \to H^1(\Gamma, \Pi_{2q-2}).$$

(b) *Let $PH_1^1(\Gamma, \Pi_{2q-2})$ be the cohomology classes that are represented by cocycles that are parabolic with respect to all parabolic elements whose fixed points are in Λ_1. There exists a linear map*

$$\varepsilon^*\colon PH_1^1(\Gamma, \Pi_{2q-2}) \to \tilde{\mathscr{R}}_q(\Lambda_1)^*,$$

such that $\varepsilon \circ \varepsilon^*$ is the identity. In particular,

$$\varepsilon(\tilde{\mathscr{R}}_q(\Lambda_1)^*) = PH_1^1(\Gamma, \Pi_{2q-2}).$$

(c) *Let N be the number of Γ-equivalence classes of non-uniqueness points in Λ_1. Then*

$$\dim \operatorname{Ker} \varepsilon = N.$$

Proof. (a) Choose $(2q-1)$ distinct points in Λ_1; call them $a_1, \ldots, a_{2q-1}$. If $l \in \tilde{\mathscr{R}}_q(\Lambda_1)^*$, then by Theorem 6.5, l is represented by a unique Eichler integral F supported on Λ_1 that vanishes at a_j, $j=1, \ldots, 2q-1$. We set $\varepsilon(l) = \operatorname{pd} F$.

(b) If $b \in \Lambda_1$ is fixed by the parabolic element $B \in \Gamma$ and $\chi = \operatorname{pd} F$, then

$$\chi(B)(b) = F(Bb)B'(b)^{1-q} - F(b) = 0,$$

since $Bb=b$, and $B'(b)=1$. Thus, by Lemma 6.7, χ is parabolic with respect to B. Conversely, let χ be a cocycle representing a cohomology class of $PH_1^1(\Gamma, \Pi_{2q-2})$. Thus $\chi(B)(b)=0$ for every fixed point $b \in \Lambda_1$ of a parabolic element $B \in \Gamma$. Proposition 4.1 and Lemma 6.9 show how to define an Eichler integral F supported on Λ_1 that vanishes at $a_1, \ldots, a_{2q-1}$ and that represents a cocycle cohomologous to χ.

To be specific, assume that there are uniqueness points in Λ_1. Then we may assume that a_j, $j=1,\dots,2q-1$, is a uniqueness point. By Proposition 4.1 and Lemma 6.9, we can find an Eichler integral F supported on the set of uniqueness points and vanishing at a_j, for $j=1,\dots,2q-1$ such that $\mathrm{pd}\,F$ is cohomologous to χ. We next choose a maximal set of inequivalent non-uniqueness points in Λ_1 (independently of χ) and set F to be zero at these points. The period of F extends F (by formula (6.9.1) with χ replaced by $\mathrm{pd}\,F$) to the Γ-orbit of this maximal set of inequivalent non-uniqueness points.

It remains to consider the case where every point of Λ_1 is a non-uniqueness point. We choose $b\in\Lambda_1$, and let $a_j\in\Gamma b$ for $j=1,\dots,2q-1$. Choose a cocycle χ representing a class in $PH^1_1(\Gamma,\Pi_{2q-2})$. Set $F_1(b)=0$ and construct an Eichler integral F_1 on Γb by (6.9.1). Next let $p\in\Pi_{2q-2}$ be such that $p(a_j)=F_1(a_j)$, $j=1,\dots,2q-1$ and define $F=F_1-p$. We now proceed as before to extend F to $\Lambda_1\setminus\Gamma b$. However, in this case the Eichler integral F depends on the cocycle χ and not just on its cohomology class $PH^1_1(\Gamma,\Pi_{2q-2})$; thus we must first choose a splitting map for the projection of cocycles onto cohomology classes.

In each case, the Eichler integral F defines a unique Γ-invariant linear functional $\varepsilon^*(\chi)$ on $\mathcal{R}_q(\Lambda_1)$. Linearity of ε^* is obvious, as is the fact that $\varepsilon\circ\varepsilon^*$ is the identity map.

(c) Let $l\in\mathcal{R}_q(\Lambda_1)^*$ and assume that $\varepsilon(l)=0$. Let F be the Eichler integral vanishing at a_j, $j=1,\dots,2q-1$, that represents the linear functional l. Then there exists a $p\in\Pi_{2q-2}$ such that

$$F(\gamma z)\gamma'(z)^{1-q}-F(z)=p(\gamma z)\gamma'(z)^{1-q}-p(z),$$

all $\gamma\in\Gamma$, all $z\in\Lambda_1$. In particular, $(F-p)\cdot\gamma=(F-p)$ for all $\gamma\in\Gamma$. Thus $F-p$ vanishes at the set of uniqueness points. If Λ_1 contains uniqueness points, then we may assume that the $a_j, j=1,\dots,2q-1$ are uniqueness points. It follows that $p=0$, and that F vanishes at the set of uniqueness points. Since the value of F at a set of inequivalent non-uniqueness points is arbitrary, the kernel of ε has the same dimension as the number of equivalence classes of non-uniqueness points in Λ_1.

Once again, it remains to consider the case where every point in Λ_1 is a non-uniqueness point. Again, we may assume that $a_1,\dots,a_{2q-1}$ are in a single Γ-equivalence class. Every coboundary χ can be represented by an N-dimensional affine space of Eichler integrals supported on Λ_1. (These need *not* vanish at the points a_j, $j=1,\dots,2q-1$.) The dimension of the vector space of Eichler integrals representing coboundaries is hence $N+(2q-1)$. The subspace of these Eichler integrals vanishing at a_j, $j=1,\dots,2q-1$ thus has dimension N.

6.11. Since a vector space and its dual have the same dimension, we have the following

THEOREM. *Let Λ_1 be a non-empty Γ-invariant subset of Λ_q consisting of non-parabolic uniqueness points and all the parabolic fixed points corresponding to the punctures on Ω/Γ. Then*

$$\dim \mathrm{A}_q(\Omega,\Gamma) \leqslant \dim PH^1_\Omega(\Gamma,\Pi_{2q-2}) \leqslant \dim \tilde{\mathscr{R}}_q(\Lambda_1).$$

Furthermore,

$$\dim \tilde{\mathscr{R}}_q(\Lambda_1) = \dim PH^1_\Omega(\Gamma,\Pi_{2q-2})+N,$$

where N is the number of Γ-equivalence classes of parabolic fixed points in Λ_1 (=the number of Γ-equivalence classes of parabolic fixed points corresponding to punctures on Ω/Γ).

Proof. In the case under consideration, $PH^1_\Omega(\Gamma,\Pi_{2q-2})=PH^1_1(\Gamma,\Pi_{2q-2})$.

Remarks. (1) The number N, above, may be smaller than the number of punctures on Ω/Γ since a parabolic fixed point may correspond to two punctures.

(2) The above theorem can be strengthened as follows to generalize Theorem 6.

COROLLARY. *Let Λ_1 consist of uniqueness points and all the parabolic fixed points. Assume $\beta^*: \mathrm{A}_q(\Omega,\Gamma)\to PH^1(\Gamma,\Pi_{2q-2})$ is surjective. Then the kernel of $\Theta_q: \tilde{\mathscr{R}}_q(\Lambda_1)\to \mathrm{A}_q(\Omega,\Gamma)$ has dimension N, where N is the number of Γ-equivalence classes of non-uniqueness points in Λ_1 ($\leqslant$the number of Γ-equivalence classes of parabolic fixed points).*

§ 7. Schottky groups (proof of Theorem 4, first part)

7.1. Let Γ be a Schottky group on $g>1$ free generators $A_1,\dots,A_g$. Let $a_1,\dots,a_{2q-1}$ be $(2q-1)$ distinct fixed points of elements of Γ. The space of cocycles for Γ, $Z^1(\Gamma,\Pi_{2q-2})$, has dimension $(2q-1)g$. The map that sends a cocycle χ to

$$(\chi(A_1),\dots,\chi(A_g))\in(\Pi_{2q-2})^g$$

is an isomorphism. The dimension of the space of coboundaries is $2q-1$, and hence

$$\dim H^1(\Gamma,\Pi_{2q-2})=(2q-1)(g-1)=\dim \mathrm{A}_q(\Omega,\Gamma).$$

Thus the hypothesis of the Theorem 3 is satisfied. We obtain a basis for the space of cocycles by assigning arbitrary polynomials to each of the generators, and thus we obtain $(2q-1)g$ functions $\hat{F}_1, \dots, \hat{F}_{(2q-1)g}$ defined at first only at the loxodromic fixed points by formulae similar to (4.1.3). Subtracting appropriate polynomials from these functions, we get $d=(2q-1)(g-1)$ functions $F_1, \dots, F_d$ that can be extended by continuity to Λ and form a basis for the space $\mathscr{F}_{1-q}(\Omega, \Gamma)$.

It follows that for $f \in \mathscr{R}_q(\Lambda)$, with poles only at (loxodromic) fixed points of elements of Γ, there is a finite algebraic algorithm for determining whether or not $\Theta_q f=0$. Theorem 4 is proven for Schottky groups.

7.2. Before proceeding to a more detailed study of Schottky groups we introduce the following general

Definition. A set $S \subset \Lambda_q$ will be called a *q-stratification* of Γ provided:

(a) each point of S is a fixed point of an element of Γ,

(b) Θ_q maps $\mathscr{R}_q(S)$ isomorphically onto $\mathbf{A}_q(\Omega, \Gamma)$, and

(c) the set S depends *only* on a presentation of Γ and at most finitely many (arbitrary) choices.

Remarks. (1) If Γ possesses a q-stratification, then we shall say that Γ is *q-stratifiable.*

(2) Every geometrically fintite non-elementary function group is 2-stratifiable by the results of [16], [17]. We will in some cases reprove this fact and provide 2-stratifications that are more useful for our purposes: the construction of a basis for the cohomology groups $H^1(\Gamma, \Pi_2)$ and $PH^1(\Gamma, \Pi_2)$ defined in § 0.2.

(3) A q-stratification of Γ always contains $\dim \mathbf{A}_q(\Omega, \Gamma)+(2q-1)$ points. If $S \subset \Lambda_q$ contains $(2q-1)$ or more points, then S is a q-stratification for Γ if and only if $e_S \colon \mathscr{F}_{1-q}(\Omega, \Gamma) \to \mathbf{C}^{|S|-(2q-1)}$ is an isomorphism, where we normalize the functions in $\mathscr{F}_{1-q}(\Omega, \Gamma)$ to vanish at $(2q-1)$ points in S.([8])

(4) The stratifications of [16] are 2-stratifications in our sense. It is not known whether 2-stratifications always are stratifications. Stratifications provide global holomorphic coordinates for deformation spaces of Kleinian groups, while 2-stratifications define global holomorphic functions that yield only local coordinates for the deformation spaces.

([8]) The evaluation map e_S was defined in § 2.3. We denote by $|S|$ the cardinality of the set S.

7.3. We return to the hypothesis of § 7.1 and assume that a_j is a fixed point of $B_j \in \Gamma$ for $j=1, \ldots, 2q-1$. Then if we consider the set of cocycles

$$Z=\{\chi \in Z^1(\Gamma, \Pi_{2q-2});\ \chi(B_j)(a_j)=0 \text{ for } j=1, \ldots, 2q-1\},$$

then $Z \cong H^1(\Gamma, \Pi_{2q-2})$. Thus from Z we construct more directly the space $\mathscr{F}_{1-q}(\Omega, \Gamma)$.

7.4. Let us assume for the moment that $q \leqslant g$. In this case, we shall take a_{2j-1} and a_{2j} to be the fixed points of A_j for $j=1, \ldots, q$. A polynomial $p \in \Pi_{2q-2}$ is uniquely determined by the vector

$$(p(a_1), \ldots, p(a_{2q-1})) \in \mathbf{C}^{2q-1}.$$

To simplify notation, we let e_j be the vector

$$\begin{array}{c}(0, \ldots, 0, 1, 0, \ldots, 0).\\ \uparrow \\ j\text{th spot}\end{array}$$

Thus e_j corresponds to the polynomial

$$p(z)=\prod_{k \neq j} \frac{z-a_k}{a_j-a_k}.$$

We shall use the above correspondence to describe the values for cocycles that will lead to a basis of $\mathscr{F}_{1-q}(\Omega, \Gamma)$. It suffices to consider only cocycles χ that satisfy

$$\chi(A_j)(a_k)=0 \quad \text{for } j=1, \ldots, q, \quad \text{and} \quad k=\begin{cases}2j-1,\ 2j, & \text{if } j<q, \\ 2j-1, & \text{if } j=q.\end{cases}$$

We consider the set

$$\{A_i(a_j);\ 1 \leqslant i \leqslant g,\ 1 \leqslant j \leqslant 2q-1\} \setminus \{a_j;\ 1 \leqslant j \leqslant 2q-1\}; \tag{7.4.1}$$

it consists of $d=(2q-1)(g-1)$ distinct points $b_1, \ldots, b_d$. By Theorem 1, Θ_q maps $\mathscr{R}_q(S)$ onto $\mathbf{A}_q(\Omega, \Gamma)$, where $S=\{a_1, \ldots, a_{2q-1}, b_1, \ldots, b_d\}$. By Theorem 2.5, there exists a basis $F_1, \ldots, F_d$ for $\mathscr{F}_{1-q}(\Omega, \Gamma)$ such that $F_j(b_k)=\delta_{jk}$. We now compute the cocycle $\chi_j=\operatorname{pd} F_j$. Assume $b_j=A_i(a_k)$. Then

$$\chi_j(A_m)=A_i'(a_k)^{1-q} e_k \delta_{im}.$$

From the knowledge of $\{\chi_j(A_m);\ 1 \leqslant j \leqslant d,\ 1 \leqslant m \leqslant g\}$, we can compute $\{\chi_j(\gamma);\ 1 \leqslant j \leqslant d\}$ for arbitrary $\gamma \in \Gamma$, and hence the values of $\{F_j;\ 1 \leqslant j \leqslant d\}$ at the fixed points of γ.

Example. For the convenience of the reader, we present the most useful example: $q=2$ and $g\geqslant 2$. Here, we can let a_1, a_2 be the fixed points of A_1, and a_3 one of the fixed points of A_2. We set

$$\begin{aligned} b_j &= A_{j+1}(a_1), \quad j=1,\dots,g-1, \\ &= A_{j-g+2}(a_2), \quad j=g,\dots,2q-2, \\ &= A_1(a_3), \quad j=2g-1, \\ &= A_{j-2g+3}(a_3), \quad j=2g,\dots,3g-3. \end{aligned}$$

For $m=1,\dots,g$, we have

$$\begin{aligned} \chi_j(A_m) &= A'_{j+1}(a_1)^{1-q} e_1 \delta_{j+1,m}, \quad j=1,\dots,g-1, \\ &= A'_{j-g+2}(a_2)^{1-q} e_2 \delta_{j-g+2,m}, \quad j=g,\dots,2g-2, \\ &= A'_1(a_3)^{1-q} e_3 \delta_{1,m}, \quad j=2g-1, \\ &= A'_{j-2g+3}(a_3)^{1-q} e_3 \delta_{j-2g+3,m}, \quad j=2g,\dots,3g-3. \end{aligned}$$

7.5. The key aspect of the computations of the previous section is finding $2q-1$ distinct points $a_1,\dots,a_{2q-1}$ in Λ with the property that the set

$$S = \{A_j(a_k);\, j=0,\dots,g,\ k=1,\dots,2q-1\} \tag{7.5.1}$$

consists of $(2q-1)g$ points (where A_0 is the identity).

LEMMA. *For any* $2q-1$ *distinct points* $a_1,\dots,a_{2q-1}$ *in* Λ, *the set* S *of* (7.5.1) *consists of at least* $(2q-1)g$ *points.*

Proof. The Poincaré series map Θ_q maps $\mathcal{R}_q(S)$ onto $\mathbf{A}_q(\Omega,\Gamma)$ by Theorem 1. Hence we have

$$\dim \mathcal{R}_q(S) \geqslant \mathbf{A}_q(\Omega,\Gamma) = (2q-1)(g-1).$$

It follows that S has at least $(2q-1)g$ points.

Thus we must show how to construct the set S with at most $(2q-1)g$ points. We fix the group Γ with generators $A_1,\dots,A_g$ (and thus g is fixed) and use induction on q. We have seen in § 7.4 how to select S for $q=2$. Assume by induction that we have selected the set S for a fixed $q\geqslant 2$. Assume that we are using the points $a_1,\dots,a_{2q-1}$. The set S of (7.5.1) hence consists, by induction, of $(2q-1)g$ points. Let us fix a j, $1\leqslant j\leqslant g$, and look at the points $\{A_j^{-1}(a);\, a\in S\}$. This set consists of $(2q-1)g$ points, and since $g\geq 2$, we can choose points a_{2q}, a_{2q+1} in this set so that $\{a_1, \ldots, a_{2q+1}\}$ consists of distinct points. The set

$$\{A_j(a_k);\, j=0, \ldots, g,\ k=1, \ldots, 2q+1\}$$

consists of at most $(2q+1)g$ points.

We summarize our results in the next

THEOREM. *Every Schottky group is q-stratifiable, for all $q\geq 2$.*

§ 8. Fuchsian and quasi-Fuchsian groups (proof of Theorem 4, conclusion)

8.1. Let Γ be a finitely generated quasi-Fuchsian group of the first kind. We can find canonical generators $A_1, B_1, \ldots, A_g, B_g, E_1, \ldots, E_n$ for Γ. The A_j, B_j are loxodromic; the E_k are parabolic (set $\nu_k=\infty$) or elliptic (set ν_k=the order of E_k). The signature of Γ is $(g, n; \nu_1, \ldots, \nu_n)$ and its type is (g, n). We write $C=[A, B]=A\circ B\circ A^{-1}\circ B^{-1}$. The defining relations for Γ are:

$$\prod_{j=1}^{g} C_j \circ \prod_{k=1}^{n} E_k = I, \tag{8.1.1}$$

$$E_k^{\nu_k} = I\,, \quad \text{all } k=1, \ldots, n \text{ with } \nu_k<\infty. \tag{8.1.2}$$

Every elliptic or parabolic element in Γ is conjugate to a power of E_k, for some $k=1, \ldots, n$.

For the group Γ, we have

$$\dim \mathrm{A}_q(\Omega, \Gamma) = 2\left\{(2q-1)(g-1)+\sum_{k=1}^{n}\,[q-q/\nu_k]\right\},$$

and

$$\begin{aligned}\dim PH^1_\Omega(\Gamma, \Pi_{2q-2}) &= \dim PH^1_\Delta(\Gamma, \Pi_{2q-2})\\ &= \dim PH^1(\Gamma, \Pi_{2q-2}) = \dim \mathrm{A}_q(\Omega, \Gamma),\end{aligned}$$

here Δ is one of the two invariant components of Γ. (For details, see [14, Chapter VII].)

8.2. If χ is a cocycle and $E\in\Gamma$ is elliptic of order ν, then

$$\chi(E^\nu) = 0.$$

It follows that (see Ahlfors [2], Eichler [7], or Weil [26] and § 8.4) $\chi(E)$ is an element of

$$(\Pi_{2q-2})_E = \{v \in \Pi_{2q-2};\ v = p \cdot E - p \text{ for some } p \in \Pi_{2q-2}\}.$$

An easy computation shows that

$$\dim(\Pi_{2q-2})_E = 2[q - q/\nu]. \tag{8.2.1}$$

Note that if χ is parabolic with respect to the parabolic element E, then we also have that $\chi(E) \in (\Pi_{2q-2})_E$.

We see that the space of parabolic cocycles χ can be identified with elements

$$(\chi(A_1), \chi(B_1), \ldots, \chi(A_g), \chi(B_g), \chi(E_1), \ldots, \chi(E_n))$$

in

$$V = (\Pi_{2q-2})^{2g} \times (\Pi_{2q-2})_{E_1} \times \ldots \times (\Pi_{2q-2})_{E_n}$$

subject to the relation forced on χ by (8.1.1). That is, if we define a map

$$h: V \to \Pi_{2q-2}$$

by

$$\begin{aligned} &h(p_1, p_2, \ldots, p_{2g-1}, p_{2g}, p_{2g+1}, \ldots, p_{2g+n}) \\ &\quad = p_1 \cdot (B_1 - 1) \cdot A_1^{-1} \cdot B_1^{-1} \cdot (C_2 \circ \ldots \circ C_g) \cdot (E_1 \circ \ldots \circ E_n) \\ &\quad\quad + p_2 \cdot (A_1^{-1} - 1) \cdot B_1^{-1} \cdot (C_2 \circ \ldots \circ C_g) \cdot (E_1 \circ \ldots \circ E_n) \\ &\quad\quad \vdots \\ &\quad\quad + p_{2g-1} \cdot (B_g - 1) \cdot A_g^{-1} \cdot B_g^{-1} \cdot (E_1 \circ \ldots \circ E_n) \\ &\quad\quad + p_{2g} \cdot (A_g^{-1} - 1) \cdot B_g^{-1} \cdot (E_1 \circ \ldots \circ E_n) \\ &\quad\quad + p_{2g+1} \cdot (E_2 \circ \ldots \circ E_n) + \ldots + p_{2g+n}, \end{aligned}$$

then

$$h(p_1, \ldots, p_{2g+n}) = 0$$

if and only if there is a parabolic cocycle χ with

$$\chi(A_1) = p_1, \quad \chi(B_1) = p_2, \quad \ldots, \quad \chi(B_g) = p_{2g},$$
$$\chi(E_1) = p_{2g+1}, \quad \ldots, \quad \chi(E_n) = p_{2g+n}.$$

Since

$$\dim V = 2g(2q-1)+2\sum_{k=1}^{n}[q-q/\nu_k],$$

it follows that the linear map h is surjective.

We have reduced the problem of constructing the values at fixed points in Λ_q of a basis for $\mathscr{F}_{1-q}(\Omega,\Gamma)$ to a problem in linear algebra (evaluating the kernel of a linear map). This completes the proof of Theorem 4 for quasi-Fuchsian groups.

8.3. If E is parabolic, then the space $(\Pi_{2q-2})_E$ consists of those polynomials $p\in\Pi_{2q-2}$ that vanish at the fixed point a of E. The kernel of the operator $(E-1)\colon\Pi_{2q-2}\to\Pi_{2q-2}$ consists of polynomials p that vanish to order $2q-2$ at a. Assume that $a\in\mathbf{C}$, then the polynomials

$$p_j(z)=(z-a)^j,\quad j=0,\dots,2q-2,$$

form a basis for Π_{2q-2}. It is easy to see that (as a consequence of 1.3.3)

$$(p_j\cdot E)(z)=(Ez-a)^jE'(z)^{1-q}=(z-a)^jE'(z)^{1-q+j/2}=\frac{(Ez-a)^{2-2q+j}}{(z-a)^{2-2q}},$$

and thus

$$p_{2q-2}\in\operatorname{Ker}(E-1).$$

An easy calculation shows that

$$\operatorname{ord}_a p_j\cdot(E-1)=j+1,\quad \text{for } j=0,\dots,2q-3.$$

Hence we conclude that

$$\operatorname{Image}(E-1)=\{p\in\Pi_{2q-2};\ p(a)\neq 0\}.$$

We have previously used this observation (see § 6.7 and [14, p. 173]).

8.4. We now carry over the above analysis to the case of loxodromic or elliptic E with fixed points at finite points a and b. Thus E can be written as

$$\frac{E(z)-a}{E(z)-b}=K\,\frac{z-a}{z-b},\quad K\neq 0,1.$$

If follows that

$$E'(a)=K,\quad E'(b)=K^{-1}.$$

The functions

$$p_j(z) = (z-a)^j (z-b)^{2q-2-j}, \quad j = 0, \dots, 2q-2,$$

form a basis for Π_{2q-2}. A calculation using (1.3.3) shows that

$$p_j \cdot E = K^{1-q+j} p_j.$$

Hence the p_j are eigenfunctions for the action of E on Π_{2q-2}. If E is loxodromic (or elliptic of infinite order), then $\mathrm{Ker}(E-1)$ is one-dimensional and is spanned by p_{q-1}. The image of $(E-1)$ is spanned by the functions $p_0, \dots, p_{q-2}, p_q, \dots, p_{2q-2}$. If E is elliptic of order ν, and $K = e^{2\pi i/\nu}$, then the kernel of $(E-1)$ is spanned by the functions p_j with $j \equiv q-1 \pmod{\nu}$ and has dimension $2[(q-1)/\nu]+1$; the image of $(E-1)$ is spanned by the functions p_j with $j \not\equiv q-1 \pmod{\nu}$ and has dimension

$$(2q-1) - 2[(q-1)/\nu] - 1 = -2[(1-q)(1-1/\nu)] = 2[q(1-1/\nu)].$$

8.5.

Example. The group $\Gamma = PSL(2, \mathbf{Z})$ is of considerable interest to number theorists. Its signature is $(0, 3; 2, 3, \infty)$ and it is generated by $A(z) = -z^{-1}$, $B(z) = -(z-1)^{-1}$. Note that $C(z) = A \circ B(z) = z - 1$. The defining relations for Γ are

$$A^2 = I = B^3,$$

$$A \circ B \text{ is parabolic;}$$

and the fixed points of A are at $\pm i$, and those of B at ω and $\bar{\omega}$, where $\omega = \frac{1}{2} + \frac{1}{2} i\sqrt{3} = e^{\pi i/3}$. The first $q \geq 2$ for which $PH^1(\Gamma, \Pi_{2q-2})$ (or, equivalently, $A_q(\Omega, \Gamma)$) is non-trivial is $q = 6$.

Assume that $q = 6$. Then

$$\dim PH^1(\Omega, \Pi_{10}) = 2.$$

We must find 2 linearly independent parabolic cocycles that are periods of potentials. One notes that

$$\dim H^1(\Gamma, \Pi_{10}) = 3.$$

The 14-dimensional space $Z^1(\Gamma, \Pi_{10})$ of cocycles is obtained by assigning to A a polynomial in the linear span of

$$\{(z-i)^j (z+i)^{10-j}; j = 0, 2, 4, 6, 8, 10\},$$

and to B a polynomial in the linear span of

$$\{(z-\omega)^j(z-\bar{\omega})^{10-j};\, j=0,1,3,4,6,7,9,10\}.$$

We notice that all the elliptic fixed points are 6-uniqueness points. We shall hence be able to choose some elliptic fixed points to be among the 11 points at which the Eichler integrals in $\mathbf{F}_{-5}(\Lambda_6^0,\Gamma)$ are required to vanish. Every cocycle for Γ is cohomologous to one that assigns the zero polynomial to B. For $p\in\Pi_{10}$, $p\in\operatorname{Ker}(B-1)$ if and only if

$$p(z)=\sum_{j=2,5,8}\beta_j(z-\omega)^j(z-\bar{\omega})^{10-j}.$$

Note that the values at $\pm i$ of such polynomials are arbitrary, and conclude that every Eichler integral on Λ_6^0 is equivalent to an F with $(\operatorname{pd}F)(B)=0$ and $F(\pm i)=0$. It follows that $(\operatorname{pd}F)(A)(\pm i)=0$. Letting $\chi=\operatorname{pd}F$, we see that we may assume that

$$\chi(A)(z)=\sum_{j=2,4,6,8}\alpha_j(z-i)^j(z+i)^{10-j}.$$

Since $\chi(A)(\pm i)=0$ and $\chi(B)=0$, it follows that

$$\chi(B^k\circ A\circ B^{-k})(B^k(\pm i))=0,\quad k=1,2,$$

because $B^k(\pm i)$ are the fixed points of $B^k\circ A\circ B^{-k}$ and

$$\chi(B^k\circ A\circ B^{-k})=\chi(A)\cdot B^{-k}.$$

Thus all the χ constructed as above are represented by Eichler integrals that vanish at the eight points:

$$\pm i,\omega,\bar{\omega},B\,(\pm i)=-\frac{1}{2}\pm\frac{i}{2},\quad B^2(\pm i)=1\pm i.$$

Since $C=A\circ B$, it follows that

$$\chi(C)=\chi(A)\cdot B. \tag{8.5.1}$$

We are interested only in parabolic cocycles; hence we require that

$$\deg\chi(C)\leq 9.$$

From (8.5.1) we see that

$$\chi(C)(z)=\sum_{j=2,4,6,8}\alpha_j[-iz+(-1+i)]^j[iz+(-1-i)]^{10-j},$$

and thus χ is a parabolic cocycle if and only if

$$\alpha_2+\alpha_4+\alpha_6+\alpha_8=0. \tag{8.5.2}$$

We must select 3 more points at which to normalize potentials. We choose

$$\begin{aligned}
&\infty = \text{fixed point of } C,\\
&0 = \text{fixed point of } B\circ A = B\circ C\circ B^{-1} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},\\
&1 = \text{fixed point of } B^2\circ C\circ B^{-2} = B^2\circ A\circ B^{-1} = \begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix}.
\end{aligned}$$

The condition that F vanish at ∞ is equivalent to $\deg\chi(C)\leq 8$. This condition is equivalent to the additional equation

$$(10-6i)\,\alpha_2+(10-2i)\,\alpha_4+(10+2i)\,\alpha_6+(10+6i)\,\alpha_8=0,$$

which, in view of (8.5.2), is equivalent to

$$-3\alpha_2-\alpha_4+\alpha_6+3\alpha_8=0. \tag{8.5.3}$$

Since $\chi(B\circ A)=\chi(A)$, the condition that F vanish at 0 is equivalent to $\chi(A)'(0)=0$, or to

$$6i\alpha_2+2i\alpha_4-2i\alpha_6-6i\alpha_8=0.$$

This last equation is equivalent to equation (8.5.3) and hence may be ignored. Finally,

$$\chi(B^2\circ A\circ B^{-1})=\chi(A)\cdot B^{-1}=\sum_{j=2,4,6,8}\alpha_j[(1-i)z-1]^j\,[(1+i)z-1]^{10-j}.$$

The condition that F vanishes at 1 is equivalent to $\chi(B^2\circ A\circ B^{-1})'(1)=0$, or to

$$(-8+6i)\,\alpha_2+(-6+2i)\,\alpha_4+(-4-2i)\,\alpha_6+(-2-6i)\,\alpha_8=0,$$

which in view of (8.5.3) can be replaced by

$$4\alpha_2+3\alpha_4+2\alpha_6+\alpha_8=0. \tag{8.5.4}$$

The three equations (8.5.2), (8.5.3), and (8.5.4) have a 2-dimensional solution space with a basis given by

$$(\alpha_2,\alpha_4,\alpha_6,\alpha_8)=(2,-3,0,1) \quad\text{and}\quad (1,-2,1,0).$$

Thus the periods $\hat{\chi}_1,\hat{\chi}_2$ of two linearly independent functions in $\mathscr{F}_{-5}(\Omega,\Gamma)$ satisfy

$$\hat{\chi}_1(A)(z) = 2(z-i)^2(z+i)^8 - 3(z-i)^4(z+i)^6 + (z-i)^8(z+i)^2,$$

$$\hat{\chi}_2(A)(z) = (z-i)^2(z+i)^8 - 2(z-i)^4(z+i)^6 + (z-i)^6(z+i)^4,$$

$$\hat{\chi}_1(B) = 0 = \hat{\chi}_2(B).$$

It is more convenient to choose

$$\chi_1 = \hat{\chi}_1 - \hat{\chi}_2,$$

$$\chi_2 = \hat{\chi}_1 - 3\hat{\chi}_2.$$

For these cocycles

$$\chi_1(A)(z) = (z-i)^2(z+i)^8 - (z-i)^4(z+i)^6 - (z-i)^6(z+i)^4 + (z-i)^8(z+i)^2,$$

$$\chi_2(A)(z) = -(z-i)^2(z+i)^8 + 3(z-i)^4(z+i)^6 - 3(z-i)^6(z+i)^4 + (z-i)^8(z+i)^2,$$

$$\chi_1(B) = 0 = \chi_2(B).$$

We now let

$$\{a_1, \dots, a_{11}\} = \left\{0, 1, \infty, \pm i, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, -\frac{1}{2} \pm \frac{1}{2} i, 1 \pm i\right\}.$$

Among the points in $\{A(a_j), B(a_j);\ j=1, \dots, 11\}$, we must be able to find two: b_1, b_2, such that $\mathscr{F}_{-5}(\Omega, \Gamma)|S$ has dimension 2, where $S = \{a_1, \dots, a_{11}, b_1, b_2\}$. Let $F_j \in \mathscr{F}_{-5}(\Omega, \Gamma)$ with $\operatorname{pd} F_j = \chi_j$, $j=1,2$. Symmetry of the cocycles under study together with the injectivity of the period map yield

$$\overline{F_1(z)} = F_1(\bar{z}), \quad z \in \Lambda_6, \tag{8.5.5}$$

$$\overline{F_2(z)} = -F_2(\bar{z}), \quad z \in \Lambda_6. \tag{8.5.6}$$

The set $\{a_j, A(a_j), B(a_j);\ j=1, \dots, 11\}$ contains in addition to $\{a_1, \dots, a_{11}\}$ only three points: $-1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\, i$. These are the fixed points of $A \circ B^2 \circ A \circ B^{-1} \circ A^{-1}$ and $A \circ B \circ A^{-1}$, respectively. Let $b = -\frac{1}{2} + \frac{1}{2}\sqrt{3}\, i = e^{2\pi i/3}$. To show that $b, \bar{b}$ may be used as b_1 b_1 and b_2, we note that in view of (8.5.5) and (8.5.6),

$$\det \begin{vmatrix} F_1(b) & F_2(b) \\ F_1(\bar{b}) & F_2(\bar{b}) \end{vmatrix} = -2 \operatorname{Re} F_1(b) \overline{F_2(b)}.$$

We must compute for $j=1,2$,

$$F_j(b)=\chi_j(\gamma)(b)[\gamma'(b)^{-5}-1]^{-1},$$

where $\gamma=A\circ B\circ A^{-1}=A\circ B\circ A$. The cocycle condition shows that

$$\chi_j(\gamma)=\chi_j(A)\cdot(B\circ A+1).$$

We note that

$$B\circ A=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},$$

and that it suffices to show that $\operatorname{Re}[(\chi_1(\gamma)(b))\overline{(\chi_2(\gamma)(b))}]\neq 0$. This follows by direct computation.(9) As a matter of fact

$$\chi_1(\gamma)(b)=48(-1+i),$$

$$\chi_2(\gamma)(b)=32(\sqrt{3}-3i).$$

It follows that the Poincaré series of the two rational functions

$$\frac{1}{\zeta+\frac{1}{2}-\frac{1}{2}\sqrt{3}\,i}f(\zeta),\quad \frac{1}{\zeta+\frac{1}{2}+\frac{1}{2}\sqrt{3}\,i}f(\zeta),$$

where

$$f(\zeta)=\frac{1}{\zeta(\zeta-1)(\zeta^2+1)(\zeta^2-\zeta+1)(\zeta^2+\zeta+\frac{1}{2})(\zeta^2-2\zeta+2)},$$

form a basis for $\mathbf{A}_6(\Omega,\Gamma)$. Similarly, the vanishing problem (for $q=6$) for this group is handled via the two cocycles χ_1 and χ_2 that we constructed above. The calculations are *tedious,* but the point to be emphasized is that as long as $f\in\mathcal{R}_6(\Lambda_6)$ has poles only at fixed points, then we can determine (*in a finite amount of time*) whether or not $\Theta_6 f=0$, by calculations over fields such as $\mathbf{Q}[i,\sqrt{3}]$.

Remark. The basis $\{\varphi_1,\varphi_2\}$ for $\mathbf{A}_6(\Omega,\Gamma)$ produced above satisfies $\varphi_1(\bar z)=\overline{\varphi_2(z)}$, all $z\in\Omega$. Set $\psi_1=\varphi_1+\varphi_2$ and $\psi_2=\varphi_1-\varphi_2$. Then $\{\psi_1,\psi_2\}$ is also a basis for $\mathbf{A}_6(\Omega,\Gamma)$ with

$$\psi_1(\bar z)=\overline{\psi_1(z)},\quad z\in\Omega,$$

$$\psi_2(\bar z)=-\overline{\psi_2(z)},\quad z\in\Omega.$$

(9) Using a micro-computer, of course, to save time.

Letting U be the upper half plane, we see that both ψ_1 and ψ_2 span the one-dimensional space $\mathbf{A}_6(U,\Gamma)$. The classical discriminant from elliptic function theory

$$\Delta(z) = (2\pi)^{12} \sum_{k=1}^{\infty} \tau(k)\, e^{2\pi i k z}, \quad z \in U,$$

where $\tau(k) \in \mathbf{Z}$ is the value at the positive integer k of the Ramanujan tau function, also spans $\mathbf{A}_6(U,\Gamma)$. It follows that there are constants $c_j \neq 0$ such that

$$\Delta(z) = c_j \psi_j(z), \quad \text{all } z \in U,\ j = 1,2. \tag{8.5.7}$$

Since it is known that $\tau(1)=1$, the evaluation of c_j involves computing the Fourier series expansion of the cusp form ψ_j. It is not clear whether the formula for Δ as a Poincaré series of a specific rational function (given by (8.5.7), for example) will help in studying the number theoretic properties of Δ and τ.

8.6. Let us assume that Γ is of type $(g,0)$, $g \geqslant 2$, and let $2 \leqslant q \leqslant 2g-1$. Let $a_1, a_2, \ldots, a_{2q-1}$ be selected from among the fixed points of the generators

$$A_1, \ldots, A_g, B_1, \ldots, B_{g-1}.$$

The set ($A_0 = I$, as usual)

$$S_0 = \{A_j(a_k);\, j = 0, \ldots, g,\ k = 1, \ldots, 2q-1\} \cup \{B_j(a_k);\, j = 1, \ldots, g-1,\ k = 1, \ldots, 2q-1\}$$

consists of $(2q-1)(2g-1)$ distinct points. If F is a potential for Γ that vanishes at each point of S_0, then the cocycle χ determined by F satisfies

$$\chi(A_j) = 0, \quad j = 1, \ldots, g,$$

$$\chi(B_j) = 0, \quad j = 1, \ldots, g-1.$$

From these two equations, we see that the defining relation (8.1.1) for Γ implies that $p = \chi(B_g)$ satisfies

$$p \cdot (A_g - 1) = 0.$$

It follows from the discussion of § 8.4, that

$$p(z) = c(z-\alpha)^{q-1}(z-\beta)^{q-1},$$

where $c \in \mathbf{C}$, and α, β are the fixed points of A_g. Thus if we define

$$S = S_0 \cup \{B_g(a_1)\},$$

and require that F vanish at all the points of S, we see that $p(a_1)=0$, and since $\alpha \neq a_1 \neq \beta$, we conclude that $p=0$.

THEOREM. *For a finitely generated quasi-Fuchsian group* Γ *of type* $(g, 0)$, $g \geq 2$, *we can always choose a set* $S \subset \Lambda$ *consisting of* $(2q-1)(2g-1)+1$ *points so that the restriction map from* Λ *to* S *on* $\mathscr{F}_{1-q}(\Omega, \Gamma)$ *is an isomorphism. In particular,*

$$\Theta_q \colon \mathscr{R}_q(S) \to \mathbf{A}_q(\Omega, \Gamma)$$

is surjective with one dimensional kernel. The set S *can be determined from the canonical generators of* Γ *and finitely many choices.*

Proof. For $q \leq 2g-1$, in particular $q=2$, the theorem has been established above. Induction on q allows us to construct the set S_0 for $q+1$ from the one for q as in § 7.5.

8.7. We would, of course, like to specify a set S, as above, with precisely $(2q-1)(2g-1)$ points. We need the following

LEMMA. *Let* A *be a loxodromic transformation with fixed points* α_1, α_2. *Let* B *be a transformation with fixed point* β_1. *Assume that* α_1 *and* α_2 *are distinct from* β_1 *and the other fixed point of* B *(if any). Let*

$$V_1 = \{p \in \Pi_{2q-2};\ p(\alpha_1) = 0 = p(\alpha_2), \quad \text{and if } q>2,\ p(A^k\beta_1) = 0 \quad \text{for } k=0, \ldots, 2q-5\},$$

$$V_2 = \{p \in \Pi_{2q-2};\ p(\beta_1) = 0\}.$$

Consider the linear operator

$$L \colon V_1 \times V_2 \to \Pi_{2q-2}$$

defined by

$$L(p_1, p_2) = p_1 \cdot (B-1) + p_2 \cdot (1-A). \tag{8.7.1}$$

Then either (a) L *is an isomorphism, or* (b) L *has a one-dimensional kernel. Case* (a) *always holds for* $q=2$, *and is equivalent to the condition that* $L(p, 0)$ *have a non-zero projection onto the kernel of* $(1-A)$, *where* p *is a non-zero element in* V_1.

Proof. Note that $\dim(V_1 \times V_2) = \dim \Pi_{2q-2} = 2q-1$. Further, L is injective on $\{0\} \times V_2$ since the kernel of $(1-A)$ is spanned by the function $p_{q-1}(z) = (z-\alpha_1)^{q-1}(z-\alpha_2)^{q-1}$ that does not vanish at β_1. It follows that $L(\{0\} \times V_2)$ is spanned by the functions $p_j(z)=(z-\alpha_1)^j(z-\alpha_2)^{2q-2-j}$, $j=0,\dots,2q-2$, $j\neq q-1$. It remains to verify that for $q=2$, L is an isomorphism. In this case, we normalize so that $\alpha_1=0$, $\alpha_2=\infty$, $\beta_1=1$. Thus the space V_1 consists of multiples of $p(z)=z$. Write $B(z)=(az+b)(cz+d)^{-1}$, $ad-bc=1$. We compute

$$(Bz)B'(z)^{-1}-z=acz^2+(ad+bc-1)z+bd.$$

The kernel of $(1-A)$ is spanned by p, and hence L fails to be an isomorphism only if $ad+bc=1$. This implies that $bc=0$, and hence B fixes either 0 or ∞. This contradiction finishes the proof of the lemma.

8.8. As a consequence of the last lemma, we can strengthen Theorem 8.6. Select as before $2q-1$ distinct points $a_1,\dots,a_{2q-1}$ so that

$$S_0=\{A_j(a_k),B_j(a_k);\, j=0,\dots,g-1,\ k=1,\dots,2q-1\}$$

consists of $(2q-1)(2g-2)$ points. Let α_1,α_2 be the fixed points of A_g and let β_1 be a fixed point of B_g. Let

$$S=S_0\cup\{\alpha_1,\alpha_2,A_g^k\beta_1;\ k=0,\dots,2q-4\}.$$

Then S consists of $(2q-1)(2g-1)$ points, and if F is an Eichler integral vanishing on S, then $\chi=\mathrm{pd}\,F$ satisfies

$$\chi(A_j)=0=\chi(B_j),\quad j=1,\dots,g-1,$$

$$\chi(A_g)(a_k)=0,\quad k=1,2,$$

$$\chi(A_g)(A_g^k\beta_1)=0,\quad k=0,\dots,2q-5,\text{ if } q>2,$$

$$\chi(B_g)(\beta_1)=0.$$

The defining relation for Γ shows that

$$\chi(C_1\circ\dots\circ C_{g-1})=0.$$

Hence

$$0=\chi(C_g)=\chi(A_g)\cdot(B_g-1)\cdot(A_g^{-1}\circ B_g^{-1})+\chi(B_g)\cdot(A_g^{-1}-1)\cdot B_g^{-1}. \tag{8.8.1}$$

Letting $p_1=\chi(A_g)$ and $p_2=\chi(B_g)$, we see that the map that restricts functions in $\mathscr{F}_{1-q}(\Omega,\Gamma)$ to S has a zero or one-dimensional kernel, and that $\mathscr{F}_{-1}(\Omega,\Gamma)|S$ is isomorphic to $\mathscr{F}_{-1}(\Omega,\Gamma)$.

Example. We consider $g=2=q$. The 9 points in S are:

$$a_1, a_2, \text{ the fixed points of } A_1,$$

$$a_3 \text{ a fixed point of } B_1,$$

$$b_1 = A_1(a_3), \quad b_2 = B_1(a_1), \quad b_3 = B_1(a_2),$$

$$b_4, b_5, \text{ the fixed points of } A_2,$$

$$b_6, \text{ a fixed point of } B_2.$$

It follows that the 6 functions defined on S by

$$F_j(b_k) = \delta_{jk}, \quad 1 \leqslant j,\ k \leqslant 6,$$

$$F_j(a_k) = 0, \quad j = 1, \ldots, 6,\ k = 1, 2, 3,$$

extend to a basis for $\mathscr{F}_{-1}(\Omega, \Gamma)$. The corresponding periods $\chi_1, \ldots, \chi_6$ can now be computed, and from these, the values of the functions $F_1, \ldots, F_6$ at an arbitrary fixed point. For example:

$$\chi_1(A_1)(a_1) = 0 = \chi_1(A_1)(a_2), \quad \chi_1(A_1)(a_3) = A_1'(a_3)^{-1},$$

$$\chi_1(B_1) = 0,$$

$$\chi_1(A_2)(b_4) = 0 = \chi_1(A_2)(b_5),$$

$$\chi_1(B_2)(b_6) = 0.$$

The known values of $\chi_1(A_1)$ and $\chi_1(B_1)$ allow us to compute $\chi_1(C_1)$ and hence $\chi_1(C_2) = -\chi_1(C_1) \cdot C_2$. Finally, we compute $p = \chi_1(C_2) \cdot (B_2 \circ A_2)$, and by formulae (8.8.1) and (8.7.1) conclude that

$$(\chi_1(A_2), \chi_1(B_2)) = L^{-1}p.$$

Similarly (the other extreme case):

$$\chi_6(A_1) = 0 = \chi_6(B_1),$$

$$\chi_6(A_2)(b_4) = 0 = \chi_6(A_2)(b_5),$$

$$\chi_6(B_2)(b_6) = B_2'(b_6)^{-1} - 1 = \alpha \neq 0.$$

As above, we see that $\chi_6(C_2) = 0$; we conclude that $p_1 = \chi_6(A_2)$ and $p_2 = \chi_6(B_2)$ satisfy

$$p_1(b_4) = 0 = p_1(b_5), \tag{8.8.2}$$

$$p_2(b_6) = \alpha, \tag{8.8.3}$$

and

$$p_1 \cdot (B_2 - 1) + p_2 \cdot (1 - A_2) = 0. \tag{8.8.4}$$

We know by Lemma 8.7 that

$$p_1 \cdot (B_2 - 1) + (p_2 - \alpha) \cdot (1 - A_2) = -\alpha \cdot (1 - A_2)$$

has a unique solution that satisfies (8.8.2) and $(p_2 - \alpha)(b_6) = 0$. Hence (p_1, p_2) solves (8.8.4) and satisfies both (8.8.2) and (8.8.3).

8.9. If F is an Eichler integral defined at a fixed point $b \in \Omega$ of an elliptic element γ of order ν, then $F(z_0)$ can be recovered from the period χ of F by formula (4.1.4) as long as b is a q-uniqueness point or equivalently $1-q \not\equiv 0 \pmod{\nu}$; in particular, whenever $q=2$.

To show that every finitely generated quasi-Fuchsian group of the first kind is 2-stratifiable, we consider various cases (compare with § 5 of [16]).

Case I. Γ has signature $(0,3;\ \nu_1, \nu_2, \nu_3)$. As remarked earlier, there is nothing to prove here since $\mathrm{A}_2(\Omega, \Gamma) = \{0\}$.

Case II. Γ has signature $(0, n; \nu_1, \dots, \nu_n)$, $n > 3$.

Let a_j be a fixed point of E_j for $j = 1, \dots, n-1$. For $j = 1, \dots, n-2$, let b_j be the other fixed point of E_j if E_j is elliptic; while $b_j = E_j(a_1)$ if $j \neq 1$ and E_j is parabolic and $b_1 = E_1(a_2)$ if E_1 is parabolic. The set

$$S = \{a_1, \dots, a_{n-1}, b_1, \dots, b_{n-2}\}$$

consists of $2n-3$ distinct points. We consider $\mathscr{F}_{-1}(\Omega, \Gamma)$ to be normalized at a_1, a_2, a_3. Let $F \in \mathscr{F}_{-1}(\Omega, \Gamma)$ and assume that F vanishes on S. Let

$$p_j = (\mathrm{pd}\, F)(E_j) = F \cdot E_j - F, \quad j = 1, \dots, n.$$

Then for $j = 1, \dots, n-2$, p_j vanishes at the two elliptic fixed points of E_j if E_j is elliptic. Such a p_j must be zero by § 8.4. For $j = 1, \dots, n-2$, and E_j parabolic, p_j vanishes at the fixed point of E_j to order two (by the parabolicity of $\mathrm{pd}\, F$ and by (5.2.3)) and at one other point (a_1 if $j > 1$, and a_2 if $j = 1$). Again p_j must be identically zero. Finally p_{n-1} vanishes at a_{n-1} (to order two if E_{n-1} is parabolic). The cocycle condition reads

$$p_{n-1}\cdot E_n = -p_n. \tag{8.9.1}$$

We consider cases. If both E_{n-1} and E_n are parabolic, then p_{n-1} vanishes also at the fixed point of E_n by (8.9.1). Hence $p_{n-1}=0=p_n$. If E_{n-1} is elliptic and E_n is parabolic, then we normalize E_{n-1} to have fixed points at $0,\infty$ (with $a_{n-1}=\infty$) and E_n to have fixed point at 1. It follows that

$$p_{n-1}(z) = a, \quad p_n(z) = b(z-1)+c(z-1)^2,$$

and hence that

$$-aE_n'(z)^{-1} = b(z-1)+c(z-1)^2.$$

Setting $z=1$, we see that $a=0$. We conclude that $b=0=c$, also.

Case III. Γ has signature $(g,0)$, $g\geqslant 2$. This case was treated in § 8.8.

Case IV. Γ has signature $(g,n;\nu_1,\dots,\nu_n)$, $n>0$ and $g\geqslant 1$. Let a_1,a_2 be the fixed points of A_g, and let a_3 be a fixed point of B_g. Let x_k,y_k be the fixed points of E_k if E_k is elliptic, and let x_k be the fixed point of E_k and $y_k=E_k(a_1)$ if E_k is parabolic, $k=1,\dots,n$. Consider

$$S = \{a_j; j=1,2,3\} \cup \{A_k(a_j), B_k(a_j); j=1,2,3,\ k=1,\dots,g-1 \text{ (provided } g>1)\}$$
$$\cup\ \{x_k, y_k;\ k=1,\dots,n\}.$$

The set S consists of $6g-3+2n=\dim \mathrm{A}_2(\Omega,\Gamma)+3$ points. We claim that the restriction of functions in $\mathscr{F}_{-1}(\Omega,\Gamma)$ to S is an isomorphism. The argument combines Case II and Case III ideas, and may be left to the reader.

We have obtained the following

THEOREM (Kra–Maskit [16]). *Every finitely generated quasi-Fuchsian group of the first kind is* 2-*stratifiable.*

Remark. In [16], the stronger result, that every geometrically finite function group is stratifiable, was established. We have reproven a weaker result for quasi-Fuchsian groups, not only because the algorithm (to obtain a 2-stratification) is simpler, but also because in proving the above theorem, we have established an algorithm for constructing the values of Eichler integrals that form a basis for $\mathscr{F}_{-1}(\Omega,\Gamma)$ at fixed points of elements of Γ, that is easier to use than the general method of § 8.2. The methods of [16] could be simplified to yield 2-stratifications and hence algorithms for the vanishing problem for $q=2$, for all geometrically finite function groups.

8.10. We consider next torsion free quasi-Fuchsian groups Γ of type $(0,n)$ with $n>2$. A q-stratification for Γ must have $(n-1)(2q-1)-n$ points. For $q=2$, we define S to consist of:

$$a_1, a_2 \text{ the fixed points of } E_1^{-1}\circ E_2,$$

$$b_1 = E_1(a_1) = E_2(a_1), \quad c_1 = E_1(a_2) = E_2(a_2),$$
$$b_3 = E_3(a_1), \quad c_3 = E_3(a_2),$$
$$\vdots$$
$$b_{n-2} = E_{n-2}(a_1), \quad c_{n-2} = E_{n-2}(a_2),$$
$$b_{n-1} = E_{n-1}(a_1).$$

In the above formulation we have assumed that $n\geq 4$. If $n=3$, b_1 is the last term to be defined. We claim that for all $q\geq 2$, we can construct a set S consisting of loxodromic fixed points with the following properties:

(a) $|S|=(n-1)(2q-1)-n$,

(b) there exist points $a_1,\dots,a_{2q-2}\in S$ so that $S\supset\{E_j(a_k);\ j=1,\dots,n-1,$ $k=1,\dots,2q-2,\ (j,k)\neq(n-1,2q-2)\}$.

For $q=2$, we have constructed such a set S_2, above. Assume S_q exists for some $q\geq 2$. The set $E_1^{-1}(S_q)$ has $(n-1)(2q-1)-n\geq 2q$ points, provided $n>3$ or $q>2$. In this case we select two points a_{2q-1}, a_{2q} in $E_1^{-1}(S_q)\setminus\{a_1,\dots,a_{2q-2}\}$. If $n=3$ and $q=2$, we let $a_3=E_1^{-1}(a_1)$, $a_4=E_1^{-1}(a_2)$. We define

$$S_{q+1} = S_q \cup \{a_{2q-1}, a_{2q}\} \cup \{E_{n-1}(a_{2q-2})\} \cup \{E_j(a_{2q-1}); j=1,\dots,n-1\}$$
$$\cup\{E_j(a_{2q}); j=1,\dots,n-2\}.$$

We note that S_{q+1} contains at most

$$(n-1)(2q-1)-n+2+1+(n-2)+(n-3) = (n-1)(2q+1)-n$$

points. We show next that property (b) implies that S is a q-stratification for Γ; hence $|S_{q+1}|=(n-1)(2q+1)-n$.

Let $F\in\mathscr{F}_{1-q}(\Omega,\Gamma)$ and let

$$p_j = (\operatorname{pd} F)(E_j), \quad j=1,\dots,n.$$

For $j=1,\dots,n-2$, p_j vanishes at a_k, $k=1,\dots,2q-2$, and the fixed point of E_j (recall that *all* the a_k are loxodromic fixed points). Hence $p_j=0$, $j=1,\dots,n-2$. Similarly p_{n-1} vanishes at the fixed point of E_{n-1} and at $a_1,\dots,a_{2q-3}$; while p_n vanishes at the fixed point of E_n. The cocycle condition $(p_{n-1}\cdot E_n+p_n=0)$ shows, as in § 8.9, that $p_{n-1}=0=p_n$.

We have established the

THEOREM. *Let Γ be a quasi-Fuchsian torsion free group of type $(0,n)$, $n\geqslant 3$. Then Γ is q-stratifiable for all $q\geqslant 2$.*

§ 9. Beltrami coefficients supported on the invariant component (proof of Theorem 5)

9.1. Let Γ be a finitely generated Fuchsian group of the first kind operating on the unit disk Δ. To prove Theorem 5, we must characterize $\mathscr{F}_{1-q}(\Delta,\Gamma)$ in $\mathscr{F}_{1-q}(\Omega,\Gamma)$. We assume that $2d=\dim\mathscr{F}_{1-q}(\Omega,\Gamma)>0$, and note that $\dim\mathscr{F}_{1-q}(\Delta,\Gamma)=d$. Let $F\in\mathscr{F}_{1-q}(\Omega,\Gamma)$, then $F\in\mathscr{F}_{1-q}(\Delta,\Gamma)$ if and only if $F|\Delta^*$ is holomorphic, where $\Delta^*=\{z\in\mathbf{C}\cup\{\infty\};\ |z|>1\}$. Here we identify F with its extension to $\mathbf{C}$ as a potential of a generalized canonical Beltrami differential. Consider the Möbius transformation $A(z)=z^{-1}$, and the group $\Gamma^*=A^{-1}\Gamma A$. The group Γ^* is also a finitely generated Fuchsian group of the first kind operating on the unit disk. For $F\in\mathscr{F}_{1-q}(\Omega,\Gamma)$, $F^*=A_q^*F$ is a potential for Γ^*. Furthermore, $F\in\mathscr{F}_{1-q}(\Delta,\Gamma)$ if and only if F^* is holomorphic on Δ. Thus $F\in\mathscr{F}_{1-q}(\Delta,\Gamma)$ if and only if

$$\int_{|z|=1} z^k F^*(z)\,dz = 0, \quad k=0,1,\dots. \tag{9.1.1}$$

(See, for example, [25, p. 361].) Simple change of argument calculations lead to the equivalence of (9.1.1) with

$$\int_0^{2\pi} e^{i(1-k-2q)\theta}F(e^{i\theta})\,d\theta=0, \quad k=0,1,\dots. \tag{9.1.2}$$

It remains to show that for $F\in\mathscr{F}_{1-q}(\Omega,\Gamma)$ condition (9.1.2) need be satisfied only for a finite set of k to guarantee that $F\in\mathscr{F}_{1-q}(\Delta,\Gamma)$. This is a consequence of the finite dimensionality of $\mathscr{F}_{1-q}(\Omega,\Gamma)$. Given $F\in\mathscr{F}_{1-q}(\Omega,\Gamma)\diagdown\mathscr{F}_{1-q}(\Delta,\Gamma)$, there is a $\bar{k}\geqslant 0$ such that the integral $I_k(F)$ in (9.1.2) does not vanish for $k=\bar{k}$.

Thus for $F\in\mathscr{F}_{1-q}(\Omega,\Gamma)\diagdown\mathscr{F}_{1-q}(\Delta,\Gamma)$ we define

$$\mu(F)=\min\{k\in\mathbf{Z};\ k\geqslant 0,\ I_k(F)\neq 0\}.$$

The space $\mathscr{F}_{1-q}(\Omega,\Gamma)$ is a direct sum

$$\mathscr{F}_{1-q}(\Omega,\Gamma)=\mathscr{F}_{1-q}(\Delta,\Gamma)\oplus\mathscr{F}_{1-q}(\Delta^*,\Gamma).$$

Every $F\in\mathscr{F}_{1-q}(\Delta^*,\Gamma)$ is holomorphic on Δ. If we let for such an F,

$$F(z)=\sum_{j=0}^{\infty}a_j z^j$$

be the Taylor series expansion of F at the origin, then for $k=0,1,\ldots,$

$$I_k(F)=2\pi a_{2q+k-1}.$$

We conclude that

$$\mu(F)=\operatorname{ord}_0\varphi,$$

where

$$\varphi=\frac{d^{2q-1}F}{dz^{2q-1}}.$$

The operator d^{2q-1}/dz^{2q-1} maps $\mathscr{F}_{1-q}(\Delta^*,\Gamma)$ isomorphically onto $\mathbf{A}_q(\Delta,\Gamma)$; see [14, pp. 210–219]. Thus the non-negative integers

$$\{\mu(F);\ F\in\mathscr{F}_{1-q}(\Omega,\Gamma)\setminus\mathscr{F}_{1-q}(\Delta,\Gamma)\}$$

are restricted to the finite set of possible orders of zeros at the origin of elements in the vector space $\mathbf{A}_q(\Delta,\Gamma)$.

9.2. We can hence obtain estimates on the number $m(\Gamma,q)$ appearing in Theorem 5.

COROLLARY. *Assume that Γ has signature $(g,n;\nu_1,\ldots,\nu_n)$. Let $\nu=1$ if $n=0$ or $\nu_j=\infty$ for $j=1,\ldots,n$; let ν be the maximum of the finite ν_j otherwise. Then*

$$m(\Gamma,q)\le q\nu\left\{(2q-2)+\sum_{j=1}^{n}(1-1/\nu_j)\right\},$$

and

$$m(\Gamma,q)=(2q-1)(g-1)+\sum_{j=1}^{n}[q-q/\nu_j]-1,$$

if 0 is not a Weierstrass point for $\mathbf{A}_q(\Delta,\Gamma)$.

Proof. The first estimate is the highest possible order of zeros of elements of $\mathbf{A}_q(\Delta, \Gamma)$; see [14, p. 114]. The hypothesis that 0 is not a Weierstrass point for $\mathbf{A}_q(\Delta, \Gamma)$ means precisely that the possible orders of vanishing at the origin for functions in $\mathbf{A}_q(\Delta, \Gamma)$ are

$$0, 1, \ldots, d-1$$

(recall that $\dim \mathbf{A}_q(\Delta, \Gamma) = d$).

Remarks. (1) We can always choose a Möbius transformation A that fixes Δ so that zero is *not* a Weierstrass point for $\mathbf{A}_q(\Delta, A\Gamma A^{-1})$.

(2) In the second of the above formulae for $m(\Gamma, q)$, we have assumed, of course, that $d>0$.

§ 10. Kleinian groups with two components (neither invariant)

10.1. Let G be a non-elementary finitely generated Kleinian group with Ω consisting of two components, neither of them invariant. It was shown in [17], that there is a (unique) finitely generated quasi-Fuchsian group Γ of the first kind so that G is a $\mathbf{Z}_2$-extension of Γ. Further

$$G = \Gamma \cup \Gamma A,$$

where $A \in G \setminus \Gamma$, $A^2 \in \Gamma$, and A is either elliptic of order 2 or loxodromic. It follows that every parabolic element in G is in Γ, and comes from a puncture on $\Omega/G \cong \Delta/\Gamma$, where Δ is one of the components of Ω. Since

$$\mathbf{E}^b_{1-q}(\Omega, G) \subseteq \mathbf{E}^b_{1-q}(\Omega, \Gamma), \tag{10.1.1}$$

it follows that $\mathbf{E}^b_{1-q}(\Omega, G) = \{0\}$, and hence

$$PH^1(G, \Pi_{2q-2}) \cong \mathscr{F}_{1-q}(\Omega, G).$$

It follows also that

$$\dim PH^1(G, \Pi_{2q-2}) = \dim \mathbf{A}_q(\Omega, G) = \dim \mathbf{A}_q(\Delta, \Gamma) = (2q-1) + \sum_{k=1}^{n} [q - q/\nu_k],$$

where $(g, n; \nu_1, \ldots, \nu_n)$ is the signature of Γ.

10.2. To show that the vanishing problem for G has an algebraic solution, we will show how to construct via linear algebra $PZ^1(G, \Pi_{2q-2})$ from $PZ^1(\Gamma, \Pi_{2q-2})$. We saw in § 8.2 that the space $PZ^1(\Gamma, \Pi_{2q-2})$ is determined from a presentation of Γ by algebraic operations. We have an obvious restriction map

$$PZ^1(G, \Pi_{2q-2}) \to PZ^1(\Gamma, \Pi_{2q-2}).$$

We must find necessary and sufficient conditions for a cocycle $\chi \in PZ^1(\Gamma, \Pi_{2q-2})$ to extend to a cocycle for G.

For $\gamma \in \Gamma$, we let $\gamma^A = A \circ \gamma \circ A^{-1}$ and observe that $\gamma^A \in \Gamma$ (since Γ is a normal subgroup of G). Let $\chi \in PZ^1(\Gamma, \Pi_{2q-2})$, and let $p \in \Pi_{2q-2}$. If A is elliptic (of order 2) we assume that $p \in (\Pi_{2q-2})_A$ (recall the definition of this subspace in § 8.2). We would like to set $p = \chi(A)$. A necessary and sufficient condition for the existence of a cocycle $\tilde{\chi} \in PZ^1(G, \Pi_{2q-2})$ with $\tilde{\chi}(A) = p$ and $\tilde{\chi}(\gamma) = \chi(\gamma)$ for all $\gamma \in \Gamma$, is that the equation

$$\chi(\gamma^A) = p \cdot (\gamma \circ A^{-1}) + \chi(\gamma) \cdot A^{-1} - p \cdot A^{-1} \tag{10.2.1}$$

be satisfied for all $\gamma \in \Gamma$. Equation (10.2.1) can be rewritten as

$$\chi(\gamma^A) \cdot A - \chi(\gamma) - p \cdot (\gamma - 1) = 0, \quad \gamma \in \Gamma. \tag{10.2.2}$$

Routine calculations show that equation (10.2.2) is satisfied for $\gamma_1 \circ \gamma_2$ and γ_1^{-1} whenever it is satisfied by γ_1 and $\gamma_2 \in \Gamma$. Hence in order for (10.2.1) to hold for all $\gamma \in \Gamma$, it is necessary and sufficient that (10.2.2) be satisfied for generators of Γ. Let $V = \Pi_{2q-2}$ if A is loxodromic, and let $V = (\Pi_{2q-2})_A$ if A is elliptic. Let $\gamma_1, \ldots, \gamma_N$ be any set of generators for Γ. Define

$$h: PZ^1(\Gamma, \Pi_{2q-2}) \times V \to \Pi_{2q-2}^N$$

by

$$h(\chi, p) = (p_1, \ldots, p_N),$$

where

$$p_j = \chi(\gamma_j^A) \cdot A - \chi(\gamma_j) - p \cdot (\gamma_j - 1), \quad j = 1, \ldots, N.$$

Then h is a **C**-linear map, and

$$(\chi, p) \in \operatorname{Ker} h$$

if and only if there exists a $\tilde{\chi} \in PZ^1(G, \Pi_{2q-2})$ with

$$\tilde{\chi} | \Gamma = \chi \quad \text{and} \quad \tilde{\chi}(A) = p.$$

We have obtained the following

THEOREM. *Let G be a non-elementary finitely generated Kleinian group with two components, neither invariant. Let $f \in \mathscr{R}_q(\Lambda_q)$ have poles only at q-uniqueness points and parabolic fixed points. There exists an algebraic algorithm for determining whether or not $\Theta_q f=0$.*

Remarks. (1) If A is elliptic, then the fixed points of A may or may not be q-uniqueness points.

(2) The results of [17] imply the existence of 2-stratifications for G.

§ 11. Vector bundles over deformation spaces of Kleinian groups

11.1. Let Γ be a finitely generated non-elementary Kleinian group. We normalize Γ so that $0, 1, \infty \in \Lambda$. Let $M(\Gamma)$ be the space of *Beltrami coefficients* for Γ; that is, the space of generalized Beltrami differentials for $q=2$ of L^∞-norm less than one. For $\mu \in M(\Gamma)$, let w^μ be the unique normalized (fixing $0, 1, \infty$) μ-conformal automorphism of $\mathbf{C} \cup \{\infty\}$. The *deformation space* of Γ, $T(\Gamma)$, may be defined (see, for example, [16]) as the set of restrictions to Λ of $\{w^\mu; \mu \in M(\Gamma)\}$. It is known to be a domain of holomorphy in $\mathbf{C}^d$, $d=\dim \mathbf{A}_2(\Omega, \Gamma)$, [16].

LEMMA. *For $z \in \Lambda_q$,*

$$\mu \mapsto w^\mu(z) \tag{11.1.1}$$

is a holomorphic mapping from $T(\Gamma)$ into $\mathbf{C}$, except possibly if $z \in \Omega$ is stabilized by a group of order 2 with the second fixed point in the same component as z.

Proof. See [16] or [17]. In [17], a point z for which (11.1.1) is a well defined function on $T(\Gamma)$ was called *sturdy*.

In addition to the trivial fiber space $T(\Gamma)\times\mathbf{C}$ over $T(\Gamma)$, we will need the *Bers fiber space*

$$F(\Gamma) = \{([\mu], z);\ [\mu] \in T(\Gamma),\ z \in w^\mu(\Omega)\},$$

where $[\mu]$ is the equivalence class in $T(\Gamma)$ of $\mu \in M(\Gamma)$. The group Γ operates on $F(\Gamma)$ in a fiber preserving way by

$$\gamma([\mu], z) = ([\mu], \gamma^\mu z), \quad z \in w^\mu(\Omega),\ \gamma \in \Gamma,\ \mu \in M(\Gamma),$$

where

$$\gamma^\mu = w^\mu \circ \gamma \circ (w^\mu)^{-1},$$

as a group of complex analytic homeomorphisms.

The natural projection of $F(\Gamma)$ onto $T(\Gamma)$ will be denoted by π; that is,

$$\pi([\mu], z) = [\mu], \quad \mu \in M(\Gamma),\ z \in w^\mu(\Omega).$$

11.2. Let f be a meromorphic function on $F(\Gamma)$ with the property that for each $\mu \in M(\Gamma)$, $f|\pi^{-1}([\mu]) \in \mathcal{R}_q(w^\mu(\Lambda_q))$. It follows that the Poincaré series

$$(\Theta_q f)([\mu], z) = \sum_{\gamma \in \Gamma} f([\mu], \gamma^\mu z)(\gamma^\mu)'(z)^q$$

converges uniformly and absolutely on compact subsets of $F(\Gamma)$ and defines a holomorphic function—an automorphic form of weight $(-2q)$ for the action of Γ on $F(\Gamma)$; that is, $\Theta_q f$ satisfies

$$(\Theta_q f)([\mu], \gamma^\mu z)(\gamma^\mu)'(z)^q = (\Theta_q f)([\mu], z), \quad \text{all } \mu \in M(\Gamma), \text{ all } z \in w^\mu(\Omega), \text{ all } \gamma \in \Gamma.$$

Let us choose $a_1, \dots, a_{2q-1}$ to be distinct sturdy points for Γ (located in Λ_q), and assume that $a_1=0$, $a_2=1$, $a_3=\infty$ (by replacing Γ by a conjugate group, if necessary). Let $\gamma_0=I$, $\gamma_1, \dots, \gamma_N$ be generators for Γ, and consider

$$\mathcal{R}_q(S, T(\Gamma)) = \{f \text{ meromorphic on } F(\Gamma);\ f|\pi^{-1}([\mu]) \in \mathcal{R}_q(w^\mu(S)), \text{ all } \mu \in M(\Gamma)\},$$

where S is defined by (0.1.8). We note that if $b \in S \setminus \{a_1, \dots, a_{2q-1}\}$, then

$$g([\mu], z)$$
$$= \left(\frac{-1}{2\pi}\right)\left(\frac{1}{z-w^\mu(b)}\right)\left(\frac{w^\mu(b)}{z}\right)\left(\frac{w^\mu(b)-1}{z-1}\right)\prod_{j=4}^{2q-1}\frac{w^\mu(b)-w^\mu(a_j)}{z-w^\mu(a_j)}, \quad \mu \in M(\Gamma),\ z \in w^\mu(\Omega),$$

defines a function in $\mathcal{R}_q(S, T(\Gamma))$. Note that the product in the above formula is to be ignored if $q=2$. As b varies in $S \setminus \{a_1, \dots, a_{2q-1}\}$, we obtain functions which on each fiber form a basis for the restrictions of $\mathcal{R}_q(S, T(\Gamma))$ to that fiber. Hence $\mathcal{R}_q(S, T(\Gamma))$ is a trivial rank s fiber bundle over $T(\Gamma)$, where

$$s = |S| - (2q-1). \tag{11.2.1}$$

Let $\mathbf{A}_q(T(\Gamma))$ be the rank d vector bundle of cusp forms of weight $(-2q)$ over $T(\Gamma)$;

that is, the fiber over $[\mu]$, $\mu \in M(\Gamma)$, consists of $\mathbf{A}_q(w^\mu(\Omega), w^\mu \Gamma (w^\mu)^{-1})$, and $d = \dim \mathbf{A}_q(\Omega, \Gamma)$.

THEOREM. *The Poincaré series map*

$$\Theta_q : \mathcal{R}_q(S, T(\Gamma)) \to \mathbf{A}_q(T(\Gamma)) \tag{11.2.2}$$

is a surjective bundle map. The kernel of Θ_q has constant dimension $s-d$ (over all the fibers). The bundle map Θ_q splits. Hence there is a rank of d subbundle of $\mathcal{R}_q(S, T(\Gamma))$ with the property that Θ_q is an isomorphism between this bundle and $\mathbf{A}_q(T(\Gamma))$.

Proof. The first part of the theorem is a direct consequence of Theorem 1 and the remarks in this paragraph. The existence of a splitting map follows from the general theory of vector bundles (or equivalently, locally free sheaves) over Stein manifolds, see, for example [9, p. 256].

Remarks. (1) It would be interesting to exhibit directly the splitting map—without relying on deep results from several complex variable theory.

(2) Many times, we can choose S so that (11.2.2) is an isomorphism.

(3) It is not known whether $\mathbf{A}_q(T(\Gamma))$ is always a trivial bundle over $T(\Gamma)$. If $T(\Gamma)$ is contractable, then every vector bundle over $T(\Gamma)$ is topologically trivial, and because $T(\Gamma)$ is Stein, analytically trivial by a general theorem of Grauert [8] (see also the exposition in [6]).

11.3.

THEOREM. *If S is a q-stratification for Γ, then Θ_q of* (11.2.2) *is a bundle isomorphism. Such q-stratifications exist whenever:*

(a) Γ *is a geometrically finite non-elementary function group and $q=2$,*

(b) Γ *is a finitely generated Kleinian group with two components, neither invariant, and $q=2$,*

(c) Γ *is a Schottky group (and q is arbitrary), or*

(d) Γ *is a torsion free quasi-Fuchsian group of type* $(0, n)$ $n \geq 3$ *(and q is arbitrary).*

Proof. Parts (a) and (b) follow from the results of [17]. (In § 8 we have also constructed 2-stratifications for finitely generated quasi-Fuchsian groups of the first kind.) Part (c) follows from § 7.5 and part (d) from § 8.10.

COROLLARY. *The vector bundle $\mathbf{A}_q(T(\Gamma))$ over $T(\Gamma)$ is analytically trivial whenever*

(a) *each component of* Ω *is simply connected, or*

(b) $q=2$, *or*

(c) Γ *is a Schottky group.*

Proof. If each component of Ω is simply connected, then $T(\Gamma)$ is contractible (see, for example, [16]) and the result follows by Grauert's theorem [8], as remarked above. Recall [16] that $T(\Gamma)$ is a domain in number space, and $A_2(T(\Gamma))$ is isomorphic to the cotangent bundle of $T(\Gamma)$. The cotangent bundle of a domain in $\mathbf{C}^d$ is always analytically trivial. The result for Schottky groups follows from the theorem.

§ 12. Appendix: Stabilizers of limit points

We have used several times the following well known result about the limit set Λ of a Kleinian group Γ. A proof is supplied for the convenience of the reader.

LEMMA. *Let* $x\in\Lambda$. *Then the stabilizer* Γ_0 *of* x *in* Γ *is always an elementary group. There are three possibilities:*

(i) Γ_0 *is a finite cyclic group of order* n, $1\leq n<\infty$,

(ii) Γ_0 *is a finite cyclic extension of a rank* 1 *or rank* 2 *parabolic group, or*

(iii) Γ_0 *is an abelian finite cyclic extension of a purely loxodromic cyclic group.*

Proof. Assume first that every element of Γ_0 is elliptic and fixes $x=\infty$. Thus the elements of Γ_0 are of the form

$$z\mapsto az+b.$$

If A and $B\in\Gamma_0$ have only ∞ as a common fixed point, then $A^{-1}\circ B\circ A\circ B^{-1}$ is parabolic and fixes ∞. Thus every element of Γ_0 also fixes another point which may be assumed to be $z=0$. Hence Γ_0 is a discrete subgroup of the group of rotations. We must be in case (i).

If Γ_0 contains a parabolic element A, then Γ_0 cannot contain a loxodromic element B. For if $A(z)=z+1$ and $B(z)=\lambda z$, $|\lambda|\neq 1$ (without loss of generality), then Γ_0 contains the element

$$B^{-n}\circ A\circ B^{n}: z\mapsto z+\lambda^{-n}, \quad n=1,2,\ldots,$$

and Γ_0 cannot be discrete.

Assume hence that Γ_0 contains a parabolic element that fixes $x=\infty$. Then Γ_0 contains only parabolic and elliptic elements and Γ_0 acts discontinuously on $\mathbf{C}$. Such a group gives us case (ii). Similarly, if Γ_0 contains a loxodromic element, then we have case (iii).

References

[1] AHLFORS, L. V., Finitely generated Kleinian groups. *Amer. J. Math.*, 86 (1964), 413–429 and 87 (1965), 759.
[2] — The structure of a finitely generated Kleinian group. *Acta Math.*, 122 (1969), 1–17.
[3] BERS, L., Inequalities for finitely generated Kleinian groups. *J. Analyse Math.*, 18 (1967), 23–41.
[4] — Poincaré series for Kleinian groups. *Comm. Pure Appl. Math.*, 26 (1973), 667–672 and 27 (1974), 583.
[5] — Automorphic forms for Schottky groups. *Adv. in Math.*, 16 (1975), 332–361.
[6] CARTAN, H., Espaces fibrés analytiques, in *Symposium Internacional de Topologia Algebraica*. Univ. Nacional de México, 1958, 97–121.
[7] EICHLER, M., Eine Verallgemeinerung der Abelschen Integrale. *Math. Z.*, 67 (1957), 267–278.
[8] GRAUERT, H., Analytische Faserungen über holomorphvollständigen Räumen. *Math. Ann.*, 135 (1958), 266–273.
[9] GUNNING, R. C. & ROSSI, H., *Analytic functions of several complex variables*. Prentice-Hall, Englewood Cliffs, New Jersey, 1965.
[10] HEJHAL, D. A., Quelques remarques à propos des séries de Poincaré sur les groupes de Schottky. *C.R. Acad. Sci. Paris Sér. I Math.*, 280 (1975), 341–344.
[11] — Monodromy groups and Poincaré series. *Bull. Amer. Math. Soc.*, 84 (1978), 339–376.
[12] KRA, I., On cohomology of Kleinian groups. *Ann. of Math.*, 89 (1969), 533–556.
[13] — On cohomology of Kleinian groups: II. *Ann. of Math.*, 90 (1969), 575–589.
[14] — *Automorphic forms and Kleinian groups*. Benjamin, Reading, Massachusetts, 1972.
[15] — On the vanishing of Poincaré series of rational functions. *Bull. Amer. Math. Soc.*, 8 (1983), 63–66.
[16] KRA, I. & MASKIT, B., The deformation space of a Kleinian group. *Amer. J. Math.*, 103 (1981), 1065–1102.
[17] — Bases for quadratic differentials. *Comment. Math. Helv.*, 57 (1982), 603–626.
[18] LJAN, G. M., The kernel of the Poincaré θ-operator. *Dokl. Akad. Nauk SSSR*, 230 (1976), 269–270. English translation: *Soviet Math. Dokl.*, 17 (1976), 1283–1285.
[19] MASKIT, B., Decomposition of certain Kleinian groups. *Acta Math.*, 130 (1973), 243–263.
[20] METZGER, T. A., The kernel of the Poincaré series operator. *Proc. Amer. Math. Soc.*, 76 (1979), 289–292.
[21] NAKADA, M., Cohomology of finitely generated Kleinian groups with an invariant component. *J. Math. Soc. Japan*, 28 (1976), 699–711.
[22] — Quasi-conformal stability of finitely generated function groups. *Tôhoku Math. J.*, 30 (1978), 45–58.
[23] PETERSSON, H., Die linearen Relationen zwischen den ganzen Poincaréschen Reihen von reeller Dimension zur Modulgruppe. *Abh. Math. Sem. Univ. Hamburg*, 12 (1938), 415–472.
[24] POINCARÉ, H., Memoire sur les fonctions fuchsiennes. *Acta Math.*, 1 (1882), 193–294.
[25] RUDIN, W., *Real and complex analysis*. McGraw-Hill, New York and Sidney, 1966.
[26] WEIL, A., Remarks on cohomology of groups. *Ann. of Math.*, 80 (1964), 149–157.
[27] WOLPERT, S., The Fenchel-Nielsen deformation. *Ann. of Math.*, 115 (1982), 501–528.

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