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A GENERAL LOCAL ERGODIC THEOREM IN L_1

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Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and let L_1 denote the usual Banach space of equivalence classes of real valued integrable functions on X. We shall not distinguish between the equivalence classes and the functions themselves. Relations between functions are assumed to hold in an a.e. sense.

Throughout this paper $\{T_t\}_{t>0}$ will denote a strongly continuous semigroup of linear contractions on L_1. That is:

(i) each T_t is a linear operator on L_1, with norm not more than 1,

(ii) $T_{s+t} = T_t T_s$ for all $t, s > 0$,

(iii) for all $f \in L_1$ and $t > 0$, $\lim_{s\to t, s>0}; \|T_s f - T_t f\| = 0$.

We will prove the pointwise local ergodic theorem for such a semi group.

THEOREM 1.1. *If $f \in L_1$, then $\lim_{t\to 0^+}(1/t)\int_0^t T_s f\, ds$ exists a.e. on X.*

Here $\int_0^t T_s f\, ds$ is defined as the strong limit of the usual Riemann sums. To give a meaning to the a.e. limit one either has to use the usual conventions in ergodic theory (p. 686 in **[4]**), or, equivalently, to avoid these conventions, has to restrict the range of t in $\lim_{t\to 0^+}$ to a countable dense subset of $(0, \infty)$, for example to the set of positive rational numbers (p. 200 in **[3]**). The same remarks also apply to Theorem 1.2 below.

Various special cases of this theorem have already been proved, going back to Wiener's local ergodic theorem **[11]**, in which $\{T_t\}$ is induced by a measure preserving flow of X. The modern form of the theory started with the results of Krengel **[6]** and Ornstein **[8]**, where the local ergodic theorem is proved under the following two additional assumptions on $\{T_t\}$:

(iv) Positivity: $T_t L_1^+ \subset L_1^+$ for all $t > 0$, where L_1^+ is the positive cone of L_1,

(v) Continuity at the origin: There is an operator T_0 on L_1 such that $\lim_{t\to 0^+}\|T_t f - T_0 f\| = 0$ for all $f \in L_1$.

Later the theorem has been proved assuming (iv) only **[1]**, or assuming (v) only **[5]**, **[7]**, **[10]**, in addition to (i), (ii) and (iii). Here we will prove the local ergodic theorem without any additional assumptions.

We will, in fact, prove Theorem 1.2 below, which generalizes both Theorem 1.1 and a weaker form of a differentiation theorem of Akcoglu-Krengel [3]. We define, as in [3], a T_t-additive process as a family $\{F_t\}_{t>0}$ of L_1 functions such that $F_t + T_t F_s = F_{t+s}$ for all $t, s > 0$. If

$$\sup_{t>0}(1/t)\|F_t\| = K < \infty$$

then $\{F_t\}$ is called a bounded additive process, and K is called the bound of the process. Note that $F_t = \int_0^t T_s f\,ds$ defines a bounded additive process for any $f \in L_1$. Another example of an additive process is $F_t = (1 - T_t) f, f \in L_1$, which may or may not be bounded.

Theorem 1.2. *If $\{F_t\}_{t>0}$ is a bounded additive process with respect to $\{T_t\}_{t>0}$ then there is a function $f \in L_1$ such that $\lim_{t\to 0^+} 1/tF_t = f$ a.e. on X. Furthermore, $\lim_{t\to 0^+} 1/t \int_0^t T_s f\,ds = f$ a.e.*

The advantage of considering additive processes is that we can then assume the continuity of $\{T_t\}$ at the origin, without any loss of generality. To see this we first collect a few results which will also be used later in the proof Theorem 1.2.

Theorem 1.3 ([10], [5]). *Given a strongly continuous semi group $\{T_t\}_{t>0}$ of L_1-contractions, there exists a strongly continuous semi-group $\{\tau_t\}_{t>0}$ of positive L_1-contractions such that $|T_t f| \le \tau_t |f|$ for any $t > 0$ and $f \in L_1$.*

Such a semi group $\{\tau_t\}_{t>0}$ will be called a linear modulus of $\{T_t\}_{t>0}$. Furthermore, if a linear modulus for $\{T_t\}$ is continuous at the origin then $\{T_t\}$ is also continuous at the origin (Lemma 1 in [9]).

Theorem 1.4 ([1]). *Given a strongly continuous semi group $\{\tau_t\}_{t>0}$ of positive L_1-contractions, there exists a unique partition $\{C, D\}$ of X into two sets such that*

(i) $\chi_D \tau_t f = 0$ *for all* $t > 0$ *and* $f \in L_1$,

(ii) *the restriction of $\{\tau_d t\}_{t>0}$ to $L_1(C)$ is a strongly continuous semi group of $L_1(C)$-contractions which is also continuous at the origin.*

Here χ denotes the characteristic function of its subscript and $L_1(C) = \{f \mid f \in L_1, \chi_D f = 0\}$.

Lemma 1.1. *If $\{F_t\}$ is a bounded T_t-additive process and if $\{C, D\}$ is the partition of X given in Theorem 1.4 with respect to a linear modulus $\{\tau_t\}$ of $\{T_t\}$ then $\chi_D F_t = 0$ a.e. for all $t > 0$.*

Proof. Let $0 < \varepsilon < t$. Then

$$|F_t| \le |F_\varepsilon| + |T_\varepsilon F_{t-\varepsilon}| \le |F_\varepsilon| + \tau_\varepsilon |F_{t-\varepsilon}|$$

shows that $\chi_D| F_t| \le \chi_D| F_\varepsilon|$, since $\chi_D \tau_\varepsilon| F_{t-\varepsilon}| = 0$. Hence $\|\chi_D| F_t| \| \le \| F_\varepsilon\| \le K\varepsilon$, where K is the bound of $\{F_t\}$.

This lemma shows that $F_t \in L_1(C)$. If $\tilde{T}_t$ is the restriction of T_t to $L_1(C)$, then F_t is also a bounded $\tilde{T}_t$-additive process. But now $\{\tilde{T}_t\}$ is continuous at the origin, since $\{\tau_t\}$ restricted to $L_1(C)$ is a linear modulus for $\{\tilde{T}_t\}$ and is continuous at the origin. Therefore we may and do assume, in the proof of Theorem 1.2, that $\{T_t\}$ is continuous at the origin. (Note that this assumption can not be made in Theorem 1.1, because f may not be in L_1C.)

THEOREM 1.5 ([3]). *Let $\{H_t\}$ be a bounded additive process with respect to a strongly continuous semigroup $\{\tau_t\}$ of positive L_1 contractions. Then there is an L_1 function h such that $\lim_{t\to 0^+}(1/t)H_t = h$ a.e. and such that $\lim_{t\to 0^+}(1/t)\int_0^t \tau_s h\, ds = h$ a.e.*

Although the final conclusion of this theorem is not explicitly stated in [3], it follows easily from (3.8) of that paper.

2. Proof of the main result. Given a bounded additive process one can construct a dominating positive additive process with respect to the linear modulus. For this the continuity at the origin is not needed.

THEOREM 2.1. *Let $\{F_t\}_{t>0}$ be a bounded T_t-additive process and let $\{\tau_t\}$ be a linear modulus of $\{T_t\}$. Then there is a τ_t-additive process $\{H_t\}$, such that* (i) $|F_t| \le H_t$ *a.e. for each* $t > 0$, (ii) *$\{H_t\}$ has the same bound as $\{F_t\}$.*

Proof. Let $K = \sup_{t>0}(1/t)\|F_t\|$ be the bound of $\{F_t\}$. To construct H_t for a certain fixed t, we consider the family $\mathscr{P}$ of partitions of $[0, t]$ of the form $P = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ with $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = t$ and $n \ge 2$.

Define

$$H_t^{(P)} = |F_{\alpha_1}| + \sum_{i=1}^{n-1} \tau_{\alpha_i} |F_{\alpha_{i+1}-\alpha_i}|.$$

The family of L_1 functions, $\{H_t^{(P)}; P \in \mathscr{P}\}$, fullfils

(1) $\sup_{P\in\mathscr{P}} \|(1/t)H_t^{(P)}\| \le K$

(2) if $P' \in \mathscr{P}$ refines $P \in \mathscr{P}$ then $H_t^{(P)} \le H_t^{(P')}$.

The validity of (1) follows immediately from the definition and boundedness of $\{F_t\}$:

$$\left\| \frac{1}{t} H_t^{(P)} \right\| \leq \|F_{\alpha_1}\| + \sum_{i=1}^{n-1} \left\| \tau_{\alpha_i} |F_{\alpha_{i+1}-\alpha_i}| \right\|$$

$$\leq \frac{1}{t} K \left(\alpha_1 + \sum_{i=1}^{n-1} (\alpha_{i+1} - \alpha_i) \right) = K.$$

In order to prove (2), let us first note that it is clearly sufficient to consider the case where $P = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ is refined by adding one point, say α. If $0 < \alpha < \alpha_1$ then, indeed,

$$|F_{\alpha_1}| = |F_\alpha + T_\alpha F_{\alpha_1 - \alpha}| \leq |F_\alpha| + \tau_\alpha |F_{\alpha_1 - \alpha}|.$$

Similarly, for the case $\alpha_i < \alpha < \alpha_{i+1}$ with $1 \leq i \leq n-1$,

$$\tau_{\alpha_i} |F_{\alpha_{i+1} - \alpha_i}| = \tau_{\alpha_i} |F_{\alpha - \alpha_i} + T_{\alpha - \alpha_i} F_{\alpha_{i+1} - \alpha}|$$

$$\leq \tau_{\alpha_i} |F_{\alpha - \alpha_i}| + \tau_{\alpha_i} \tau_{\alpha - \alpha_i} |F_{\alpha_{i+1} - \alpha}|$$

$$= \tau_{\alpha_i} |F_{\alpha - \alpha_i}| + \tau_\alpha |F_{\alpha_{i+1} - \alpha}|.$$

Now, since for any two partitions there is one that refines both (take their union), there is a sequence $P_i \in \mathscr{P}$ such that $H_t^{(P_i)}$ is increasing and $\lim_{i\to\infty} \|H_t^{(Pi)}\| = \sup_{P\in\mathscr{P}} \|H_t^{(P)}\|$. We define

$$H_t = \lim_{i\to\infty} H_t^{(P_i)} \quad \text{a.e.}$$

Clearly any other such sequence would yield the same limit. Since for any partition of the form $P = \{0, \alpha, t\}$ we have

$$|F_t| = |F_\alpha + T_\alpha F_{t-\alpha}| \leq |F_\alpha| + \tau_\alpha |F_{t-\alpha}| = H_t^{(P)}$$

(i) is proved. Considering (1) above, we also have (ii). The proof of the theorem will be completed by showing the additivity of $\{H_t\}$ with respect to $\{\tau_t\}$.

Fix $t > 0$, $s > 0$ and consider arbitrary partitions $P' = \{\alpha_0, \ldots, \alpha_n\}$ and $P'' = \{\beta_0, \ldots, \beta_m\}$ with $n \geq 2$, $m \geq 2$, of $[0, t]$ and $[0, s]$ respectively. Denote by $P'P''$ the partition $\{\alpha_0, \ldots, \alpha_n, t + \beta_i, \ldots, t + \beta_m\}$ of $[0, t+s]$. Then

$$H_{t+s}^{(P'P'')} = |F_{\alpha_1}| + \sum_{i=1}^{n-1} \tau_{\alpha_i} |F_{\alpha_{i+1} - \alpha_i}| + \tau_{\alpha_n} |F_{t+\beta_1 - \alpha_n}|$$

$$+ \sum_{i=1}^{m-1} \tau_{t+\beta_i} |F_{t+\beta_{i+1} - t - \beta_i}| = H_t^{(P')} + \tau_t H_s^{(P'')}.$$

Let then $\{P_i'\}$ and $\{P_i''\}$ be sequences of partitions such that the sequences $H_t^{(P_i')}$ and $H_s^{(P_i'')}$ are increasing and converge to H_t and H_s respectively. Since τ_t is a positive operator, $\tau_t H_s^{(P_i'')} \underset{i\to\infty}{\to} \tau_t H_s$ a.e. Let us take limits, as $i \to \infty$, in

$$H_{t+s}^{(P_i'P_i'')} = H_t^{(P_i')} + \tau_t H_s^{(P_i'')}.$$

Since the left-hand side is increasing, by our definition $H_{t+s} \geq \lim_{i\to\infty} H_{t+s}^{(P_i'P_i'')}$, and we obtain $H_{t+s} \geq H_t + \tau_t H_s$. On the other hand, given a partition P of $[0, t+s]$, refine it (if necessary) to include the point t, and consider P' and P'', on $[0, t]$ and $[0, s]$ respectively: P' is the partition induced by P on $[0, t]$ and P'' is the one induced by P on $[t, t+s]$ and "shifted" to $[0, s]$. A similar argument, choosing a sequence P_i such that P_{i+1} refines P_i, $t \in P_i$ and $H_{t+s}^{(P_i)}$ converges to H_{t+s} a.e., shows

$$H_{t+s} \leq \lim_{i\to\infty} H_t^{(P_i')} + \tau_t \lim_{i\to\infty} H_s^{(P_i'')} \leq H_t + \tau_t H_s.$$

Note that in the construction above we actually use only the process $\{|F_t|\}_{t>0}$ and the semigroup $\{\tau_t\}$. The additivity of $\{F_t\}$ with respect to $\{T_t\}$ makes $\{|F_t|\}$ subadditive with respect to $\{\tau_t\}$; that is $|F_{t+s}| \leq |F_t| + \tau_t |F_s|$ for all $t > 0, s > 0$. Thus we have the following theorem.

THEOREM 2.2. *A bounded positive process, subadditive with respect to a strongly continuous semigroup of positive linear contractions, has a dominating, positive and additive process, with the same bound.*

In proving Theorem 1.2 we shall find it convenient to use the following notation: $f_t = (1/t)F_t$, $h_t = (1/t)H_t$; also, w-lim and s-lim will denote the weak and strong limits in L_1. The L_1 function f in Theorem 1.2 shall be obtained as the limit of a weakly convergent sequence. It is known that a bounded sequence in L_1 which is dominated by a fixed L_1 positive function is weakly sequentially compact (see Theorem IV.8.9 in [4]). For our purposes a certain sharpening of that result is needed.

LEMMA 2.1. *Let $\phi_n \in L_1$ and $|\phi_n| \leq \psi_n \in L_1^+$ such that there exists $\psi \in L_1^+$ with $\|\psi_n - \psi\| \underset{n\to\infty}{\to} 0$. Then $\{\phi_n\}$ is weakly sequentially compact.*

Proof. Let $\psi_n' = \psi_n \wedge \psi$; then $0 \leq \psi_n' \leq \psi_n$. Now write ϕ_n as $\phi_n = \phi_n' + \phi_n''$ where

$$\phi_n' = (-\psi_n') \vee (\phi_n \wedge \psi_n')$$

$$\phi_n'' = \phi_n - \phi_n'.$$

For the sequence ϕ_n' we have $|\phi_n'| \leq \psi$ and by the theorem mentioned above is weakly sequentially compact. As for ϕ_n'', by definition

$$|\phi_n''| = |\phi_n - \phi_n'| \leq \psi_n - \psi_n' \leq |\psi_n - \psi|.$$

This implies that $\|\phi_n''\| \underset{n\to\infty}{\to} 0$.

We shall also need the following fact; here again continuity at the origin is not needed.

LEMMA 2.2. *For any $t > 0$, $F_t = \text{s-lim}_{\varepsilon\to 0^+} \int_0^t T_s f_\varepsilon \, ds$.*

Proof.

$$\begin{aligned}\int_0^t T_s f_\varepsilon \, ds &= \frac{1}{\varepsilon}\int_0^t T_s F_\varepsilon \, ds = \frac{1}{\varepsilon}\int_0^t (F_{s+\varepsilon} - F_s)\, ds \\ &= \frac{1}{\varepsilon}\int_\varepsilon^{t+\varepsilon} F_s \, ds - \frac{1}{\varepsilon}\int_0^t F_s \, ds.\end{aligned}$$

For $\varepsilon < t$ we get

$$\int_0^t T_s f_\varepsilon \, ds = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} F_s \, ds - \frac{1}{\varepsilon}\int_0^\varepsilon F_s \, ds.$$

Now, since the process is bounded, the first term converges in norm to F_t, whereas the second term converges to zero.

Proof of Theorem 1.2. Let H_t be the dominating positive process for F_t, constructed in Theorem 2.1. Let $\lim_{t\to 0^+}(1/t)H_t = \lim_{t\to 0^+} h_t = h$ a.e., as given in Theorem 1.5. Define a process H_t' by

$$H_t' = \int_0^t \tau_s h \, ds$$

and consider the decomposition

$$H_t = H_t' + H_t''.$$

Then the following holds:

$$H_t'' \text{ is positve and } \lim_{t\to 0^+} (1/t) H_t'' = 0 \text{ a.e.}$$

To see that, we take any sequence $\varepsilon_n \to 0$ and consider the sequence $\psi_n = h_{\varepsilon_n} \wedge h$. Then $0 \leq \psi_n \leq h$ and, obviously, $\psi_n \to h$ a.e. Being bounded by $h \in L_1$, by the dominated convergence theorem it also converges in norm:

$$\|h - \psi_n\| = \int (h - \psi_n)\, d\mu \underset{n\to\infty}{\to} 0.$$

Since $\int_0^t \tau_s g\, ds$, for a fixed t, acting on $g \in L_1$, is a bounded linear operator in L_1, we also got

$$\underset{n\to\infty}{\text{s-lim}} \int_0^t \tau_s \psi_n\, ds = \int_0^t \tau_s h\, ds.$$

Therefore, using Lemma 2.2,

$$H_t = \underset{n\to\infty}{\text{s-lim}} \int_0^t \tau_s h_{\varepsilon_n}\, ds \geq \underset{n\to\infty}{\text{s-lim}} \int_0^t \tau_s \psi_n\, ds = \int_0^t \tau_s h\, ds = H_t'.$$

Pointwise convergence of $(1/t)H_t''$ to zero is given in Theorem 1.5.

Now we obtain the L_1 function f in Theorem 1.2. Let

$$\phi_n = (-\psi_n) \vee (f_{\varepsilon_n} \wedge \psi_n).$$

Then $|\phi_n| \leq \psi_n$, so that the sequences ϕ_n and ψ_n fullfil the condition of Lemma 2.1 (with ψ in the Lemma equal to h). Thus, by passing to a subsequence, if necessary, we may assume that ϕ_n converges weakly, say to $f^* \in L_1$. Put $f = T_0 f^*$. Define a process $F_t' = \int_0^t T_s f\, ds$ and consider the decomposition $F_t = F_t' + F_t''$. By the results in **[5]**, **[7]** and **[10]**, $\lim_{t\to 0^+}(1/t)F_t' = T_0 f = f$ a.e. Hence the proof shall be completed by showing that $\lim_{t\to 0^+}(1/t)F_t'' = 0$ a.e. This will follow from $|F_t''| \leq H_t''$, which we now prove.

Observe, first, as in the proof of Lemma 2.1, that we have $|f_{\varepsilon_n} - \phi_n| \leq h_{\varepsilon_n} - \psi_n$. To evalaute F_t'' express F_t, F_t', H_t and H_t' as the limits of integrals. From Lemma 2.2:

$$F_t = \underset{n\to\infty}{\text{s-lim}} \int_0^t T_s f_{\varepsilon_n}\, ds \quad \text{and} \quad H_t = \underset{n\to\infty}{\text{s-lim}} \int_0^t \tau_s h_{\varepsilon_n}\, ds;$$

actually only weak convergence will be needed. Since $\int_0^t T_s g\, ds$ (or $\int_0^t \tau_s g\, ds$) applied to $g \in L_1$ is a bounded linear operator,

$$f^* = \underset{n\to\infty}{\text{w-lim}}\, \phi_n \quad \text{implies}$$

$$\int_0^t T_s f\, ds = \int_0^t T_s f^*\, ds = \underset{n\to\infty}{\text{w-lim}} \int_0^t T_s \phi_n\, ds, \quad \text{and}$$

$$h = \underset{n\to\infty}{\text{s-lim}}\, \psi_n \quad \text{implies}$$

$$\int_0^t \tau_s h\, ds = \underset{n\to\infty}{\text{s-lim}} \int_0^t \tau_s \psi_n\, ds \quad \left(= \underset{n\to\infty}{\text{w-lim}} \int_0^t \tau_s \psi_n\, ds\right).$$

Now

$$F_t'' = F_t - \int_0^t T_s f\, ds = \underset{n\to\infty}{\text{w-lim}} \int_0^t T_s f_{\varepsilon_n}\, ds - \underset{n\to\infty}{\text{w-lim}} \int_0^t T_s \phi_n\, ds$$

$$= \underset{n\to\infty}{\text{w-lim}} \int_0^t T_s (f_{\varepsilon_n} - \phi_n)\, ds.$$

Since $|f_{\varepsilon_n} - \phi_n| \le h_{\varepsilon_n} - \psi_n$, this gives

$$|F_t''| \le \operatorname*{w-lim}_{n\to\infty} \int_0^t \tau_s(h_{\varepsilon_n} - \psi_n) = \operatorname*{w-lim} \int_0^t \tau_s h_{\varepsilon_n}\, ds$$

$$= \operatorname*{w-lim}_{n\to\infty} \int_0^t \tau_s \psi_n \, ds = H_t - \int_0^t \tau_s h \, ds = H_t''.$$

This completes the proof.

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